Theoretical Relations Between Risk Premiums and Conditional Variances

David K. Backus
Stern School of Business, New York University, New York, NY 10012

Allan W. Gregory
Department of Economics, Queen's University, Kingston, Ontario, Canada K7L 3N6

Many statistical models of time-varying risk premiums, including the autoregressive conditional heteroscedasticity (ARCH)-in-mean, attempt to exploit a relation between risk premiums and conditional variances or covariances of asset returns. We examine this relation in numerical versions of a dynamic asset-pricing theory and show that it can be increasing, decreasing, flat, or nonmonotonic. Its shape depends on both the preferences of the representative agent and the stochastic structure of the economy. Without additional structure, the theory does not provide either a general foundation for ARCH-in-mean specifications or a simple interpretation of their parameters.

KEY WORDS: ARCH-in-mean; Bond and stock returns; Dynamic asset-pricing theory.

In the history of financial economics, perhaps the longest chapter concerns the relation between risk and return. Over the last few decades, this relation has been refined theoretically and used in a wide range of academic and business applications. In the capital asset-pricing model (CAPM) risk and return for an asset \( i \) are related by

\[
E_{r_i} = k \text{cov}(r_i, r_m),
\]

where \( r_i \) is the excess return on the asset, \( r_m \) is the excess return on the market portfolio, and \( k \) is a factor of proportionality. In early applications the expected return, its covariance with the market, and \( k \) were assumed, at least implicitly, to be constant. More recently, relations similar to (1) with constant \( k \) have been crossbred with econometric methods for estimating time-varying conditional second moments to produce models with the potential to explain time variation in expected excess returns. Their maintained hypothesis is that risk premiums can be expressed as increasing functions of the conditional covariance of the asset with the market.


This work has proceeded largely on the strength of convenient econometric specifications for the dynamics of conditional second moments. The theoretical justification is generally the static CAPM, as in Equation (1), with time variation in conditional variances or covariances accounting for changes in risk premiums over time. For the most part, there is little connection with the dynamic asset-pricing theory associated with Cox, Ingersoll, and Ross (1985), Lucas (1978), and others.

In this article, we ask whether dynamic theory can be used to deduce a relation between risk premiums and conditional second moments that can be exploited in empirical studies. This relation might be proportional, as in (1), or simply increasing as in most of the ARCH-M literature. We argue that, although theory may lead to a monotonic relation between risk premiums and conditional variances, it does not guarantee it. In a series of numerical examples, we show that the relation between the risk premium and the conditional...
variance of the excess return can have virtually any shape: It can be increasing, decreasing, flat, or even nonmonotonic. The shape depends on both the preferences of the representative agent and the probability structure across states. Put somewhat differently, the question is whether conditional second moments are likely to be good proxies for risk premiums. In theory the answer may be either yes or no, or even both, depending on the asset and the stochastic structure of the economy. Evidence from less restrictive specifications than ARCH-M, including works by Glosten, Jagannathan, and Runkle (1989) and Harvey (1989, 1991), suggests that the monotonic relation between risk premiums and conditional variances presumed in most ARCH-M applications is not uniformly supported by the behavior of actual asset prices.

We should make clear that we are not considering whether conditional variances are better modeled as ARCH processes à la Engle (1982) and Engle and Bollerslev (1986); by moving averages as in the work of Merton (1980), Pindyck (1984), and Poterba and Summers (1986); by regressions on observable “instruments” as in the work of Campbell (1987) and Harvey (1989); or by seminonparametric techniques as in the work of Gallant and Tauchen (1989), Harvey (1991), Pagan and Hong (1988), and Pagan and Ullah (1988). We focus instead on the specific issue of whether the conditional variance, or other conditional second moments, can be used to account for time variation in expected returns in theoretical economies. This issue must be addressed regardless of the method used to estimate conditional variances and covariances and in that sense is more fundamental.

We proceed as follows. In Section 1, we review the empirical evidence associating expected returns with conditional variances. Since most of this research employs some variant of the ARCH-M model, we concentrate on this specification and review its track record as a statistical model of time-varying expected returns. In Section 2, we build a theoretical economy, and in Section 3, we explore the relation in this economy between expected returns and conditional variances and covariances. We also consider the implications of more general preference relations designed to moderate empirical anomalies associated with representative-agent theory with additively separable preferences. We show that nonmonotonic relations can also be generated in economies that match many of the moments of consumption and asset returns in the United States. We conclude with suggestions for future research.

1. ARCH-M MODELS OF RISK PREMIUMS

The ARCH-M model has become one of the workhorses of econometric research on time-varying expected returns, as evidenced by the bibliography of Bollerslev et al. (1992). In this section we review the model and describe some prominent applied studies in empirical finance documenting its ability to account for expected returns on several assets. In the standard specification, for which the article by Engle et al. (1987) served as a prototype, the return \( r_{t+1} \) on an asset held from \( t \) to \( t+1 \) is decomposed into its conditional mean \( m_t \) and a forecast error \( \epsilon_{t+1} \):

\[
r_{t+1} = m_t + \epsilon_{t+1},
\]

where \( E \) is the expectations operator conditional on the date-\( t \) information set, \( m_t = E_{t-1} r_{t+1} \), and \( \epsilon_{t+1} = r_{t+1} - E_{t-1} r_{t+1} \). If \( r_{t+1} \) is the excess return (the return net of the risk-free rate) then the conditional mean, \( m_t \), is generally referred to as a risk premium. Both \( m_t \) and the conditional variance of \( r_{t+1} \) and \( \epsilon_{t+1} \), which we denote by \( h^2_t \), can vary with time. One of the strengths of ARCH models is that they provide a convenient econometric specification for \( h_t \) that appears to correspond reasonably well with time variation in the conditional variances of several economic time series. In its simplest form, the conditional variance is first-order ARCH, or ARCH(1),

\[
h^2_t = \alpha_0 + \alpha \epsilon^2_{t-1},
\]

but longer lags of \( \epsilon^2_t \) (higher order ARCH), lags of \( h^2_t \) (generalized ARCH, or GARCH) and exponential functional forms (EGARCH) also have been used in some applications.

For our purpose the critical element of the analysis is the conditional mean. The ARCH-M model of the risk premium postulates dependence of \( m_t \) on both a vector of explanatory variables \( x_t \) and an increasing function \( g \) of the conditional standard deviation:

\[
m_t = \beta' x_t + \gamma g(h_t).
\]

If \( r \) is the return on the market portfolio, then where \( g(h) = h^2 \) and \( \beta = 0 \), Equation (4) is identical to (1). The parameter \( \gamma \) measures the effect of the conditional variance on the mean return and, when \( r \) is an excess return, on the risk premium. In principle, Equation (4) could be combined with any estimator of the conditional variance, but the tractability of the ARCH specification has made it by far the most common choice. Several recent works, however, focused specifically on the specification of the conditional variance and suggested that the ARCH process, Equation (3), may be inadequate (e.g., see Nelson 1991; Pagan and Hong 1990; Pagan and Ullah 1988). Whatever the merits of this argument, we think the more fundamental issue is whether \( h_t \) is a suitable proxy for the conditional mean. But before we address this concern let us document some of the empirical evidence using the ARCH-M specification.

Variants of Equations (2)–(4) have been used, with some success, to characterize returns on short- and long-term bonds, on common stock, and on foreign currency forwards and futures. Engle et al. (1987) let \( x_t \) be a constant and used a fourth-order ARCH process [see their eqs. (19) and (20)]. They estimated this model using excess returns of six-month (over three-month) U.S. treasury bills and 20-year corporate bonds at quar-
Quarterly intervals and of two-month (over one-month) treasury bills at monthly intervals. Citing significant coefficient estimates for $\alpha$ and $\gamma$ and results from a battery of diagnostic tests, they argued that their model accounts for the observed variability of the term premium. Their applications use $g(h) = h$ or $\log h$ and yield positive estimates of $\gamma$. Their preferred model for excess returns on two-month treasury bills is

$$m_t = 0.355 \times 0.135 \log h_t$$

(4.38) (3.36)

$$h_t^2 = 0.005 + 1.58 \sum_{k=1}^{t} \alpha_k e_{t-k}^2,$$

(2.22) (5.56)

where $\alpha_k = (5 - k)/10$. The numbers in parentheses are $t$ statistics, so there is statistically significant evidence (at conventional test sizes) of a conditional variance effect on the expected return.

Subsequent studies have extended this work to other assets and data sets. French et al. (1987) found that risk premiums on monthly excess stock returns from 1928 to 1984 have been positively correlated with their conditional variances and, using $g(h) = h^2$, cited Merton (1980) to interpret the parameter $\gamma$ as the coefficient of relative risk aversion. Their estimated model for the entire sample is (see their table 5)

$$m_t = 0.201 \times 10^{-3} + 2.410 h_t^2$$

(2.54) (2.58)

$$h_t^2 = 0.063 \times 10^{-5} + 0.918 h_{t-1}^2$$

(10.5) (306.)

$$+ 0.121 e_{t-1}^2 + 0.043 e_{t-2}^2.$$  

(17.3) (6.14)

Note the significant mean effect and the plausible estimate ($\gamma = 2.41$) of the risk-aversion parameter. Baillie and DeGennaro (1990), however, found for the same data that, although GARCH(1,1) works quite well in explaining the dynamics of the conditional variance, the mean effect is negligible when the conditional distribution is $t$ rather than normal. Bodurtha and Mark (1991) built an ARCH-inspired CAPM and examined risk premiums on monthly returns of five size-based stock portfolios. They modeled the market excess return as, in one case, an ARCH-M process and specified the “betas” on the portfolios as ARCH-like functions of lagged forecast errors on market and portfolio returns. They found strong evidence in this specification of a relation between conditional variances and covariances and time variation in the risk premium on the market return. In another application, Nelson (1991) found empirical support for his EGARCH specification using daily U.S. stock returns.

A sizable body of work on foreign currencies has had mixed success relating risk premiums to conditional variances. Domowitz and Hakko (1985) were the first, and they found little evidence of ARCH-M effects for monthly returns from holding five foreign currencies. Asymptotic $t$ statistics for the hypothesis $\gamma = 0$ were typically around 1. Diebold and Pauly (1988) provided slightly stronger evidence of a relation between $m_t$ and $h_t$ for the dollar/Deutschemark rate in a multiequation macroeconomic model of the exchange rate. McCurdy and Morgan (1987, 1988) extended this work to foreign currency futures but failed to uncover any evidence that mean returns are related to conditional variances, modeled as univariate GARCH, despite strong indications otherwise that risk premiums vary over time. Baillie and Bollerslev (1990) examined forward rates at weekly intervals and also found strong evidence of ARCH but little that ARCH-M helps to explain risk premiums. Mark (1988), however, used a somewhat different specification and found stronger evidence that risk premiums on forward foreign-exchange contracts are related to movements in conditional second moments. As in the work of Bodurtha and Mark (1991), the market return is ARCH-M and the betas on forward contracts are related, as in ARCH models, to lagged forecast errors.

The most recent innovation has been to extend ARCH to vector processes and explain simultaneously risk premiums on several assets. Bollerslev et al. (1988) used a trivariate model to explain quarterly returns on treasury bills, 20-year treasury bonds, and equity. Their richer specification for the conditional variance vector permits lagged forecast errors on each asset to influence estimated conditional covariances for all three assets. They also modify the mean effect [Eq. (4)] so that the conditional covariance with the market portfolio, rather than the own variance, enters the risk premium. Since the market portfolio never has less than 80% equity, this is close to using the conditional covariance with the equity return to explain risk premiums. They also constrained the effect of a change in the conditional covariance to be the same for each asset and, like French et al. (1987), interpreted this parameter as the coefficient of relative risk aversion. The estimated value is .499, which is a plausible value for the coefficient of relative aversion but considerably different from the French et al. estimate. In related work, Friedman and Kuttner (1988) examined quarterly excess returns on long bonds and equity using two estimators, including a GARCH(1,1), of conditional second moments. With the GARCH(1,1)-M specification they found a significant relation between risk premiums and conditional variances and covariances and estimated the coefficient of relative risk aversion to be 2. Engle et al. (1990) applied a more parsimonious factor ARCH covariance structure to monthly returns on 2- to 12-month treasury bills and found that two factors capture most of the relevant conditional covariances in this case.

The evidence, in short, is that in some cases ARCH-M and related models reveal strong correlations, typically positive, between mean excess returns and conditional variances, but in others the correlation is slight. There is also a suggestion in the literature (Bollerslev
et al. 1988; French et al. 1987; Friedman and Kuttner 1988), that the coefficient \( \gamma \) in Equation (4) can be interpreted, when \( g(h) = h^2 \), as the coefficient of relative risk aversion, and therefore that positive estimates provide support for equilibrium asset-pricing theory. We return to these issues in Section 3.

2. A THEORETICAL FRAMEWORK

We want to examine the relation between risk premiums and conditional variances of asset returns in a theoretical model and ask what we might expect the relation to look like. To make this discussion concrete, we report numerical examples using Mehra and Prescott's (1985) dynamic exchange economy. This economy shares many of the important features of Breeden (1985), Brock (1982), Cox et al. (1985), LeRoy (1973), Lucas (1978), Merton (1973), and Rubinstein (1976). Mehra and Prescott's use of finite-state Markov chains gives us both analytical tractability and flexibility in specifying the stochastic structure. Equilibrium asset prices are readily computed recursively from solutions to linear systems of equations. We compute, in particular, the equilibrium values of risk premiums and conditional second moments for excess returns on equity.

In the theoretical economy, a representative agent consumes a stochastic endowment of a single good. The endowment \( y \) varies over time according to \( y_{t+1} = x_{t+1} y_t \). Rates of growth \( x \) take on a finite number of values, denoted \( \lambda_i \) for \( i = 1, \ldots , I \), and their evolution is described by a stationary Markov chain with transition probabilities \( \pi_{ij} = \Pr(x_{t+1} = \lambda_j \mid x_t = \lambda_i) \), which we collect in the matrix \( \Pi \). We will refer to \( i \) as the state of the economy at date \( t \) if \( x_t = \lambda_i \). The representative agent maximizes the expected utility function

\[
E_t \sum_{k=0}^{\infty} \beta^k u(c_{t+k}),
\]

with \( u(c) = (c^{1-\alpha} - 1)/(1 - \alpha) \), \( \alpha > 0 \), and \( 0 < \beta < 1 \). An equilibrium is then characterized by a set of state-contingent prices for which \( c_t = y_t \) in all periods and states.

Prices of pure state-contingent claims are computed as ratios of marginal utilities, evaluated at \( c = y \), and prices of assets as sums of prices of pure contingent claims. Asset returns are then derived from prices. Both prices and returns can be characterized as functions of the current state, which in this economy consists of vectors of values corresponding to the states of the economy.

We start with prices of risk-free bonds—the prices, that is, of claims to one unit of the commodity in all states \( k \) periods in the future. As described by Backus, Gregory, and Zin (1989), the price of a \( k \)-period bond in state \( i \) is \( q_i^k = \Sigma \beta^j b_{ij}^k \), where \( B \) is the matrix of elements \( b_{ij} = \pi_{ij} \beta \lambda_j^{-\alpha} \) and \( b_{ij}^k \) is the \( ij \)th element of \( B^k \). The return from holding a \( k \)-period bond for one period, from state \( i \) at date \( t \) to state \( j \) at date \( t + 1 \), is \( 1 + r_{ij}^k = q_{ij}^k - q_{ij}^{k-1} \), with \( q_{ij}^0 = 1 \). The return \( r_{ij}^k \) on a one-period bond is the short rate, the risk-free one-period return on a one-period bond, and does not depend on \( j \). Holding-period yields on longer bonds, however, are not certain, and we label the difference between their expected returns and the risk-free or short rate a bond premium, \( b_{ij}^k = \Sigma \pi_{ij} q_{ij}^k - r_{ij}^k \). Thus the evolution of prices and returns is governed by the process generating evolution of the state.

One share of equity in state \( i \) at date \( t \) is a claim to the entire stream of future endowments, starting with period \( t + 1 \). As described by Mehra and Prescott (1985, pp. 151–153), the price per unit of output, denoted by the vector \( w \), is the solution to the linear system \( (I - A)w = d \), where \( A \) is the matrix with elements \( a_{ij} = \pi_{ij} \beta \lambda_j^{-\alpha} \) and \( d \) is the vector with elements \( d_i = \Sigma \mu_{ij} \). This defines a value \( w_i \) for each state \( i \). A necessary and sufficient condition for expected utility to exist in this economy is that \( A \) be stable (all its eigenvalues are less than 1 in absolute value), which guarantees that each \( w_i \) is positive. The return from holding equity for one

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<th>Conditional variance of equity return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.995</td>
<td>-.517 \times 10^{-3}</td>
<td>.172 \times 10^{-3}</td>
<td>-.155 \times 10^{-3}</td>
<td>.156 \times 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>1.015</td>
<td>-.833 \times 10^{-3}</td>
<td>.280 \times 10^{-4}</td>
<td>-.249 \times 10^{-3}</td>
<td>.250 \times 10^{-4}</td>
</tr>
</tbody>
</table>

Table 2. Risk Premiums and Conditional Variances for Example 2

NOTE: Parameter values are as in Table 1, except that \( \rho = 3 \). See text on page 182.
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Persistence Parameter

Figure 1. Slopes of Risk/Return Regressions.

period, for a transition from state $i$ to state $j$, is $1 + r_{ij}^* = \lambda_j(w_i + 1)/w_i$. We define the **equity premium** as the expected difference between this return and the short rate, $e_p = \Sigma_i \pi_i(r_{ij}^* - r_i)$. The equity premium varies, in general, across states $i$.

These relations define returns and risk premiums in each state. Conditional variances follow from the distribution of returns, given by their values in each state and the probability structure $\Pi$. The relevant conditional variances for comparison with the ARCH-M model are the conditional variances of asset returns. In state $i$ these conditional variances are

$$\text{var}_i r_{ij}^* = \Sigma_j \pi_i(r_{ij}^* - bp_k)^2$$

$$\text{var}_i r_{ij}^* = \Sigma_j \pi_i(r_{ij}^* - e_p)^2,$$

where var$_i$ denotes the conditional variance if the current state is $i$.

Both risk premiums and conditional variances of excess returns, then, are functions of the current state. Their evolution through time is described by the Markov chain governing evolution of the state. The question for Section 3 is how these two sets of functions of the state are related.

### 3. RISK PREMIUMS AND CONDITIONAL VARIANCES

We can now approach the question of this article: Can risk premiums be expressed as increasing functions of conditional second moments, as assumed in the static CAPM and in recent applications based on time-varying conditional variances and covariances? In our theoretical economy (in fact in any economy that admits an equilibrium in which returns are stationary functions of an underlying state vector) both risk premiums and conditional second moments are functions of the state of the economy and are, in that sense, time varying. We have yet to see, however, that there is any functional relation between them. We will show with a series of numerical examples that the relation can have virtually any form.

Many of the features of the relation between risk premiums and conditional variances can be illustrated most simply in a two-state version of the model. With two states we can, without loss of generality, parameterize the transition probabilities as

$$\pi_{ij} = (1 - \rho) \pi_i + \rho \delta_{ij},$$

where $[\pi_j]$ is the long-run equilibrium, or unconditional, distribution, $\delta_{ij}$ is the Kronecker delta, and $\rho$ is the autocorrelation of any random variable adapted to $i$.

Example 1 in Table 1 reports risk premiums and conditional variances for a particular choice of parameter values. We have dropped the $k$ superscripts on one-period bond returns with the understanding that from now on the bond premium $bp$ is that of a two-period bond (i.e., $bp^2$). In the example, we have set $\alpha = 5$, $\beta = .9$, and $\lambda = 1.005 \pm .010$. Because risk premiums and conditional variances of bond period returns are (at least approximately) constant when the equilibrium distribution is symmetric (see Backus et al. 1989, pp. 392–393), we set $[\pi_j] = [.7, .3]$. The persistence parameter is $\rho = -.3$.

These parameter values imply that in population regressions of risk premiums on conditional variances the slopes are positive—3.01 and 2.39 for bond premiums and equity, respectively. We compute the slopes from regressions of risk premiums on conditional variances, with the equilibrium probabilities used to weight the states. With two states, however, the slopes are simply ratios of differences between risk premiums in the two states to differences between conditional variances. The positive numbers in this example are in rough accord with what Engle et al. (1987), French et al. (1987), and Bollerslev et al. (1988) found in empirical work with ARCH-M models.

We get a much different picture, however, when the persistence parameter $\rho$ is positive. In Example 2 of

<table>
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<tr>
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<th>Conditional variance of equity return</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.9900</td>
<td>$-.786 \times 10^{-3}$</td>
<td>$202 \times 10^{-9}$</td>
<td>$-.462 \times 10^{-5}$</td>
<td>$.706 \times 10^{-7}$</td>
</tr>
<tr>
<td>2</td>
<td>.9975</td>
<td>$-.603 \times 10^{-3}$</td>
<td>$155 \times 10^{-9}$</td>
<td>$618 \times 10^{-6}$</td>
<td>$.582 \times 10^{-7}$</td>
</tr>
<tr>
<td>3</td>
<td>1.0050</td>
<td>$-.555 \times 10^{-3}$</td>
<td>$143 \times 10^{-9}$</td>
<td>$821 \times 10^{-6}$</td>
<td>$.720 \times 10^{-7}$</td>
</tr>
<tr>
<td>4</td>
<td>1.0125</td>
<td>$-.623 \times 10^{-3}$</td>
<td>$163 \times 10^{-9}$</td>
<td>$814 \times 10^{-6}$</td>
<td>$.595 \times 10^{-7}$</td>
</tr>
<tr>
<td>5</td>
<td>1.0200</td>
<td>$-.824 \times 10^{-3}$</td>
<td>$213 \times 10^{-9}$</td>
<td>$960 \times 10^{-6}$</td>
<td>$.733 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

**NOTE:** Parameter values are $\alpha = 5$, $\beta = .9$, $[\pi_j] = [2, 2, 2, 2, 2]$, and $\rho = .2265$. 
Table 2, where \( \rho = .3 \), the slopes are negative: \(-2.93\) (bond premium) and \(-9.92\) (equity premium). For bond prices this may not be surprising: We know (Backus et al. 1989, pp. 382, 385) that the sign of the bond premium is the opposite of the sign of \( \rho \), and this coincides with the switch in sign of the regression. A similar thing happens with equity: As \( \rho \) varies between .2 and .25, equity premiums change sign and the regression slope changes sign as well. A graph of the slopes against \( \rho \) reveals a discontinuity in the regression slopes at \( \rho = 0 \) for the bond premiums and between .20 and .25 for the equity premiums; see Figure 1.

These patterns give us some idea of what we can expect of the theoretical relation between risk premiums and conditional variances. Note, for example, that the slope can have either sign. In Example 1 assets have largest risk premiums in those states in which their returns were least predictable, but in Example 2 it is the reverse. There is no presumption, in other words, that risk premiums are increasing in the conditional variances of their returns. Note too that these features are not driven in any obvious way by the preference parameters or the unconditional distribution over states, since we changed only the persistence parameter \( \rho \). A fortiori there is no direct connection between the slope and the coefficient of relative risk aversion, a connection suggested by French et al. (1987). The same statement applies to the analysis of Bollerslev et al. (1988), who related risk premiums to conditional covariances of returns with the market portfolio: In our theoretical economy the market portfolio is equity, and the covariance of its return with the market is its own variance. This suggests too that the difficulty does not lie in using the conditional variance rather than, as in Equation (1), the covariance with the market portfolio.

Nevertheless, the correlation between risk premiums and conditional variances in the two-state examples is either +1, −1, or, when the conditional variance is constant, undefined. The conditional variance is, in this sense, a very good proxy for the risk premium in these examples. The perfect correlation between risk premiums and conditional variances is an artifact of having only two states, since any two points always lie on a straight line. One question we might ask is whether with more states the risk premium is still highly correlated with the conditional variance or if the relation is, say, a nonlinear but monotonic function. We show that neither is required by the theory.

Table 3 reports various features of a third example for an economy with five states, with growth rates equally spaced and a uniform equilibrium distribution. As before, transition probabilities are characterized by simple persistence [Eq. (5)]. In the example reported, the persistence parameter \( \rho \) was chosen so that the correlation between the risk premium on equity and its conditional variance is only .098. The theoretical \( R^2 \) for a regression of the equity premium on equity and its conditional variance is only .098. It is clear, therefore, that there is no necessary linear relation between the two variables. In Figure 2 we see that it need not be monotonic either, since the relation in the example is U-shaped.

In Table 4 we report a fourth example, in which the economy alternates between two regimes, each consisting of two states. The regimes are both similar to Example 1, but they differ in the variance of the consumption growth rate. Within each regime there is negative serial correlation, in the sense that the probability of staying in a particular state in that regime is less than

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<tr>
<td>1</td>
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<td>.464 \times 10^{-3}</td>
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<td>.767 \times 10^{-2}</td>
</tr>
<tr>
<td>3</td>
<td>.9650</td>
<td>.763 \times 10^{-2}</td>
<td>.237 \times 10^{-2}</td>
<td>.104 \times 10^{-1}</td>
<td>.720 \times 10^{-2}</td>
</tr>
<tr>
<td>4</td>
<td>1.0350</td>
<td>.221 \times 10^{-2}</td>
<td>.724 \times 10^{-3}</td>
<td>.241 \times 10^{-2}</td>
<td>.329 \times 10^{-2}</td>
</tr>
</tbody>
</table>

NOTE: Parameter values are \( \alpha = 5, \beta = .9, \) and

\[
\Pi = \begin{bmatrix}
.80 & .38 & .01 & .01 \\
.89 & .09 & .01 & .01 \\
.01 & .01 & .60 & .38 \\
.01 & .01 & .69 & .09
\end{bmatrix}
\]

which has an equilibrium distribution of \([.35, .15, .35, .15] \).
the unconditional probability. Regimes themselves are positively autocorrelated. The effect on the equilibrium is to give us a sawtoothing relation between the risk premium and the conditional variance, illustrated in Figure 3. The relation between the bond premium and the conditional variance of the return on a two-period bond (not shown) is also nonmonotonic, but not sawtoothing. Unlike the examples of Tables 2 and 3, the absence of monotonicity for bond returns is not associated with negative risk premiums.

There has been speculation in the literature that the relation between the risk premium and the conditional variance must be increasing if the risk aversion parameter is low (e.g., see Abel 1988). Our Examples 1 to 4 do not contradict this suggestion, since the risk-aversion parameter is 5 in all of them. Table 5 reports an example with logarithmic utility \( \alpha = 1 \) in which a small equity premium is nevertheless associated with a large conditional variance. This suggests that, although low-risk aversion, logarithmic utility in particular, may play a role in the relation between risk and return, it is not sufficient to guarantee a positive association between the expected excess return and the conditional variance of equity.

All of these examples are based on time and state additive utility functions, a class of preferences that has been found inadequate in accounting for many of the observed properties of asset prices. With respect to the relation between risk premiums and conditional variances, extending the preference structure to include nonseparable utility (like the habit persistence model of Constantinides 1990) or nonexpected utility (Epstein and Zin 1989) does not restrict the ability of the theory to generate nonmonotonic relations between risk premiums and conditional second moments. In a series of examples (available on request) using these two preference orderings we have found that the same kinds of reversals between risk premiums and conditional variances can occur, often with less extreme values of parameters. In one three-state example with habits, we match all of the Mehra–Prescott (1985) moments and thus “solve” the equity premium puzzle, yet find that the relation between the risk premium and the conditional variance is not monotonic. The essentials are reported in Table 6 with preferences

\[
U_t = \sum_{k=0}^{\infty} \beta^k u(d_{t+k}), \quad u(d) = \frac{1}{\alpha} [d^{1-\alpha} - 1],
\]

and

\[
d_t = c_t - \delta c_{t-1}.
\]

The relationship between the equity premium and the conditional variance for this example is illustrated in Figure 4.

In short, the theory can produce relations between risk premiums and conditional variances of returns of virtually any shape. This lack of theoretical structure between risk premiums and conditional variances may help to explain why such modeling strategies have worked fairly well for some assets, but not at all for others, as we saw in Section 1. It may also account for the apparent

4. FINAL REMARKS

We have seen that theory can generate a variety of relations between risk premiums and conditional variances. Depending on the parameters of the economy the relation can be increasing, decreasing, U-shaped, or sawtoothed. In this sense the statistical relations exploited by ARCH-M models, as well as their close relatives, are not a general consequence of dynamic asset-pricing theory. An outstanding question for the theory then is whether there exist simple restrictions that guarantee monotonic relations between risk premiums and expected excess returns. If so, examples for which ARCH-M specifications indicate strong relations between risk premiums and conditional variances may also provide us with information about the structure of the economy.

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