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STRUCTURAL LAPLACE TRANSFORM AND COMPOUND AUTOREGRESSIVE MODELS

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Abstract. This paper presents a new general class of compound autoregressive (Car) models for non-Gaussian time series. The distinctive feature of the class is that Car models are specified by means of the conditional Laplace transforms. This approach allows for simple derivation of the ergodicity conditions and ensures the existence of forecasting distributions in closed form, at any horizon. The last property is of particular interest for applications to finance and economics that investigate the term structure of variables and/or of their nonlinear transforms. The Car class includes a number of time-series models that already exist in the literature, as well as new models introduced in this paper. Their applications are illustrated by examples of portfolio management, term structure and extreme risk analysis.

Keywords. Nonlinear dynamics; Laplace transform; nonlinear canonical analysis; affine term structure; value at risk; Wishart autoregressive process.
JEL classification numbers. C14, C22.

1. INTRODUCTION

In finance and economics, the dependence betwen the forecasted values of a process and the forecast horizon, called the term structure, is an important topic of theoretical and applied research. In finance, considerable attention is given to term structures of variables such as interest rates, volatility of financial assets and optimal portfolio allocations. The forecasts generally concern the exponential transforms of variables of interest, and therefore, require the computation of real conditional Laplace transforms. This paper introduces a new class of non-Gaussian time-series models, called the compound autoregressive (Car)[1]. models. The Car models are distinguished from other processes by their specifications based on the conditional Laplace transforms. The advantage of this approach, compared with the commonly used conditional mean- or variance-based specification, is simple derivation and verification of stationarity and ergodicity conditions. Another advantage, compared with the conditional density (distribution)-based specification is that it provides a unified treatment for discrete, continuous and mixed variables. Furthermore, the conditional Laplace transform specification ensures that any forecast of a Car process, or its nonlinear transforms, can be found easily, and written in a closed form. This characteristic is very important in the term structure analysis of financial and economic variables, and their nonlinear transforms. In particular, the use of discrete time Car models facilitates the term structure analysis of interest rates, commonly conducted in a continuous time setup that entails restrictive time coherency conditions (see Duffie *et al.*, 2003; Gouriéroux *et al.*, 2005).

The class of Car processes is very large and includes a variety of linear and nonlinear time series. It differs from other extensions of autoregressive moving-average (ARMA) models that appeared recently in the literature, such as the generalized linear GARMA models (see Li, 1994; Kuchler and Sorensen, 1997; Benjamin *et al.,* 2003; Fokianos and Kedem, 2004). The advantage of GARMA models is simple estimation by partial likelihood. However, the strength of Car models lies in the availability and closed form of any forecasting distribution.

The paper is organized as follows. Section 2 defines the conditional Laplace transform of a Car process and derives the closed-form expressions of forecasting distributions at any horizon. Examples of Car processes are described in Section 3. Weak and geometric ergodicity are characterized in Section 4. A comprehensive classification of univariate time-reversible Car processes is given in Section 5. Section 6 illustrates the application of Car to portfolio allocations, term structure of interest rates, and introduces a new, and so far unexplored, application to extreme risks. Section 7 concludes the paper. The proofs are given in the Appendices.

2. THE COMPOUND AUTOREGRESSIVE MODEL

The dynamics of a Car process are defined in terms of the conditional Laplace transform which is the conditional expectation $E[\exp(-z'Y_t) \mid \underline{Y_{t-1}}]$, where z is a complex vector argument, and $\underline{Y_{t-1}}$ denotes the σ-algebra generated by the lagged values of the process. Below, we define the Car process and show that the nonlinear forecasts based on Car and standard Gaussian AR(1) have similar expressions.

2.1. Definition

DEFINITION 1. *The vector process Y of dimension n is compound autoregressive of order p[Car(p)], if and only if the conditional distribution of Y_t given $\underline{Y_{t-1}}$ admits the conditional Laplace transform:*

$$E\left[\exp(-z'Y_t) \mid \underline{Y_{t-1}}\right] = \exp[-a_1'(z)Y_{t-1} - \cdots - a_p'(z)Y_{t-p} + b(z)], \tag{1}$$

where $a_p \neq 0$, *for any multivariate* $z = u + \mathrm{i}w$ *with complex components, such that the conditional expectation exists.*

The conditional log-Laplace transform in eqn (1) is an affine function of the p most-recent lagged values of the process. This implies that Y is a Markov process of order p.

The coefficients in the series expansion of the conditional Laplace transform allow for identification of the conditional moments (if these exist). In particular, the logarithm of the conditional Laplace transform admits an expansion with coefficients related to the conditional multivariate cumulant moments (see McCullagh, 1987). Proposition 1 follows.

Proposition 1. *If the conditional moments of all orders exist and the conditional distribution is characterized by the sequence of conditional moments, then the process admits a* Car(p) *representation if and only if the conditional multivariate cumulant moments are affine functions of* $Y_{t-1}, \ldots, Y_{t-p}$.

In particular, the conditional mean is affine. Therefore, when the conditional mean exists, a Car process is a conditional linear AR(1) [CLAR(1)] model, as defined in Grunwald *et al.* (2000). Like any Markov process, a Car process of order p is equivalent to a Car process of order 1 obtained by stacking the lagged values into a vector of dimension p.

Proposition 2. *The process* Y *is a* Car(p) *process if and only if the process* $(\widetilde{Y}_t) = (Y_t', Y_{t-1}', \ldots, Y_{t-p+1}')'$ *is a* Car(1) *process.*

Proof. See Appendix A. □

Therefore, without loss of generality only Car(1) processes are considered in the sequel.

2.2. Invariant distributions

Proposition 3. *The log-Laplace transform of an invariant distribution of a* Car(1) *process is a function* c *such that:* $b(z) = c(z) - c[a(z)]$.

Proof. By the invariance property and the iterated expectation theorem, we get:

$$\begin{aligned}\exp[c(z)] &= E[\exp(-z'Y_t)] = E\big[E\big[\exp(-z'Y_t) \mid \underline{Y_{t-1}}\big]\big] \\ &= E\big[\exp\big[-a(z)'Y_{t-1} + b(z)\big]\big] \\ &= \exp[c[a(z)] + b(z)].\end{aligned}$$ □

The conditions of existence and uniqueness of function c are not always satisfied (see Section 4 for a discussion of ergodicity). When Y_t has an invariant distribution, the conditional Laplace transform can be parameterized either by a and b, or by a and c. In the first case, functional parameters a and b represent the time dependence and the distribution of innovations, respectively (see Section 3.1); in the second case, functional parameters a and c represent the serial dependence and the marginal distribution, respectively. Under the latter functional parameterization, we get:

$$E\left[\exp(-z'Y_t) \mid \underline{Y_{t-1}}\right] = \exp\left[-a(z)'Y_{t-1} + c(z) - c[a(z)]\right]. \tag{2}$$

2.3. Nonlinear Forecasting

An important characteristic of Car processes is the existence of closed form of the forecast of any nonlinear transform of a Car process at any horizon. For comparison, Grunwald *et al*. (1996, 2000) focus on linear forecasts, which are less often used in financial applications. The forecasting distribution of a Car(1) process at any horizon h is easily obtained by recursive substitution.

Proposition 4. *For a* Car(1) *process we get:*

$$E\left[\exp(-z'Y_{t+h}) \mid \underline{Y_t}\right] = \exp\left[-a^{\circ h}(z)'Y_t + \sum_{k=0}^{h-1} b[a^{\circ k}(z)]\right], \tag{3}$$

where $a^{\circ h}$ denotes function a compounded h times with itself.

The nonlinear forecast function of a Car extends the forecast function of an autoregressive Gaussian model of order 1 where $a^{\circ h}(u) = \rho^h u$ and ρ is the autoregressive coefficient. The forecasting distribution has a simple form for a stationary process Y_t which has a stationary (invariant) distribution with log-Laplace transform c, since:

$$\sum_{k=0}^{h-1} b[a^{\circ k}(z)] = c(z) - c[a^{\circ h}(z)].$$

For a Gaussian autoregressive process, this expression defines the distribution of innovations at horizon h given the marginal distribution and the autocorrelation coefficient ρ.

Corollary 1. *For a* Car(1) *process with log-Laplace transform c, we get:*

$$E\left[\exp(-z'Y_{t+h}) \mid \underline{Y_t}\right] = \exp\left[-a^{\circ h}(z)'Y_t + c(z) - c[a^{\circ h}(z)]\right]. \tag{4}$$

Similarly, it is possible to derive a closed-form expression of the joint conditional Laplace transform for $Y_{t+1}, \ldots, Y_{t+h}$.

Proposition 5. *We get:*

$$E\left[\exp(z'_{t+1}Y_{t+1} + \cdots + z'_{t+h}Y_{t+h}) \mid \underline{Y_t}\right] = \exp\left[A(t,t+h)'Y_t + B(t,t+h)\right],$$

where coefficients A and B satisfy the backward recursions:

$$A(t+j,t+h) = a\left[z_{t+j+1} + A(t+j+1,t+h)\right],$$

and

$$B(t+j,t+h) = b\left[z_{t+j+1} + A(t+j+1,t+h)\right] + B(t+j+1,t+h),$$

for $j < h$, *with terminal conditions* $A(t+h,t+h) = 0$, $B(t+h,t+h) = 0$.

Proof. See Appendix B. □

3. EXAMPLES OF CAR PROCESSES

A theoretical advantage of a Laplace transform model compared with a density model is that this approach provides a unified treatment for discrete, continuous and mixed variables. Thus, Car processes can represent the dynamics of count data, non-negative continuous variables, dichotomous variables taking values 0 and 1, and so on, depending on the specification of functions a and b. In this section, we show some examples of Car models and comment on the relation between the Car family and the class of affine processes in continuous time introduced by Duffie *et al.* (2003).

3.1. Example 1. compound Poisson process

In risk analysis, the variable of interest Y_t is often integer-valued and measures the number of claims in period $(t, t+1)$. Therefore, one may be interested in an autoregressive specification for (Y_t), that has the marginal Poisson distribution, for instance. It is not possible to use the standard linear AR(1) model such as: $Y_t = \rho Y_{t-1} + \varepsilon_t$, where $|\rho| < 1$ and (ε_t) is a strong white noise, independent of Y_{t-1}. A stationary, integer-valued Y_t would require integer-valued ε_t and ρY_{t-1}. This condition can hold only under the absence of serial correlation $\rho = 0$. To circumvent this difficulty, the deterministic autoregression can be replaced by the following stochastic autoregression:

$$Y_t = \sum_{i=1}^{Y_{t-1}} Z_{i,t} + \varepsilon_t, \tag{5}$$

where ε_t, $Z_{i,t}$, i varying, are independent and independent of Y_{t-1}, and the variables $Z_{i,t}$ follow a Bernoulli distribution $\mathcal{B}(1,\rho)$.

In particular, the invariant distribution of a Poisson-distributed ε_t, $\varepsilon_t \sim \mathcal{P}(\lambda)$, is $\mathcal{P}(\lambda/(1-\rho))$. The processes in this class are called (Integer-Valued autoregressive INAR) and have been explored in the time series and insurance literature (see, e.g. McKenzie, 1985, 1988, and Al-Osh and Alzaid, 1987 for the definition, Brannas and Hellstrom, 2001 for survey, and Edwards and Gurland, 1961 and Gouriéroux and Jasiak, 2002a for applications to car insurance and insurance premium updating). The INAR(1) process is a Car process with:

$$a(z) = -\ln[\rho\exp(-z) + 1 - \rho], \quad b(z) = -\lambda[1 - \exp(-z)],$$
$$c(z) = -\frac{\lambda}{1-\rho}[1 - \exp(-z)].$$

3.2. Example 2. compounding processes with non-negative continuous values

A process Y_t with non-negative values is obtained from two components: an infinitely divisible distribution on $\mathbb{R}^+$ with Laplace transform $\exp[-a(u)]$ and a distribution on $\mathbb{R}^+$ with Laplace transform $\exp[-b(u)]$. Then the function:

$$\Psi(u, y_t) = \exp[-a(u)y_t + b(u)],$$

is a well-defined conditional Laplace transform. Thus, under compounding, it is equivalent to specify: (1) a Car process or (2) function a as an infinitely divisible distribution and function b as a continuous distribution. The set of alternatives is quite large. For instance, among the infinitely divisible distributions are compound Poisson distributions, gamma distributions, stable distributions, mixtures of exponential distributions, and so on. Moreover, it is possible to specify a (resp. b) as a discrete distribution and b (resp. a) as a continuous distribution, or to select distributions a and b with different tail behaviours (see Section 6.3). In some sense, the Car processes are defined by extending the definition of the thinning operator. Let Z_i denote independent and identically distributed (i.i.d.) Bernoulli variables with common distribution $\mathcal{B}(1, p)$, $p \in [0, 1]$. The thinning operator associated with the sequence $Z = (Z_i)$, and applied to a (deterministic or stochastic) integer-valued function of Y denoted by $N(Y)$, yields the variable defined as:

$$Z * N(Y) = \sum_{i=1}^{N(Y)} Z_i,$$

(see, e.g. Sim, 1990 for such a model for continuous positive variables Z_i). Loosely speaking, the extension consists in replacing the arithmetic sum by a stochastic integral $\int_0^{N(Y)} Z_i \,\mathrm{d}i$, where N is continuously valued. In Car processes, we have

$N(y) = y$. This shows that Car processes considerably extend the class of 'thinning models defined on the positive real line', in Grunwald *et al.* (2000).

3.3. Example 3. Continuous time affine processes

Continuous time affine processes have been considered independently by Duffie *et al.* (2003). A continuous time Markov process is affine if and only if:

$$E_t[\exp(-z'Y_{t+h})] = \exp\left[-a_h(z)'Y_t + b_h(z)\right] \quad \forall t \in R, \forall h \in R^+, \forall z \in \mathcal{D}.$$

Since the requirement that the log-Laplace transform has an affine form has to be satisfied for any real positive horizon, it is in particular satisfied for any integer horizon. Thus, any-time discretized continuous time affine process is a Car (but there exist many Car processes without a continuous time counterpart). For example, the autoregressive gamma process (Gouriéroux and Jasiak, 2005) is the time-discretized Cox–Ingersoll–Ross diffusion process (Cox *et al.*, 1985). It is defined in two steps by introducing an intermediate latent variable X_t. We assume that: (1) the conditional distribution of Y_t given the latent variable X_t is gamma with degree of freedom $\delta + X_t$: $Y_t \mid X_t \sim \gamma(\delta + X_t)$, and (2) the conditional distribution of X_t given Y_{t-1} is Poisson with parameter βY_{t-1}: $X_t \mid Y_{t-1} \sim \mathcal{P}(\beta Y_{t-1})$, where $\beta \geq 0$, $\delta \geq 0$. Then,

$$a(z) = \frac{\beta z}{1+z}, \quad b(z) = -\delta \ln(1+z), \quad c(z) = -\delta \ln\left(1 + \frac{z}{1-\beta}\right).$$

In particular, the marginal distribution is such that: $(1 - \beta)Y_t \sim \gamma(\delta)$.

3.4. Example 4. Wishart autoregressive process

Let us consider a Gaussian VAR(1) process: $X_t = AX_{t-1} + \epsilon_t, \quad \epsilon_t \sim N(0, \Sigma)$, and the matrix process defined by $Y_t = X_tX_t'$. For any symmetric matrix Γ we get: (Gouriéroux *et al.*, 2004; Gouriéroux, 2006)

$$E[\exp(X_{t+1}'\Gamma X_{t+1})|X_t] = \frac{\exp[X_t'\Sigma^{-1}(\Sigma^{-1} - 2\Gamma)^{-1}\Gamma A X_t]}{\det(Id - 2\Sigma\Gamma)^{1/2}}.$$

Equivalently, this relation can be written:

$$E\left[\exp\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{ij}Y_{ij,t+1}\right)\middle| X_t\right] = \frac{\exp(\sum_{i=1}^{n}\sum_{j=1}^{n}(\Sigma^{-1}(\Sigma^{-1} - 2\Gamma)^{-1}\Gamma A)_{ij}Y_{ij,t})}{\det(Id - 2\Sigma\Gamma)^{1/2}},$$

where B_{ij} denotes the $(i,j)th$ element of a matrix B. Therefore, the matrix process (Y_t) is a Markov process with an exponential affine conditional Laplace transform. The process (Y_t) can be considered as a dynamic extension of the Wishart distribution. The Wishart autoregressive processes (WAR) are dynamic models for stochastic, symmetric matrices which extend the autoregressive gamma process to a multivariate framework.

4. ERGODICITY

The ergodicity and mixing properties of a Car can be derived by using the well-known sufficient conditions based on Lyapunov functions (see Tong, 1990; Meyn and Tweedie, 1992, 1999). This section discusses two types of ergodicity conditions. The condition $\lim_{h\to\infty} a^{\circ h}=0$ ensures that the transition distribution at horizon h tends to a limiting distribution. The second condition concerns the stability of the Jacobian matrix $(\partial a/\partial u')\,(0)$ and ensures geometric ergodicity and β-mixing.

4.1. Weak ergodicity

Let us consider a Car process with marginal log-Laplace transform c. We get:

$$E[\exp(-z'Y_{t+h}) \mid \underline{Y_t}] = \exp\big(-a^{\circ h}(z)'Y_t + c(z) - c\big[a^{\circ h}(z)\big]\big).$$

Since the weak convergence is equivalent to the convergence of Laplace transforms, we get the following property.

Proposition 6. *If there exists an invariant distribution with Laplace transform* $\exp\, c(z)$, *where c is continuous at zero, then the conditional distribution of* Y_{t+h} *given* $\underline{Y_t}$ *weakly converges to the invariant distribution if and only if:* $\lim_{h\to\infty} a^{\circ h}(z) = 0$, $\forall z \in \mathcal{D}$.

Proof. Indeed we get

$$\begin{aligned}\lim_{h\to\infty} E\big[\exp(-z'Y_{t+h}) \mid \underline{Y_t}\big] &= \exp\left(-\lim_{h\to\infty} a^{\circ h}(z)'Y_t + c(z) - c\left[\lim_{h\to\infty} a^{\circ h}(z)\right]\right)\\ &= \exp c(z).\end{aligned}$$

□

Under the condition $\lim_{h\to\infty} a^{\circ h}=0$, $\forall z \in \mathcal{D}$, the invariant distribution is necessarily unique and, at an infinite horizon, the conditional distribution does not depend on the initial value of the process.

4.2. Geometric ergodicity

Under the condition $\lim_{h\to\infty} a^{\circ h}=0$, we have:

$$\lim_{h\to\infty} |P\big[Y_{t+h} \in A \mid \underline{Y_t}\big] - \pi_\infty(A)| = 0,$$

for any set A such that the invariant distribution π_∞ has mass zero on the boundary ∂A of set A. Asymptotically valid statistical inference requires a stronger ergodicity condition such as geometric ergodicity, which is met if and only if there exists a scalar $r > 1$ such that:

$$\sum_{h=1}^{\infty} r^h \sup_A |P[Y_{t+h} \in A \mid Y_t = y] - \pi_\infty(A)| < \infty,$$

for any y, where the supremum is taken over all measurable sets. The condition above implies a geometric rate of convergence of the transition distributions. Proposition 7 is proved in Appendix C.

PROPOSITION 7. *Let us assume:*

(A.1.) The conditional Laplace transform admits a series expansion in the neighbourhood of 0;
(A.2.) The process is a Car process, which satisfies one of the following conditions:
(i) a and b are either distributions on $\mathbf{N}^n$ *or on* $\mathbf{Z}^n$.
(ii) if a and b are both distributions on $\mathbf{R}^n$ *(resp.* $\mathbf{R}^{+n}$*), a is infinitely divisible.*

Then the process (Y_t) *is geometrically ergodic and β-mixing with a geometric decay rate if the largest eigenvalue of the Jacobian matrix* $(\partial a/\partial u')(0)$ *has modulus* < 1*, that is if:*

$$\lim_{h\to\infty} \frac{\partial a}{\partial u'}(0)^h = 0.$$

For the univariate Car, the stability condition becomes:

$$\left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right| < 1.$$

4.3. Relation between weak and geometric ergodicity (for a univariate process)

4.3.1. Non-negative processes

Let us consider a non-negative process and function a which is the log-Laplace transform of an infinitely divisible distribution. Function a is increasing, concave and satisfies $a(0) = 0$. In particular, we get the Lipschitz condition:

$$a(u) \le \frac{\mathrm{d}a}{\mathrm{d}u}(0) \cdot u, \quad \forall u \in \mathbf{R}^+.$$

Thus the stability condition $\mathrm{d}a/\mathrm{d}u\,(0) < 1$ (geometric ergodicity) implies $\lim_{h\to\infty} a^{\circ h}(u) = 0$, $\forall u \in \mathbf{R}^+$ (weak ergodicity).

4.3.2. The autoregressive gamma process

For the autoregressive gamma process, we have $a(z) = \beta z/1+z$. We see that: if

$$\beta \neq 1 : a^{\circ h}(z) = \beta^h z \left[1 + \frac{1-\beta^h}{1-\beta} z\right]^{-1},$$

and if

$$\beta = 1 : a^{\circ h}(z) = z[1 + hz]^{-1}.$$

The weak ergodicity condition is satisfied if

$$\lim_{h\to\infty} a^{\circ h}(z) = 0, \quad \forall z \in \mathcal{D} \Longleftrightarrow \beta \leq 1.$$

Similarly, the geometric ergodicity is satisfied if:

$$\left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right| = \beta < 1.$$

In the limiting case $\beta=1,(Y_t)$ is a martingale and, as h tends to infinity, we get:

$$\begin{aligned} &E[\exp(-uY_{t+h}) \mid Y_t] - E[\exp(-uY_{t+h})] \\ &\quad = \exp\left(-a^{\circ h}(u)Y_t + c(u) - c\left[a^{\circ h}(u)\right]\right) - \exp\, c(u) \\ &\quad = \exp\left(-\frac{u}{1+hu}Y_t + c(u) - c\left[\frac{u}{1+hu}\right]\right) - \exp\, c(u) \\ &\quad \sim -\frac{1}{h}\left(Y_t + \frac{\mathrm{d}c}{\mathrm{d}u}(0)\right)\exp c(u). \end{aligned}$$

We observe a hyperbolic decay rate of the conditional expectation written as a function of h, which creates a long memory effect.

4.4. Spectral decomposition of the conditional expectation operator

When the geometric ergodicity condition is satisfied, the rate of convergence of a prediction at horizon h, $E[g(Y_{t+h}) \mid Y_t]$, to the stationary level $E\,[g(Y_{t+h})]$ is less than or equal to a geometric rate, but for some well-chosen transformations g the rate of convergence can be strictly smaller. Proposition 8 provides insights into basic transformations that yield different geometric decay rates.

PROPOSITION 8. *Let us assume that the Laplace transform exists for any real argument and that*

$$\left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right| < 1.$$

Then the conditional expectation operator $\varphi \to T\varphi$, defined by:

$$T\varphi(y) = E[\varphi(Y_{t+1}) \mid Y_t = y],$$

admits a spectral decomposition. The eigenvalues are real, and given by:

$$\lambda_n = \left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right|^n, \quad n \geq 0,$$

and the eigenfunction associated with λ_n is polynomial P_n of degree n.

PROOF. See Appendix D. □

In particular, the β-mixing coefficient is equal to $|(\mathrm{d}a/\mathrm{d}u)\,(0)|$, and the transformation with the lowest rate of convergence is an affine transformation. This extends the well-known result for AR(1) Gaussian process with Hermite polynomials as eigenfunctions.

5. CLASSIFICATION OF REVERSIBLE CAR(1) PROCESSES

In this section, we provide a comprehensive classification of univariate reversible Car(1) processes and their properties.

5.1. Definition and characterization

The process Y is reversible if its dynamic properties in calendar and reversed time are identical. Since the process is Markov, the reversibility condition is equivalent to the symmetry of joint distribution of (Y_t, Y_{t-1}) with respect to both arguments. The Laplace transform for Y_t and Y_{t-1} associated with the joint distribution can be written as:

$$\begin{aligned} E[\exp(-uY_t - vY_{t-1})] &= E\left[\exp(-vY_{t-1})E\left[\exp(-uY_t) \mid \underline{Y_{t-1}}\right]\right] \\ &= E[\exp(-(a(u)+v)Y_{t-1} + c(u) - c[a(u)])] \\ &= \exp(c[a(u)+v] + c(u) - c[a(u)]) \\ &= \exp[\Phi(u,v)], \text{say}. \end{aligned}$$

PROPOSITION 9. *The* Car(1) *process Y is reversible if and only if* $\Phi(u,v) = c[a(u) + v] + c(u) - c[a(u)]$ *is a symmetric function of u and v.*

The condition above implies some restrictions on functions c and a [see Appendix F (i), (ii), (iii)].

PROPOSITION 10. *When the process Y is reversible:*

(i)

$$a(u) = \left(\frac{dc}{du}\right)^{-1}\left[\frac{da(0)}{du}\left\{\frac{dc(u)}{du} - \frac{dc(0)}{du}\right\} + \frac{dc(0)}{du}\right];$$

(ii) the function

$$\gamma(u) = \frac{d^2c}{du^2} \circ \left(\frac{dc}{du}\right)^{-1}(u)$$

is quadratic.

Therefore the log-Laplace transform of the marginal distribution of a reversible Car process satisfies necessarily a Ricatti differential equation:

$$\frac{d^2c}{du^2}(u) = \beta_0 + \beta_1 \frac{dc}{du}(u) + \beta_2\left(\frac{dc}{du}(u)\right)^2. \quad (6)$$

This equation is solved in Section 5.3. Once the marginal distribution is found, the dynamics of a reversible Car process is characterized by the single parameter $(da/du)(0)$ equal to the β-mixing coefficient.

5.2. Nonlinear canonical decomposition

Let us assume

$$\left|\frac{da}{du}(0)\right| < 1.$$

For a reversible Car(1) process, the eigenfunctions P_n, $n \geq 0$, of the conditional expectation operator are orthogonal with respect to the innerproduct associated with the invariant probability density function f. We can derive the nonlinear canonical decomposition of the transition probability (see Lancaster, 1958).

PROPOSITION 11. *If*

$$\left|\frac{da}{du}(0)\right| < 1,$$

and the stationary Car(1) *process is reversible, we have:*

$$f(y_t \mid \underline{y_{t-1}}) = f(y_t)\left[1 + \sum_{n=1}^{\infty}\left[\frac{da}{du}(0)\right]^n P_n(y_t)P_n(y_{t-1})\right], \quad (7)$$

where P_n, n varying, is an orthonormal basis of polynomial eigenfunctions of the conditional expectation operator.

By recursions, we derive the forecasting distribution at any horizon h:

$$f_h(y_t \mid \underline{y_{t-h}}) = f(y_t)\left[1 + \sum_{n=1}^{\infty}\left[\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right]^{hn} P_n(y_t)P_n(y_{t-h})\right].$$

5.3. The classification

This section describes all univariate reversible Car(1) processes. To proceed, we need to find the expression of function c by solving the Ricatti equation in (6) (see Appendix E), and to infer function a from Proposition 10. The processes given below are distinguished with respect to the roots of the characteristic equation: $\beta_0 + \beta_1 x + \beta_2 x^2 = 0$, which can be of degrees 0, 1 or 2. We review their distributional properties, ergodicity conditions and canonical decomposition (for the eigenpolynomials, see Wong and Thomas, 1962).

5.3.1. Class 1: autoregressive Gaussian process

The Gaussian processes are obtained when $\beta_1 = \beta_2 = 0$, that is when the γ-function is constant. Then $Y_t = \rho Y_{t-1} + \varepsilon_t$, where (ε_t) is a standard Gaussian white noise, and we get:

- Conditional distribution: $\mathcal{N}(\rho y_{t-1}, 1)$; Marginal distribution: $\mathcal{N}(0, \frac{1}{1-\rho^2})$;
- Log-Laplace transforms:

$$a(u) = u\rho, \quad b(u) = \frac{u^2}{2}, \quad c(u) = \frac{u^2}{2(1-\rho^2)};$$

- Geometric ergodicity condition:

$$\left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right| = |\rho| < 1;$$

- Polynomial eigenfunctions: Hermite polynomials;
- Forecasting distribution at horizon h:

$$\mathcal{N}\left(\rho^h y_{t-h}, \frac{1-\rho^{2h}}{1-\rho^2}\right);$$

- Compound function a : $a^{\circ h}(u) = \rho^h u$;
- γ-function:

$$\gamma(u) = \frac{1}{1-\rho^2};$$

- Joint log-Laplace transform:

$$\Psi(u, v) = \frac{1}{2(1-\rho^2)}(u^2 + v^2 + 2\rho uv).$$

5.3.2. Class 2: compound Poisson process

This process is obtained when $\beta_2 = 0$, $\beta_1 \neq 0$, that is when the γ-function is affine.

- Conditional distribution: $\mathcal{B}(y_{t-1},\alpha) * \mathcal{P}(\lambda(1-\alpha))$; Marginal distribution: $\mathcal{P}(\lambda)$;
- Log-Laplace transforms:

$$a(u) = -\log[\alpha\exp(-u) + 1 - \alpha],\ b(u) = -\lambda(1-\alpha)[1 - \exp(-u)],$$
$$c(u) = -\lambda[1 - \exp(-u)];$$

- Geometric ergodicity condition: $0 < \alpha < 1$;
- Polynomial eigenfunctions: Charlier polynomials;
- Forecasting distribution at horizon h: $\mathcal{B}(y_{t-h},\alpha^h) * \mathcal{P}(\lambda(1-\alpha^h))$;
- Compound function a:

$$a^{\circ h}(u) = -\log\left[\alpha^h \exp(-u) + 1 - \alpha^h\right];$$

- γ-function: $\gamma(u) = -u$;
- Joint log-Laplace transform:

$$\Psi(u, v) = -\lambda(2 - \alpha) + \lambda\alpha \exp(-u - v) + \lambda(1 - \alpha)[\exp(-u) + \exp(-v)].$$

5.3.3. Class 3: autoregressive gamma process

This process (see Gouriéroux and Jasiak, 2005) is obtained when the γ-function is quadratic and has a double root, i.e. for $\beta_2 \neq 0$, $\beta_1^2 - 4\beta_0\beta_2 = 0$.

- Conditional distribution: $\gamma(\delta,\beta y_{t-1})$; marginal distribution: $(1 - \beta)Y_t \sim \gamma(\delta)$;
- Log-Laplace transforms:

$$a(u) = \frac{\beta u}{1 + u}, \quad b(u) = -\delta \log(1 + u), \quad c(u) = -\delta \log\left(1 + \frac{u}{1 - \beta}\right);$$

- Geometric ergodicity condition:

$$\left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right| = |\beta| < 1;$$

- Polynomial eigenfunctions: Laguerre polynomials
- Forecasting distribution at horizon h:

$$\frac{1 - \beta}{1 - \beta^h} Y_t \sim \gamma\left(\delta, \beta^h \frac{1 - \beta}{1 - \beta^h} y_{t-h}\right);$$

- Compound function a:

$$a^{\circ h}(u) = \beta^h u \left[1 + \frac{1 - \beta^h}{1 - \beta} u\right]^{-1};$$

- γ-function:

$$\gamma(u) = \frac{u^2}{\delta};$$

- Joint log-Laplace transform:

$$\Psi(u, v) = -\delta \log\left[1 + \frac{uv + u + v}{1 - \beta}\right].$$

5.3.4. Class 4: Bernoulli process with switching regimes

This process is obtained when the γ-function is quadratic with two distinct real roots, i.e. for $\beta_2 \neq 0, \beta_1^2 - 4\beta_0\beta_2 > 0$. The process is a Markov chain with two states 0 an 1.

- Conditional distribution: $\mathcal{B}(1, \alpha(1 - \gamma) + \gamma y_{t-1})$; Marginal distribution: $\mathcal{B}(1, \alpha)$;
- Log-Laplace transforms:

$$\begin{aligned}
a(u) &= -\log\left[\frac{(1 - (1 - \alpha)(1 - \gamma))\exp(-u) + (1 - \alpha)(1 - \gamma)}{\alpha(1 - \gamma)\exp(-u) + 1 - \alpha(1 - \gamma)}\right], \\
b(u) &= \log(1 - \alpha(1 - \gamma) + \alpha(1 - \gamma)\exp(-u)), \\
c(u) &= \log(\alpha\exp(-u) + 1 - \alpha);
\end{aligned}$$

- Geometric ergodicity condition:

$$\left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right| = |\gamma| < 1;$$

- Polynomial eigenfunctions: Two polynomials only, which are the first Krawtchouk polynomials (see Abramowitz and Stegun, 1970, 22.17).
- Forecasting distribution at horizon h: $\mathcal{B}(1, \alpha(1 - \gamma^h) + \gamma^h y_{t-1})$;
- Compound function a:

$$a^{\circ h}(u) = -\log\left[\frac{(1 - (1 - \alpha)(1 - \gamma^h))\exp(-u) + (1 - \alpha)(1 - \gamma^h)}{\alpha(1 - \gamma^h)\exp(-u) + 1 - \alpha(1 - \gamma^h)}\right];$$

- γ-function: $\gamma(u) = -u(1+u)$;
- Joint log-Laplace transform:

$$\begin{aligned}
\Psi(u, v) = \log[&(1 - \alpha)(1 - \alpha(1 - \gamma)) + \alpha(1 - \alpha)(1 - \gamma)(\exp(-u) + \exp(-v)) \\
&+ \alpha(1 - (1 - \alpha)(1 - \gamma))\exp(-(u + v))].
\end{aligned}$$

5.3.5. Class 5

Is formed by processes, with a quadratic γ-function with conjugate complex roots, obtained for $\beta_2 \neq 0$, $\beta_1^2 - 4\beta_0\beta_2 < 0$.

- Log-Laplace transforms:

$$a(u) = \arctan[\gamma \tan u],$$
$$b(u) = -\log \cos u + \log \cos \arctan[\gamma \tan u], \quad c(u) = -\log \cos u;$$

- Geometric ergodicity condition:

$$\left|\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right| = |\gamma| < 1;$$

- Compound function a:

$$a^{\circ h}(u) = \arctan\left[\gamma^h \tan u\right];$$

- γ-function: $\gamma(u) = 1+u^2$;
- Joint log-Laplace transform:

$$\Psi(u, v) = -\log[\cos(u+v) + (1-\gamma) \sin u \sin v].$$

6. STRUCTURAL MODELLING USING REAL CONDITIONAL LAPLACE TRANSFORM

A number of problems encountered in finance, insurance and duration analysis involve the real conditional Laplace transform as the functional parameter of interest. This is due to common use of exponential transformations for modelling positively valued functions such as utility functions, intensity functions, stochastic discount factors and so on. Some typical examples of models based on conditional real Laplace transforms are presented in this section. Their common feature is the importance of the term structure (forecast function) of some variables, such as prices, interest rates, defaults, extreme risks and so on, which defines the dependence of the forecast function on horizon h.

6.1. Portfolio management

The optimal strategy of portfolio management is usually considered in the conditional mean–variance framework based on the conditional normal distribution of returns and constant absolute risk aversion (CARA) utility function (see Markovitz, 1976). The CARA utility function is defined as $U(w) = -\exp(-Aw)$, and is increasing and concave. It depends on the positive risk aversion parameter A. A natural extension consists in relaxing the assumption of conditional normality. Then the optimal portfolio at horizon h is determined by maximizing:

$$\max_{\alpha_0, \alpha} E_t\left\{-\exp -A\left[\alpha' p_{t+h} + \alpha_0(1 + r_{t,t+h})\right]\right\},$$

subject to the budget constraint $\alpha' p_t + \alpha_0 = w_t$, where $\alpha = (\alpha_1, \ldots, \alpha_n)'$ is the allocation in the n risky assets, $p_{i,t}$, $i = 1, \ldots, n$, the price per share of asset i, α_0 the quantity of risk-free assets, $r_{t,t+h}$ the risk-free rate at horizon h, and w_t the amount invested at time t. After eliminating the quantity invested in the risk-free asset α_0, we obtain the new optimization:

$$\max_{\alpha} E_t\{-\exp[-A\alpha' Y_{t+h}]\}, \tag{8}$$

where $Y_{t+h} = p_{t+h} - (1 + r_{t,t+h})p_t$ denotes the excess gain. This optimization depends directly on the conditional real Laplace transform of the excess gain Y. In the stationary case, this problem is equivalent to minimizing:

$$\frac{1}{A}\ln E_t\{\exp[-A\alpha' Y_{t+h}]\} = \frac{1}{A}c(A\alpha) - \frac{1}{A}\left\{a^{\circ h}(A\alpha)' Y_t + c[a^{\circ h}(A\alpha)]\right\},$$

where $Y_{t+h} = p_{t+h} - (1 + r_{t,t+h})p_t$ denotes the excess gain. The objective function is a sum of two terms. The first term does not depend on horizon h, corresponds to the static (time-invariant) allocation, and represents the marginal expected utility. The second term provides the adjustment to the price dynamics through function a. In the i.i.d. case, or when horizon h tends to infinity, only the first term in the criterion function matters, and the optimal allocation is static.

As an illustration, let us consider a portfolio that includes a risk-free asset with a constant risk-free interest rate r and a risky asset, with a stable conditional distribution of the price per share, such that:

$$E_t[\exp(-uY_{t+1})] = \exp[-|u|^{\beta} Y_t],$$

with $0 < \beta < 1$. The process Y takes positive values, and its conditional moments do not exist because of the heavy tails of the the stable distribution. In this case, $a^{\circ h}(u) = u^{\beta^h}$, and the optimal allocation in risky asset:

$$\alpha_h^* = \left[\frac{(1+r)^h}{\beta^h A^{\beta^h - 1}}\right]^{\frac{1}{\beta^h - 1}},$$

does not depend on the current price level. In the long run ($h \to \infty$), the allocation in the risky asset α_h^* tends to $+\infty$, and the allocation in the risk-free asset tends to $-\infty$ to satisfy the budget constraint. Thus the investor will borrow the risk-free asset to acquire more of the risky asset, despite the risk associated with fat tails.

6.2. Term structure of interest rates

A common element in asset pricing models in discrete time is a stochastic discount factor (sdf), which accommodates both temporal discounting and risk adjustment (see, e.g. Gouriéroux and Jasiak, 2001). Let us consider the price at t of a zero-coupon bond with residual maturity h which pays \$1 at date $t + h$. Its price at t, denoted $B(t, t + h)$, can be written as:

$$B(t, t+h) = E_t\left[M_{t,t+1} \cdots M_{t+h-1,t+h}\right],$$

where $M_{t,t+1}$ denotes the sdf between t and $t+1$. The sdf depends on the information available at date $t+1$ and is positive due to the arbitrage-free constraints. The stochastic discount factor is often specified as an exponential affine function of underlying factors (see, e.g. Gouriéroux and Monfort, 2006):

$$M_{t,t+1} = \exp(\alpha' Y_{t+1} + \beta), \text{ say.}$$

Then the price of the zero-coupon bond becomes:

$$B(t, t+h) = \exp(\beta h) E_t[\exp(\alpha' Y_{t+1} + \cdots + \alpha' Y_{t+h})], \tag{9}$$

A closed-form expression of this price is easily derived when the factor process is Car. From Proposition 5, it follows that:

$$B(t, t+h) = \exp(\beta h) \exp\left[A(h)' Y_t + B(h)\right], \tag{10}$$

where $A(h)$, $B(h)$ satisfy the recursive equations $A(h) = a[\alpha + A(h-1)]$, $B(h) = b[\alpha + A(h-1)] + B(h-1)$, $h \geq 1$, with $A(0)$=0, $B(0)$=0. In particular, the geometric yield defined by:

$$B(t, t+h) = \exp[-hr(t, t+h)],$$

is such that:

$$r(t, t+h) = -\beta - \frac{A(h)'}{h} Y_t - \frac{B(h)}{h}, \tag{11}$$

h varying. We get an affine term structure driven by factor Y_t and characterized by functions A and B. The discrete time approach presented above is much more flexible than the standard affine term structure analysis in continuous time (see Duffie and Kan, 1996).

6.3. Extreme risk

The approach based on conditional Laplace transform is applicable to extreme risk analysis. Indeed, by the Karamata's Tauberian theorem the asymptotic behaviour of a cumulative density function (cdf) at infinity is related to the behaviour of the Laplace transform at the origin. We have the proposition 12 (see Feller (1971), Chapter 13, Bingham, Goldie, Teugels (1987), Corollary 8.1.7).

Proposition 12. *For* $0 \leq \delta(Y_t) \leq 1$, *the following relations are equivalent:*

(i)

$$E\left[\exp(-uY_{t+1}) \mid \underline{Y_t}\right] \sim 1 - u^{\delta(Y_t)} l\left(\frac{1}{u}, Y_t\right), \quad \textit{for } u \approx 0,$$

(ii)

$$P\left[Y_{t+1} < y \mid \underline{Y_t}\right] \sim 1 - \frac{l(y, Y_t)}{y^{\delta(Y_t)}\Gamma(1-\alpha(Y_t))}, \quad \text{for } y \to \infty,$$

where l is a slowly varying function and Γ denotes the gamma function.

In finance, the extreme risk analysis concerns the conditional Value-at-Risk (VaR), which is a conditional quantile associated with a small risk level α (see Gouriéroux and Jasiak, 2002b). Typically, a conditional VaR for horizon h and risk level α is:

$$\mathrm{VaR}_t(h, \alpha) = F_{t,h}^{-1}(\alpha),$$

where $F_{t,h}$ is the conditional cdf of Y_{t+h} given Y_t.

Let us consider a non-negative univariate Car process and assume that $a(u) \approx c_0 u^{\delta_0}$, $c(u) \approx c_1 u^{\delta_1}$, for $u \approx 0$, where $0 \le \delta_0 \le 1$, $0 \le \delta_1 \le 1$. Thus we allow for different behaviours of a and c in a neighbourhood of the origin or equivalently of the tails of the conditional and marginal distributions of the process. Then

$$a^{\circ h}(u) \approx c_{0,h} u^{\delta_0^h}, \quad c(u) \approx c_1 u^{\delta_1}, \quad c\left[a^{\circ h}(u)\right] \approx c_{1,h} u^{\delta_1 \delta_0^h}.$$

Therefore the conditional real Laplace transform at horizon h is equivalent to:

$$\begin{aligned} E\left[\exp(-uY_{t+h}) \mid \underline{Y_t}\right] &= \exp\left[-a^{\circ h}(u)Y_t + c(u) - c\left[a^{\circ h}(u)\right]\right] \\ &\approx 1 - c_{0,h} u^{\delta_0^h} Y_t + c_1 u^{\delta_1}, \end{aligned}$$

when $u \approx 0$. When

$$\delta_0^h > \delta_1, \quad E[\exp(-uY_{t+h}) \mid \underline{Y_t}] \approx 1 - c_{0,h} u^{\delta_0^h} Y_t.$$

By applying Tauberian Theorem 6.1, we get:

$$P\left[Y_{t+h} < y \mid \underline{Y_t}\right] \approx 1 - \frac{c_{0,h} Y_t}{y^{\delta_0^h}\Gamma\left(1-\delta_0^h\right)},$$

and the VaR:

$$\mathrm{VaR}_t(h, \alpha) \approx \left[\frac{c_{0,h} Y_t}{(1-\alpha)\Gamma\left(1-\delta_0^h\right)}\right]^{\frac{1}{\delta_0^h}}.$$

When $\delta_0^h < \delta_1$, the application of Tauberian theorem 6.1 implies:

$$\mathrm{VaR}_t(h, \alpha) \approx \left[\frac{c_1}{(1-\alpha)\Gamma(1-\delta_1)}\right]^{\frac{1}{\delta_1}}.$$

Let us, for instance, discuss the case $\delta_0 > \delta_1$. There exists the smallest integer H such that $\delta_0^H < \delta_1$. For $h < H$, the conditional VaR depends on the lagged value of Y and its behaviour is driven by function a. If $h > H$, the conditional VaR is equivalent to the marginal VaR computed from function c. In conclusion, the behaviour of functions a and c in a neighbourhood of 0 provides information on the dependence of VaR on the risk level and residual maturity for small α.

7. CONCLUDING REMARKS

Several models in finance, credit management and insurance concern the term structures and are specified in terms of conditional Laplace transforms. This paper proposed a family of dynamic processes for the analysis of term structures, called the Car. The Car model represents the conditional Laplace transform as an affine function of lagged values of a process. The class of dynamic Car models is quite large. It comprises continuous as well as discrete or qualitative processes, and is easy to use for nonlinear forecasting at any horizon.

APPENDICES

A. PROOF OF PROPOSITION 2

Let us consider a Car(p) process. The conditional Laplace transform associated with the conditional p.d.f. of $\widetilde{Y}$ is:

$$\begin{aligned}
&E[\exp(-z'\widetilde{Y}_t) \mid \underline{\widetilde{Y}_{t-1}}] \\
&\quad = E[\exp(-z_1'Y_t - z_2'Y_{t-1} - \cdots - z_p'Y_{t-p+1}) \mid \underline{Y_{t-1}}] \\
&\quad = \exp(-[a_1'(z_1) + z_2']Y_{t-1} - \cdots - [a_{p-1}'(z_1) + z_p']Y_{t-p+1} - a_p'(z_1)Y_{t-p} + b(z_1)).
\end{aligned}$$

The above expression of Laplace transform defines a Car(1) process with:

$$a(z) = \left[a_1(z_1) + z_2, ..., a_{p-1}(z_1) + z_p, a_p(z_1)\right]'.$$

B. JOINT LAPLACE TRANSFORM

We get:

$$\begin{aligned}
&E[\exp(z_{t+j+1}'Y_{t+j+1} + \cdots + z_{t+h}'Y_{t+h}) \mid \underline{Y_{t+j}}] \\
&= E\{E[\exp(z_{t+j+1}'Y_{t+j+1} + \cdots + z_{t+h}'Y_{t+h}) \mid \underline{Y_{t+j+1}}] \mid \underline{Y_{t+j}}\} \\
&= E\{\exp[z_{t+j+1}'Y_{t+j+1} + A(t+j+1, t+h)'Y_{t+j+1} + B(t+j+1, t+h)] \mid \underline{Y_{t+j}}\} \\
&= \exp\{a[z_{t+j+1} + A(t+j+1, t+h)]'Y_{t+j+1} + B(t+j+1, t+h) \\
&\quad + b[z_{t+j+1} + A(t+j+1, t+h)]\}.
\end{aligned}$$

The backward recursion follows by comparing the first and last expressions. The terminal conditions are satisfied, since

$$A(t+h-1,t+h) = a(z_{t+h}), \quad B(t+h-1,t+h) = b(z_{t+h}).$$

C. GEOMETRIC ERGODICITY

The proof of geometric ergodicity of a Markov process is a multistep procedure which is well known in the literature (see, e.g. Tong, 1990; Meyn and Tweedie, 1999) and adapted to Car processes below.

C.1. Choice of a basic σ-finite measure

First, we have first to select a σ-finite measure Φ, such as a Lebesgue measure λ_n, λ_n^+, or a counting measure $\sum_{\mathbf{N}^n}\delta_j$, $\sum_{\mathbf{Z}^n}\delta_j$, depending on the domain of the conditional distribution which can be $\mathbf{R}^n$, $\mathbf{R}^{+n}$, $\mathbf{N}^n$, or $\mathbf{Z}^n$ (see Assumption A.2).

C.2. Φ-irreductibility

The chain is Φ-irreductible if and only if $\sum_{h=1}^{\infty} P[Y_{t+h} \in A \mid Y_t] > 0$, $\forall A$ with $\Phi(A) > 0$. This condition is satisfied by Assumption A.1 of Proposition 7 since $P[Y_{t+h} \in A | Y_t] > 0$.

C.3. Definition of the small sets

By the result established by Feigin and Tweedie (1985), which says that every compact set C such that $\Phi(C) > 0$ is small, if $E[g(Y_{t+1}) \mid Y_t = y]$ is a continuous function of y for any bounded continuous function g. Since $E\,[\exp(-\mathrm{i}wY_{t+1}) \mid Y_t = y] = \exp[-a(\mathrm{i}w)y + b(\mathrm{i}w)]$, is continuous in y for any w, the result follows since the set of exponential functions ($\exp(-\mathrm{i}wy)$, w varying) is dense in the set of bounded continuous functions and the dominated convergence theorem applies.

C.4. Φ-aperiodicity

The chain is aperiodic if and only if there exists a small set C and a positive integer h such that

$$P[Y_{t+h} \in C \mid Y_t = y] > 0 \quad \text{and} \quad P[Y_{t+h+1} \in C \mid Y_t = y] > 0, \quad \forall y \in C.$$

This condition is satisfied for $h = 1$ and any compact set C such that $\Phi(C) > 0$ by Assumption A.2.

C.5. Drift condition

Proposition C.1 (Drift property, Tong, 1990, p. 457). *Let (Y_t) be Φ-irreductible and Φ-aperiodic. Suppose that there exists a small set C, a non-negative measurable function g and constant $0 < r < 1$, $\gamma > 0$, $B > 0$ such that:*

$$E[g(Y_{t+1}) \mid Y_t = y] < r[g(y) - \gamma], \quad y \notin C, \quad \textit{and} \quad E[g(Y_{t+1}) \mid Y_t = y] < B, y \in C,$$

then (Y_t) is geometrically ergodic.

The drift condition is applied for a compact set $C = \{y : \|y\| < K\}$. Then, the Lyapunov function g can be quadratic. This choice is due to the special form of the first- and second-order conditional moments. Indeed we have:

$$E[Y_{t+1} \mid Y_t = y] = -\frac{\partial a}{\partial u'}(0)y + \frac{\partial b}{\partial u'}(0),$$

$$\begin{aligned} &E[Y_{t+1}Y'_{t+1} \mid Y_t = y] \\ &= \left(\frac{\partial^2}{\partial u \partial u'} E[\exp(-uY_{t+1}) \mid Y_t = y]\right)_{u=0} \\ &= \left(\frac{\partial^2}{\partial u \partial u'} \exp(-a(u)y + b(u))\right)_{u=0} \\ &= \sum_{j=1}^{n} y_j \frac{\partial^2 a_j(0)}{\partial u \partial u'} + \frac{\partial^2 b(0)}{\partial u \partial u'} + \left[\frac{\partial a(0)}{\partial u'} y + \frac{\partial b(0)}{\partial u'}\right]\left[-y' \frac{\partial a'(0)}{\partial u} + \frac{\partial b'(0)}{\partial u}\right]. \end{aligned}$$

For a large value of y, these conditional moments are equivalent to:

$$E[Y_{t+1} \mid Y_t = y] \sim -\frac{\partial a}{\partial u'}(0)y, \quad E[Y_{t+1}Y'_{t+1} \mid Y_t = y] \sim \frac{\partial a}{\partial u'}(0)yy'\frac{\partial a'}{\partial u}(0).$$

The conditional moments of a Gaussian vector autoregressive model

$$Y_t = \Psi Y_{t-1} + u_t, \quad u_t \sim N(0, \Omega),$$

are obtained by substituting $\Psi = (\partial a/\partial u')(0)$ in the last expression. This explains the condition given in Proposition 7. More precisely, let us first assume that the Jacobian matrix $(\partial a/\partial u')(0)$ can be diagonalized with eigenvectors $e_j, j = 1, \ldots, n$ and eigenvalues λ_j, $j = 1, \ldots, n$. The following form of the Lyapunov function can be used:

$$g(y) = \max_j \|e'_j y\|^2.$$

The conditional expectation $E[g(Y_{t+1}) \mid Y_t = y]$ is quadratic in y and continuously valued. Thus the second condition of the drift property is satisfied. Moreover:

$$E[g(Y_{t+1}) \mid Y_t = y] \leq \max_j E\left(\|e'_j Y_{t+1}\|^2 \mid Y_t = y\right) \sim \max_j |\lambda_j|^2 y,$$

for large y. Thus it follows from Feigin and Tweedie (1985; Theorem 1), that the first condition of the drift property is satisfied if $\max_j|\lambda_j|^2 < 1$. When the matrix $(\partial a/\partial u')(0)$ cannot be diagonalized, the Lyapunov function can be modified following the lines in Tong (1990), proof of Theorem A.1.7.

C.6. Mixing conditions

The geometric ergodicity implies β-mixing with geometric decay (see Davidov, 1973; or Doukhan, 1994, Chap. 2.4).

D. SPECTRAL DECOMPOSITION OF THE CONDITIONAL EXPECTATION OPERATOR

D.1. Preliminary lemma

LEMMA D.1. *We have:*

$$E\left[Y_t^n \mid \underline{Y_{t-1}}\right] = P_n(Y_{t-1}),$$

where P_n is a polynomial of degree n, and its coefficient of the highest degree is: $[(\mathrm{d}a/\mathrm{d}u)(0)]^n$.

PROOF. We have just to compare the series expansions of:

$$E\left[\exp(-uY_t) \mid \underline{Y_{t-1}}\right] = \sum_{n=0}^{\infty} \frac{u^n}{n!} E\left[Y_t^n \mid \underline{Y_{t-1}}\right],$$

and of:

$$\begin{aligned}\exp[-a(u)Y_{t-1} + b(u)] &= \sum_{n=0}^{\infty} \frac{[-a(u)Y_{t-1} + b(u)]^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=1}^{\infty} \left[-\frac{\mathrm{d}a^k(0)}{\mathrm{d}u^k} \frac{u^k}{k!} Y_{t-1} + \frac{\mathrm{d}b^k(0)}{\mathrm{d}u^k} \frac{u^k}{k!} \right] \right)^n,\end{aligned}$$

to prove the proposition. □

In particular, the Car processes are such that

$$E\left[Y_t | \underline{Y_{t-1}}\right] = \alpha Y_{t-1} + \beta,$$

that is they satisfy a weak linear AR(1) model as defined by Grunwald *et al.* (2000).

D.2. Spectral decomposition

From Lemma D.1, the space of polynomial functions of degree less than or equal to n is invariant with respect to the conditional expectation operator. The operator restricted to this space can be represented by a diagonal matrix, with diagonal elements λ_j, $j = 0, \ldots, n$. The result follows directly.

E. REVERSIBILITY CONDITION

(i) The reversibility condition can be written as:

$$c[a(u)+v]-c[a(u)]-\{c[a(v)+u]-c(u)\}-c(v)+c[a(v)]=0, \quad \forall u,$$
$$\iff \sum_{j=1}^{\infty}\frac{1}{j!}\left\{\frac{\mathrm{d}^j c}{\mathrm{d}u^j}[a(u)]-\frac{\mathrm{d}^j c}{\mathrm{d}u^j}(0)\right\}v^j=\sum_{j=1}^{\infty}\frac{1}{j!}\left\{\frac{\mathrm{d}^j c}{\mathrm{d}u^j}(u)-\frac{\mathrm{d}^j c}{\mathrm{d}u^j}(0)\right\}a(v)^j,$$

By equating the coefficients of the term v^j, we deduce that there exist constants d_{jk} such that:

$$\forall j:\frac{\mathrm{d}^j c}{\mathrm{d}u^j}[a(u)]=\sum_{k=1}^{j}d_{jk}\frac{\mathrm{d}^k c}{\mathrm{d}u^k}(u)+d_{j0}, \quad \forall u. \tag{E.1}$$

(ii) Let us first consider the condition corresponding to $j=1$. We get:

$$\frac{\mathrm{d}c}{\mathrm{d}u}[a(u)]-\frac{\mathrm{d}c}{\mathrm{d}u}(0)=\frac{\mathrm{d}a}{\mathrm{d}u}(0)\left[\frac{\mathrm{d}c}{\mathrm{d}u}(u)-\frac{\mathrm{d}c}{\mathrm{d}u}(0)\right].$$

Thus, if $\mathrm{d}c/\mathrm{d}u$ is invertible, we get an expression of function a:

$$a(u)=\left(\frac{\mathrm{d}c}{\mathrm{d}u}\right)^{-1}\left[\frac{\mathrm{d}a}{\mathrm{d}u}(0)\left[\frac{\mathrm{d}c}{\mathrm{d}u}(u)-\frac{\mathrm{d}c}{\mathrm{d}u}(0)\right]+\frac{\mathrm{d}c}{\mathrm{d}u}(0)\right]. \tag{E.2}$$

(iii) The condition written for $j=2$ implies a constraint on function c. Indeed this condition can be written as:

$$\frac{\mathrm{d}^2c}{\mathrm{d}u^2}[a(u)]-\frac{\mathrm{d}^2c}{\mathrm{d}u^2}(0)=\left(\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right)^2\left[\frac{\mathrm{d}^2c}{\mathrm{d}u^2}(u)-\frac{\mathrm{d}^2c}{\mathrm{d}u^2}(0)\right]+\frac{\mathrm{d}^2a}{\mathrm{d}u^2}(0)\left[\frac{\mathrm{d}c}{\mathrm{d}u}(u)-\frac{\mathrm{d}c}{\mathrm{d}u}(0)\right], \quad \forall u.$$

Let us introduce the function:

$$\gamma(u)=\frac{\mathrm{d}^2c}{\mathrm{d}u^2}\circ\left(\frac{\mathrm{d}c}{\mathrm{d}u}\right)^{-1}(u),$$

the change of variable $v=\mathrm{d}c/\mathrm{d}u(u)$, and use equation E.2, so that the condition becomes:

$$\gamma\left[\frac{\mathrm{d}a}{\mathrm{d}u}(0)v+\frac{\mathrm{d}c}{\mathrm{d}u}(0)\left(1-\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right)\right]-\frac{\mathrm{d}^2c}{\mathrm{d}u^2}(0)=\left[\gamma(v)-\frac{\mathrm{d}^2c}{\mathrm{d}u^2}(0)\right]\left(\frac{\mathrm{d}a}{\mathrm{d}u}(0)\right)^2+\frac{\mathrm{d}^2a}{\mathrm{d}u^2}(0)\left[v-\frac{\mathrm{d}c}{\mathrm{d}u}(0)\right], \forall v.$$

Thus there exist scalars α_j, $j=1,\ldots,4$ such that:

$$\gamma(\alpha_1 v+\alpha_2)=\alpha_1^2\gamma(v)+\alpha_3 v+\alpha_4, \quad \forall v. \tag{E.3}$$

We deduce that function γ is quadratic.

(iv) Case $\beta_1=\beta_2=0$

The function c is quadratic, whereas the function a is linear by E.2: $c(u)=\delta_1 u+\delta_2 u^2, a(u)=\gamma_1 u$, and it is easily checked that the joint log-Laplace transform:

$$\Phi(u,v)=\delta_2(u^2+v^2)+2\delta_2\gamma_1 uv+\delta_1(u+v),$$

is symmetric in u and v. We get a Gaussian process with mean $m=-\delta_1$ variance $2\delta_2$ and autocorrelation $\rho=\delta_2/\delta_1$.

(v) Case $\beta_1 \neq 0$, $\beta_2 = 0$

By integrating the differential eqn (6), we get the necessary form:

$$c(u) = \delta_1 u + \delta_2(1 - \exp\, \delta_3 u),$$

and, by eqn (E.2), we deduce the necessary form for function a:

$$a(u) = \frac{1}{\delta_3}\log[\alpha_0\, \exp(\delta_3 u) + (1 - \alpha_0)].$$

Then we get:

$$\Phi(u, v) = \delta_1(u + v) + \delta_2(2 - \alpha_0) - \delta_2\alpha_0\, \exp[\delta_3(u + v)] - \delta_2(1 - \alpha_0)(\exp\, \delta_3 u + \exp\, \delta_3 v).$$

(vi) Case $\beta_1^2 - 4\beta_0\beta_2 = 0$, $\beta_2 \neq 0$

By integrating the differential eqn (6) we get the necessary form:

$$c(u) = \delta_1 u + \delta_2\, \log(1 + \delta_3 u),$$

and by eqn (E.2), the necessary form for function a:

$$a(u) = \frac{\alpha_1 u}{1 + (1 - \alpha_1)\delta_3 u}.$$

Then the joint log-Laplace transform is symmetric:

$$\Phi(u, v) = \delta_1(u + v) + \delta_2\, \log\left\{1 + \delta_3(u + v) + \delta_3^2(1 - \alpha_1)uv\right\}.$$

Up to a change of scale and location, we get the autoregressive gamma process.

(vii) Case $\beta_1^2 - 4\beta_0\beta_2 > 0$, $\beta_2 \neq 0$

The necessary form for function c is:

$$c(u) = \delta_1 u + \delta_2 \log[\alpha \exp(\delta_3 u) + 1 - \alpha],$$

and by eqn (E.2), the necessary form of function a:

$$a(u) = \frac{1}{\delta_3}\log\left[\frac{(1 - (1 - \alpha)(1 - \gamma))\exp(\delta_3 u) + (1 - \alpha)(1 - \gamma)}{\alpha(1 - \gamma)\exp(\delta_3 u) + 1 - \alpha(1 - \gamma)}\right].$$

Then the joint log-Laplace transform is symmetric:

$$\begin{aligned}\Phi(u, v) = {} & \delta_1(u + v) + \delta_2 \log[(1 - \alpha)(1 - \alpha(1 - \gamma)) \\ & + \alpha(1 - \alpha)(1 - \gamma)(\exp(\delta_3 u) + \exp(\delta_3 v)) + \alpha(1 - (1 - \alpha)(1 - \gamma))\exp(\delta_3(u + v))].\end{aligned}$$

This case is associated with the Bernoulli process with switching regimes.

(viii) Case $\beta_1^2 - 4\beta_0\beta_2 < 0$, $\beta_2 \neq 0$

The necessary form for function c is:

$$c(u) = \delta_1 u + \delta_2\, \log[\cos(\delta_3 u + \delta_4)] - \delta_2\, \log\, \cos(\delta_4),$$

and by eqn E.2, a necessary form for function a:

$$a(u) = \frac{1}{\delta_3}[\arctan(\gamma\, \tan(\delta_3 u + \delta_4) + (1 - \gamma)\tan\delta_4) - \delta_4].$$

Then the joint log-Laplace transform is symmetric:

$$\Phi(u,v) = \delta_1(u+v) - \delta_2 \cos \delta_4 + \delta_2 \log\left[\cos(\delta_3(u+v)+\delta_4) + (1-\gamma)\frac{\sin \delta_3 u \sin \delta_3 v}{\cos \delta_4}\right].$$

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NOTES

1. The acronym Car has to be distinguished from CAR which stands for continuous autoregressive process in the time series literature (Hyndman, 1993; Brockwell, 1994).

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