Disasters Implied by Equity Index Options

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Summary

The idea

- Problem: disasters infrequent ⇒ hard to estimate their distribution
- Solution: infer from option prices

What we find

- Disasters apparent in options data
- More modest than disasters in macro data

Why this is harder than we thought

- Barro data gives us “true” distribution of consumption growth
- Option prices give us “risk-neutral” distribution of returns
Outline

Preliminaries: entropy, cumulants, plan

Disasters in macroeconomic models

Digression: risk-neutral probabilities

Disasters in option models

Comparing models
Hans-Otto Georgii (quoted by Hansen and Sargent):

*When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: “Call it entropy. It is already in use under that name and, besides, it will give you a great edge in debates because nobody knows what entropy is anyway.”*
Entropy bound

Entropy is a measure of dispersion: for $x > 0$

$$L(x) \equiv \log E_x - E \log x \geq 0$$

Pricing relation: there exists $m > 0$ such that

$$E_t (m_{t+1} r_{t+1}) = 1$$

Entropy bound

$$L(m) \geq E (\log r - \log r^1)$$
Cumulants

Cumulant generating function

\[ k(s; x) = \log E e^{sx} = \sum_{j=1}^{\infty} \kappa_j(x) s^j / j! \]

Cumulants are almost moments

- Mean = \( \kappa_1 \)
- Variance = \( \kappa_2 \)
- Skewness = \( \kappa_3 / \kappa_2^{3/2} \)
- (Excess) Kurtosis = \( \kappa_4 / \kappa_2^2 \)
Entropy and cumulants

Entropy of pricing kernel

\[ L(m) = \log E e^{\log m} - E \log m \]

\[ = k(1; \log m) - E \log m = \sum_{j=2}^{\infty} \kappa_j(\log m)/j! \]

Zin’s “never a dull moment” conjecture

\[ L(m) = \frac{\kappa_2(\log m)}{2!} + \frac{\kappa_3(\log m)}{3!} + \frac{\kappa_4(\log m)}{4!} + \cdots \]

(log)normal term

high-order cumulants (incl disasters)
Plan of attack

Modeling assumptions

- iid
- Tight link between consumption growth and equity returns
- Representative agent with power utility [if needed]

Parameter choices

- Match mean and variance of log consumption growth
- Ditto log equity return
- Base “disasters” on Barro’s macroeconomic evidence
- Or on equity index options

Compare macro- and option-based examples
Macro disasters: environment

Consumption growth and “equity” return are iid

\[
g_{t+1} = \frac{c_{t+1}}{c_t}
\]

\[
d_t = c_t^\lambda
\]

\[
\log r_{t+1}^e = \text{constant} + \lambda \log g_{t+1}
\]

Power utility

\[
\log m_{t+1} = \log \beta - \alpha \log g_{t+1}
\]
Macro disasters: the bazooka

Cumulant generating functions

\[ k(s; \log m) = k(-\alpha s; \log g) \]

Yaron’s “bazooka”

\[ \kappa_j(\log m)/j! = \kappa_j(\log g)(-\alpha)^j/j! \]
Macro disasters: Poisson-normal mixture

Consumption growth

\[ \log g_{t+1} = w_{t+1} + z_{t+1} \]

\[ w_{t+1} \sim \mathcal{N}(\mu, \sigma^2) \]

\[ z_{t+1} | j \sim \mathcal{N}(j\theta, j\delta^2) \]

\[ j \geq 0 \text{ has probability } e^{-\omega j} / j! \]

Parameter values

- Match mean and variance of log consumption growth
- Jump probability (\( \omega = 0.01 \)), mean (\( \theta = -0.3 \)), and variance (\( \delta^2 = 0.15^2 \)) [similar to Barro, Nakamura, Steinsson, and Ursua]
Macro disasters: entropy

Cumulant generating functions

\[
k(s; \log g) \equiv \log E e^{s \log g} = k(s; w) + k(s; z)
\]

\[
k(s; w) \equiv \log E e^{sw} = s\mu + (s\sigma)^2/2
\]

\[
k(s; z) \equiv \log E e^{sz} = \omega \left(e^{s\theta} + (s\delta)^2/2 - 1\right)
\]

Entropy

\[
L(m) = (-\alpha\sigma)^2/2 + \omega \left(e^{-\alpha\theta} + (\alpha\delta)^2/2 - 1\right) + \alpha\omega\theta,
\]
Macro disasters: entropy
Macro disasters: entropy

![Graph showing entropy of pricing kernel L(m) as a function of risk aversion α. The graph includes two lines: one for the Alvarez-Jermann lower bound and another for normal disasters. The y-axis represents the entropy of pricing kernel L(m) from 0 to 0.4, and the x-axis represents risk aversion α from 0 to 12.]
Macro disasters: entropy

Risk Aversion $\alpha$

Entropy of Pricing Kernel $L(m)$

Alvarez–Jermann lower bound

disasters

booms

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Macro disasters: cumulants

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Macro disasters: cumulants

<table>
<thead>
<tr>
<th>Model (( \alpha = 10 ))</th>
<th>Entropy</th>
<th>Variance/2</th>
<th>High-Order Cumulants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0.0613</td>
<td>0.0613</td>
<td>0</td>
</tr>
<tr>
<td>Poisson disaster</td>
<td>0.5837</td>
<td>0.0613</td>
<td>0.2786</td>
</tr>
<tr>
<td>Poisson boom</td>
<td>0.0266</td>
<td>0.0613</td>
<td>-0.2786</td>
</tr>
</tbody>
</table>
Macro disasters: equity premium

![Graph showing entropy and equity premium as functions of risk aversion α. The sample mean is depicted as a lower bound for the equity premium.](image-url)
Risk-neutral probabilities

Notation: states $x$ have (true) probabilities $p(x)$

Risk-neutral probabilities $p^*$

$$p^*(x) = \frac{p(x)m(x)}{q^1}$$
$$m(x) = \frac{q^1 p^*(x)}{p(x)}$$
$$q^1 = Em \quad (1\text{-period bond price})$$

Entropy (aka “relative entropy” or “Kullback-Leibler divergence”)

$$L(m) = L(p^*/p) = E \log(p/p^*)$$
Risk-neutral probabilities: power utility

Normal log consumption growth
- If $\log g \sim \mathcal{N}(\mu, \sigma^2)$ (true distribution)
- Then risk-neutral distribution also lognormal with $\mu^* = \mu - \alpha \sigma^2, \sigma^* = \sigma$

Poisson log consumption growth
- Jumps have probability $\omega$ and distribution $\mathcal{N}(\theta, \delta^2)$
- Risk-neutral distribution has same form with $\omega^* = \omega \exp[-\alpha \theta + (\alpha \delta)^2 / 2], \theta^* = \theta - \alpha \delta^2, \delta^* = \delta$
Option disasters: overview

Options an obvious source of information ...
  ... about risk-neutral distribution of equity returns

Critical ingredients
  - Option prices
  - Merton model
  - Estimated parameters
  - Implied volatilities
Option disasters: information in option prices

Put option (bet on low returns)

\[ q^p_t = q^1_t E_t^*(b - r^e_{t+1})^+ \]

Strategy

- Estimate \( p^* \) by varying strike price \( b \) (cross section)

Black-Scholes-Merton benchmark

- Quote prices as implied volatilities (high price \( \Leftrightarrow \) high vol)
- Horizontal line if lognormal
- “Skew” suggests disasters
Option disasters: Merton model

Equity returns iid

\[ \log r_{t+1}^e = \log r^1 + w_{t+1} + z_{t+1} \]

\[ w_{t+1} \sim \mathcal{N}(\mu, \sigma^2) \]

\[ z_{t+1}|j \sim \mathcal{N}(j\theta, j\delta^2) \]

\[ j \geq 0 \text{ has probability } e^{-\omega} \omega^j / j! \]

Risk-neutral distribution: ditto with *s
Option disasters: parameter values

Set \((\omega^*, \theta^*, \delta^*)\) to match option prices

- Jumps: \(\omega^* = \omega, \theta^* = -0.0482, \delta^* = 0.0981\)
- Set \(\sigma^* = \sigma\)
- Set \(\mu^*\) to satisfy pricing relation \((q^1 E^* r^e = 1)\)

Later: choose \((\mu, \sigma, \omega, \theta, \delta)\) to match distribution of equity returns

- Jumps: \(\omega = 1.512, \theta = -0.0259, \delta = 0.0229\)
- Equity premium: \(\mu + \omega \theta\)
- Variance of equity returns: \(\sigma^2 + \omega(\theta^2 + \delta^2)\)

All from Broadie, Chernov, and Johannes (JF, 2007)
Option disasters: implied volatility

- Estimated Merton model
- Smaller jump mean $\theta^*$
- Smaller jump standard deviation $\delta^*$

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Comparing macro and option models

Approach 1: compare pricing kernels
  ▶ Required: estimated $p$ from daily data on equity returns

Approach 2: compare consumption growth distributions
  ▶ Required: connections between $g$ and $r^e$, $p$ and $p^*$

Approach 3: compare option prices
  ▶ Required: connections between $g$ and $r^e$, $p$ and $p^*$
Comparing pricing kernels: components of entropy

<table>
<thead>
<tr>
<th>Model</th>
<th>Entropy</th>
<th>Variance/2</th>
<th>Odd</th>
<th>Even</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Consumption-based models</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal ($\alpha = 10$)</td>
<td>0.0613</td>
<td>0.0613</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Poisson ($\alpha = 10$)</td>
<td>0.5837</td>
<td>0.0613</td>
<td>0.2786</td>
<td>0.2439</td>
</tr>
<tr>
<td>Poisson ($\alpha = 5.38$)</td>
<td>0.0449</td>
<td>0.0177</td>
<td>0.0173</td>
<td>0.0099</td>
</tr>
<tr>
<td><strong>Option-based model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Option model</td>
<td>0.7647</td>
<td>0.4699</td>
<td>0.1130</td>
<td>0.1819</td>
</tr>
</tbody>
</table>
Comparing pricing kernels: cumulants

Cumulants of equity return

Contributions to entropy of pricing kernel

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Comparing consumption growth

<table>
<thead>
<tr>
<th>Consumption Process Based on</th>
<th>Cons Growth</th>
<th>Option Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>5.38</td>
<td>10.07</td>
</tr>
<tr>
<td>( \omega )</td>
<td>0.0100</td>
<td>1.3864</td>
</tr>
<tr>
<td>( \theta )</td>
<td>-0.3000</td>
<td>-0.0060</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.1500</td>
<td>0.0229</td>
</tr>
<tr>
<td>Skewness</td>
<td>-11.02</td>
<td>-0.31</td>
</tr>
<tr>
<td>Excess Kurtosis</td>
<td>145.06</td>
<td>0.87</td>
</tr>
<tr>
<td>Tail prob ( \leq -3 \text{ st dev} )</td>
<td>0.0090</td>
<td>0.0086</td>
</tr>
<tr>
<td>Tail prob ( \leq -5 \text{ st dev} )</td>
<td>0.0079</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

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Comparing option prices

![Graph showing implied volatility as a function of moneyness for an option-based model.]

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Comparing option prices

![Graph comparing option-based and consumption-based models](image-url)

- **Option-based model**: Implied Volatility (annual) versus Moneyness.
- **Consumption-based model**: Dashed line shows the decrease in implied volatility as moneyness increases.

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Comparing models

All of these comparisons point in the same direction
  ▶ Macro disasters more pronounced than option disasters
Reconsidering ...

Our models are based on

- iid
- Tight link between consumption growth and equity returns
- Representative agent with power utility

Let’s take a closer look at the last two
Reconsidering power utility

“Risk aversion” implied by arbitrary pricing kernel

\[ RA \equiv -\frac{\partial \log m}{\partial \log g} = -\frac{\partial \log (p^* / p)}{\partial \log r^e} \cdot \frac{\partial \log r^e}{\partial \log g} \]
Reconsidering power utility
Reconsidering the link between $g$ and $r^e$
Reconsidering the link between $g$ and $r^e$
Recapitulation

Barro, Longstaff & Piazzesi, Rietz

- Disasters contribute to equity premium [entropy]
- Evident in macro data
- Range of opinion on magnitude

We look at options

- Smile/smirk suggests something like disasters
- Prices available even for outcomes that don’t occur in sample
- Implied disasters less severe than macro data
- High entropy from options suggests it’s not enough to match equity premium