Portfolio Choice and Permanent Income*

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Abstract

We solve the optimal saving/portfolio-choice problem in an intertemporal recursive utility framework. To date, progress on this problem has been constrained by both the lack of analytical solutions and the computational burdens inherent in large-scale stochastic dynamic programs of this type. Our solution to this problem is sufficiently general to allow (i) risk aversion to vary independently of intertemporal substitution, (ii) many risky assets with stochastic properties that can exhibit very general dynamics, (iii) stochastic labor income that may be correlated with asset returns and/or follow life-cycle patterns, (iv) portfolio adjustment costs, and (v) time-nonseparabilities in preferences (e.g., habit formation and consumption durability). We use the Linear Exponential Quadratic Gaussian (LEQG) model as a starting point. We use perturbation methods around this analytical solution to derive decision rules for portfolios. Unlike previous models that have been solved by these methods, our baseline case is explicitly stochastic, which greatly enhances the accuracy of our approximations without imposing additional computational costs.

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1 Introduction

One of the most basic problems in financial economics is the optimal consumption and portfolio choice of a long-lived investor facing time-varying investment opportunities. As one would expect, the solution to this problem is specific to the investor’s preferences and the stochastic properties of asset payoffs. What is surprising, however, is the difficulty one encounters when solving this problem for even very specific, and restrictive, parametric classes of preferences and investment opportunities. Generally, analytical solutions are available only for cases where the dynamics of the problem are trivialized either through severely myopic restrictions on intertemporal preferences, it e.g., assuming a quadratic, or perhaps logarithmic, expected utility index defined over end-of-period wealth, or through severe restrictions on the stochastic properties of the opportunity set, e.g., iid environments. Absent restrictions like these, analysis of the problem typically proceeds with numerical approximations of the solutions of specific numerical examples of the general problem. Even though our capacity for solving these types of numerical problems is advancing over time, both through improvements in hardware and computational algorithms, analytical solutions to problems in this class would prove extremely useful.

Recently progress has been made on this problem by Campbell and Viceira (1999). They endow the investor with the Kreps-Porteus specification of Recursive Utility (see Epstein and Zin (1989)) that allows separation of static risk aversion from deterministic intertemporal substitution. The investor chooses between an asset with a deterministic, ie., riskless, return and an asset with a random and autocorrelated return. The state variable is a first-order mean-reverting autoregressive process with iid Gaussian innovations. By approximating the first-order conditions of the investor’s maximization problem with a second-order Taylor series, and by using a log-linear approximation of the budget constraint, they obtain closed-form solutions for the approximate problem.

Our approach builds on this work and our approximations are similar in
spirit to those of Campbell and Viceira. The most obvious difference is in how and where we make our approximations. We also model intertemporal preferences with recursive utility, but we assume that the period-t utility function for deterministic consumption is quadratic (rather than the Euler equation). These preferences along with linear, Gaussian state space dynamics result in a model in the class of Linear Exponential Quadratic Gaussian (LEQG) specification of recursive utility. Like the Kreps-Porteus model, they allow a separation of risk aversion from intertemporal substitution, where static risk preferences are of the constant-absolute-risk-aversion form. Therefore, our analysis can draw on the methods of recursive risk sensitive control developed for the discounted LEQG problem by Hansen and Sargent (1995) and applied in Hansen, Sargent, and Tallarini (1999). We can, therefore, greatly expand the class of consumption/portfolio-choice problems that we can solve. This framework allows for multiple risky assets, a state space with higher order stationary dynamics, portfolio adjustment costs, stochastic labor income, and time nonseparabilities in the form of durable goods and/or habits.

We use the LEQG specification as a starting point. Our baseline model is a permanent income model as in Hansen, Sargent, and Tallarini (1999), but we extend this basic model to allow for stochastic returns and a non-trivial portfolio choice problem. We do this by using perturbation methods as described by Judd (1998), Judd and Guu (2001), Mrkaic (1998), and others. The idea is to specify a general problem, find a special case for which an analytical solution is known, and then perturb the problem around this special case. The solution to the general problem can then be approximated using Taylor series approximations of the decision rules evaluated at the analytical special case.

One major difference between our implementation of this procedure and others is that our special case is stochastic. Generally, the only special case that has an analytical solution is a deterministic version of the problem and all randomness is then introduced via the perturbation. Since our baseline model is linear quadratic with Gaussian disturbances, we can approximate around a
model with random shocks. There are two inherent sources of uncertainty in our model: labor income and asset returns. With non-quadratic preferences, the baseline case would have deterministic income. In our framework, we can have as much or as little income variation as we wish without having to make it part of the approximation. Figure 1 demonstrates the benefits of this approach. Looking at the left panel first, standard methods approximate around the origin. Our method is centered at the point $(\sigma_y, 0)$. If the income risk is larger than return risk, the effect is even greater, as captured in the right panel. In fact, when we set risk aversion to be greater than with expected utility, we obtain a solution that exhibits precautionary saving.

We first consider the case of two assets, one with riskless return $R_f$ and the other with risky return $R^e_t$. The idea here is to apply perturbation methods by parameterizing the stochastic return $R^e_t$ as

$$R^e_t = R_f + \epsilon z_t + \epsilon^2 \pi$$

where $z_t$ is a random variable with mean zero and variance $\sigma_z^2$, $\pi$ represents a risk premium, and $\epsilon$ is our perturbation parameter. When $\epsilon = 0$ we know the solution of our problem exactly. When $\epsilon \neq 0$ we can approximate the solution using the Implicit Function for Analytic Operators as in Judd (1998), and Mrkaic (1998). The idea is to expand the decision rule for wealth around the solution with a constant return. Solving the for the portfolio share poses an
additional problem: Now, under certainty, the portfolio choice is indetermi-
nate. So we use the bifurcation method explained in detail in Judd and Guu
(2001) to perform our perturbation around the solution we derived above.

The basic model is detailed in section 2. Section 3 describes our solution
algorithm. Numerical examples of the general consumption/portfolio choice
model are presented in section 4. Section 5 discusses possible extensions.

2 The Model

We start with a linear quadratic expected utility version of the Hall (1978)
permanent income model as in Hansen, Sargent, and Tallarini (1999). The
basic form of the model has one riskless asset that a consumer can use to
smooth consumption relative to income. We augment the model by allowing
the consumer to allocate his savings across several assets: the original riskless
one and additional risky ones. The consumer will now choose a portfolio
allocation in addition to making a consumption-savings decision.

Preferences are quadratic:

$$U_t = -\frac{1}{2} E_t \left[ \sum_{j=0}^{\infty} \beta^j (c_{t+j} - b)^2 \right]$$

where \(E_t\) is the expectation operator conditional on information available at
time \(t\), \(c_t\) is consumption in date \(t\), and \(b\) is a bliss point for consumption.

Our consumer supplies labor inelastically and receives a stochastic stream
of labor income, \(\{y_t\}\), which may be serially correlated. We will assume that
the process for labor income is covariance stationary.

At each date \(t\), the consumer chooses how much wealth, \(a_t\) to carry over
into the next period. The return on this wealth is realized at the beginning
of the next period and depends on the return on the riskless and risky assets
as well as the portfolio allocation chosen in period \(t\). The period \(t\) budget
constraint is
\[ a_t = R^p_t a_{t-1} + y_t - c_t \]  \hspace{1cm} (2)

with \( R^p_t \), the return on the portfolio, given by
\[ R^p_t = \sum_{n=0}^{N} \omega^n_{t-1} R^n_t \]  \hspace{1cm} (3)

where \( \omega^n_{t-1} \) is the fraction of the portfolio allocated to asset \( n \) in period \( t - 1 \). \( R^0 \) is the return on the riskless asset. We make the standard assumption that \( \beta R^0 = 1 \). The portfolio weights must sum to one: \( \sum_{n=0}^{N} \omega^n_{t-1} = 1 \).

We will assume that the risky returns and labor income are jointly Normally distributed and can be written in a state space representation:
\[
\begin{bmatrix}
y_t \\
R^1_t \\
\vdots \\
R^N_t
\end{bmatrix} = Hv_t
\]  \hspace{1cm} (4)

and
\[ v_t = A_v v_{t-1} + G_v w_t \]  \hspace{1cm} (5)

where \( H \) is a selector matrix and \( v_t \) is the underlying state vector that can accommodate linear dynamics of arbitrary order and \( w_t \) is a vector of innovations, Normally distributed with mean zero, and an identity covariance matrix.

It is worth noting the generality in the dynamic environment that this approach allows. Adding higher-order dynamics will enlarge the state space and, as we will see shortly, will imply that the optimal consumption/portfolio choice is the solution to a larger system of equations. However, it does not otherwise alter the form of the solution. Complex dynamics in asset returns or labor income are as easy to incorporate into the model as simple dynamics. Moreover, the computational burden of solving problems with higher-order dynamics is relatively small. The primary limitation of this approach is the need for Gaussian innovations. Assets with cash flows that are not well ap-
proximated with Gaussian stochastic processes, \( e.g., \) options, are difficult to handle in this setup.

## 3 Solution Method

When the return on wealth is constant, \( i.e., \) there is only the riskless asset, we have the permanent income model of Hall (1978) and Hansen, Sargent, and Tallarini (1999). We can solve the model in multiple ways including standard linear-quadratic control methods. With risky assets these methods are no longer sufficient to solve the model. In fact, if we allow wealth to be invested in a single risky asset, and eliminate the portfolio allocation problem, the linear-quadratic nature of the problem evaporates and a different approach is necessary.

A common approach to approximating models without analytic solutions is to find a steady state and linearize. Campbell and Viceira (1999) use a quadratic approximation to the log linearized Euler equation. We follow a different approach, that of Judd and Guu (2001) and Mrkaic (1998). In particular, we will use perturbation methods to derive our approximate solution. The general idea of perturbation methods is to find a simple case with an exact solution and then parameterize the general case as a perturbation treating the approximate solution of the general case as a Taylor expansion around the exact solution of the special case. For stochastic models, this generally means solving the deterministic case and treating randomness as a perturbation. Our choice of the permanent income model with quadratic preferences allows us to make our approximation even closer to the true solution in the sense that our special case is explicitly stochastic. We do not need to shut down the income risk in order to derive an exact solution as a starting point.

Given our exact solution, we expand around it to find our approximate solution taking advantage of the Implicit Function for Analytic Operators as
in Judd (1998) and Mrkaic (1998). For wealth we have

\[ a(x_t, \epsilon) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} a^k(x_t) \]  

(6)

where \( \epsilon \) is the perturbation parameter and \( x_t \) is a state vector that includes the forcing process \( v_t \) as well as any endogenous state variables. For the portfolio weights we have

\[ \omega^n(x_t, \epsilon) = \sum_{k=0}^{\infty} \frac{\epsilon^k}{k!} \omega^{n,k}(x_t) \]  

(7)

This allows us to approximate the true solution with arbitrary accuracy by choosing the number of terms we compute in the expansion.

We will parameterize the risky assets as perturbations around the riskless asset:

\[ R^n_t = R^0 + \epsilon z^n_t + \epsilon^2 \pi^n \]  

(8)

where \( R^0 \) is the riskless asset, \( z^n_t \) is the stochastic component of asset \( n \), \( \pi^n > 0 \) is a measure of the risk premium for asset \( n \), and \( \epsilon \) is the perturbation parameter. The variance of asset \( n \) is \( \epsilon^2 \sigma^2_n \). By multiplying the risk premium by \( \epsilon^2 \) we have a risk premium that is proportional to the variance. We use the same perturbation parameter, \( \epsilon \), for all of the risky assets.

For the remainder of the paper, we will assume that there is only one risky asset. (Adding more is a straightforward extension.) We can therefore dispense with the superscripts on the portfolio weights. We will use \( \omega_t \) to denote the fraction of wealth allocated to the risky asset, with beginning of period return \( R^e_t \), at the end of period \( t \). Consequently, \( 1 - \omega_t \) is the fraction allocated to the riskless asset, with beginning of period return \( R^f \). The perturbation is \( R^e_t - R^f = \epsilon z_t + \epsilon^2 \pi \). There are two decision rules to approximate: end of period wealth and the portfolio weight. There are two optimality conditions that we will use to derive the coefficient functions for our approximation. The intertemporal Euler equation is

\[ b - c_t = \beta E_t[(b - c_{t+1})(\omega_t(R^e_{t+1} - R^f) + R^f)]. \]  

(9)
The optimality condition for the portfolio weight is

\[ E_t[\beta (b - c_{t+1})(R^e_{t+1} - R^f)] = 0. \]  

(10)

The perturbation method proceeds by substituting our expansions into these equations, differentiating with respect to \( \epsilon \), setting \( \epsilon \) to 0 and then solving for the first term in the expansion. We repeat this procedure to solve for higher order terms.

The zero-th-order term in the expansion for \( a \) is the decision rule from the linear quadratic permanent income model. The zero-th-order term in the expansion for \( \omega \) is not nearly as straight forward. When \( \epsilon = 0 \), the portfolio choice is indeterminate. From Eq(8), when \( \epsilon = 0 \) \( R^e_t = R^0 \). So there is no obvious choice for \( \omega^0 \). We closely follow Judd and Guu (2001) and apply bifurcation methods to find \( \omega^0 \) and then proceed as above.

### 3.1 The Portfolio Problem and Bifurcations

The idea behind bifurcation methods is that when \( \epsilon \neq 0 \), \( \omega(\epsilon) \) is well defined, but when \( \epsilon = 0 \), any value of \( \omega \) satisfies the optimality conditions. Graphically it looks like Figure 2.

We find the bifurcation point, \( \omega^0 \), by starting with the optimality condition for \( \omega \), Eq(10), and deriving a function \( H(\omega, \epsilon) \) that implicitly defines \( \omega(\epsilon) \):

\[ H(\omega(x_t, \epsilon), \epsilon) = E_t[\beta (b - c(x_{t+1}, \epsilon))(z_{t+1} + \epsilon \pi)] = 0 \]  

(11)

Note that we have factored out an \( \epsilon \) from the optimality condition. Since \( \epsilon \) is a constant, this does not affect the expectation operation. Consumption in period \( t + 1 \) is derived from the evolution equation for wealth, Eq(2), and can be substituted for so we can focus on the approximations for \( a \) and \( \omega \). In what follows, dependence on the state vector \( x_t \) will be represented by a time
Figure 2: The optimal portfolio is a function of the perturbation parameter, $\omega(\epsilon)$. When $\epsilon = 0$, the portfolio choice is indeterminate.

We want to compute a Taylor series expansion for $\omega(\epsilon)$. We can implicitly differentiate Eq(12) to obtain

$$H \omega + H \epsilon = 0$$  \hspace{1cm} (13)

When $\epsilon = 0$, $H \omega = 0$ for all $\omega$ so we cannot apply the implicit function theorem to compute $\omega'$. Rearranging Eq(13) as

$$\omega' = -\frac{H \epsilon}{H \omega}$$  \hspace{1cm} (14)

The denominator is 0 when $\epsilon = 0$, but if there is a point $\omega_0$ such that
$H_t(\omega_t^0, 0) = 0$ then we can apply L'Hospital's rule to compute $\omega^1$:

$$\omega^1_t = -\frac{dH_t}{d\epsilon}$$

which is well defined as long as $H_{\omega\epsilon} \neq 0$. We find $\omega_t^0$ by setting $H(\omega_t^0, 0) = 0$. Constraining $z_t$ to be iid, we have:

$$\omega_t^0 = \frac{\beta E_t[a_{t+1}^1 z_{t+1}] + \pi \beta E_t[b - c_t^0]}{a_t^1 \sigma_z^2}.$$  

(16)

The unperturbed permanent income model generates $c_{t+1}^0$. When $\epsilon = 0$ and recalling the assumption that $\beta R_f = 1$, it also implies that

$$b - c_t^0 = E_t[\beta R_f (b - c_{t+1}^0)] = E_t[b - c_t^0]$$

(17)

so

$$\omega_t^0 = \frac{\beta E_t[a_{t+1}^1 z_{t+1}] + \pi \beta [b - c_t^0]}{a_t^1 \sigma_z^2}.$$  

(18)

Now we have a centering point for our Taylor approximation. We compute $a^0$ from the unperturbed permanent income model and $a^1$ is the first-order coefficient in the Taylor expansion for $a$ which will be derived below.

We derive the first-order coefficient, $\omega^1_t$ by implicitly differentiating $H(\omega, \epsilon)$ a second time,

$$H_{\omega\omega} \omega^1 + 2H_{\omega\epsilon} + H_{\epsilon\epsilon} = 0,$$

(19)

and rearranging to get

$$\omega^1 = -\frac{1}{2} \frac{H_{\epsilon\epsilon}}{H_{\omega\omega}}$$

(20)

since $H_{\omega} = 0$ for all $\omega$ when $\epsilon = 0$ and likewise for $H_{\omega\epsilon}$. For our problem we have

$$\omega^1_t = -\frac{1}{2} \frac{H_{\epsilon\epsilon}}{H_{\omega\omega}} = \frac{\pi (E_t[a_{t+1}^1] - R_f a_t^1)}{a_t^1 \sigma_z^2} - \omega_t^0 a_t^1 \sigma_z^2 + \frac{1}{2} E_t[a_{t+1}^2 z_{t+1}].$$

(21)

where $a^2$ is the second-order coefficient for wealth, again to be derived below.
3.2 The Decision Rule for Wealth

The zero-th-order term, $a_t^0$, is derived from the unperturbed permanent income model. This rule is linear in the state variable $x_t$. To derive the higher order terms, we return to the intertemporal Euler Equation, Eq(9), substituting in for $c_t$ and $c_{t+1}$:

$$b - [\omega_{t-1}(R_t^e - R_t^f) + R_t^f]a_{t-1} - y_t + a_t = \beta(\omega_t(R_{t+1}^e - R_{t+1}^f) + R_{t+1}^f)a_{t+1} - y_{t+1} + a_{t+1}).$$

We substitute our Taylor expansion into this equation for $a_{t-1}, a_t, a_{t+1}, \omega_{t-1}, \omega_t, R_t^e - R_t^f$, and $R_{t+1}^e - R_{t+1}^f$. Then we differentiate with respect to $\epsilon$ and set $\epsilon = 0$. The result is a second order stochastic difference equation in $a_1^1$:

$$R_t^f a_{t-1}^1 - (1 + \beta R_t^f)^2 a_1^1 x_t + \beta R_t^f E_t[a_{t+1}^1]$$

$$= -\omega_{t-1}^0 a_{t-1}^0 z_t + 2R_t^f w_t a_t^0 E_t[z_{t+1}] - \beta b \omega_t^0 E_t[z_{t+1}]$$

$$+ \beta \omega_t^0 E_t[y_{t+1} z_{t+1}] - \beta \omega_t^0 E_t[a_{t+1}^0 z_{t+1}].$$

Let $\delta_t$ be equal to the right hand side of Eq(23). Recalling that $\beta R_t^f = 1$ we can rewrite this difference equation and ignoring expectation for the time being we can write this as

$$(R_t^f L - (1 + R_t^f) + L^{-1})a_1^1 = \delta_t.$$

Not coincidentally, the lag polynomial in this difference equation is that same as in the second-order stochastic difference equation for wealth in the unperturbed permanent income model. We can factor it the same way to get

$$(1 - L)a_1^1 = -\frac{1}{R_t^f} (1 - \frac{1}{R_t^f} L^{-1})^{-1} \delta_t$$

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which, after we reintroduce expectations, gives us

\[ a_t^1 - a_{t-1}^1 = -\beta E_t[\sum_{j=0}^{\infty} \beta^j \delta_{t+j}] . \]  

(25)

Taking advantage of the assumption that \( z_t \) is iid allows us to simplify \( \delta_t \):

\[ \delta_t = -\omega_t^0 a_{t-1}^0 z_t + \beta \omega_t^0 E_t[(y_{t+1} - a_{t+1}^0)z_{t+1}] \]  

(26)

Now we will begin to consolidate the approximations of the decision rules for wealth and the portfolio weight. To clarify the analysis, we will make one final assumption that we will maintain for the remainder of the paper: \( z_t \) is uncorrelated with all other exogenous random variables at all leads and lags which implies that it is uncorrelated with the endogenous state variable in the solution to the unperturbed permanent income model \((a^0)\). The bifurcation point, \( \omega_0^0 \) simplifies to

\[ \omega_0^0 = \frac{\pi}{\sigma_z^2} \frac{b - c_{t-1}^0}{a_{t-1}^0} . \]  

(27)

Looking at \( \delta_t \):

\[ \delta_t = -\omega_{t-1}^0 a_{t-1}^0 z_t = -\beta \frac{\pi}{\sigma_z^2} (b - c_{t-1}^0) z_t \]  

(28)

and

\[ E_t[\delta_{t+1}] = -\beta \frac{\pi}{\sigma_z^2} E_t[(b - c_t^0)z_{t+1}] = 0 \]  

(29)

due to the assumption that \( z \) is uncorrelated with the solution of the unperturbed permanent income model. This implies that that the first-order correction for \( a \) is:

\[ a_t^1 - a_{t-1}^1 = -\beta \delta_t = \beta^2 \frac{\pi}{\sigma_z^2} z_t (b - c_{t-1}^0) . \]  

(30)

The first-order coefficient for the portfolio weight simplifies to

\[ \omega_t^1 = \frac{\pi[(1 - \beta) \delta_t - (R - 1) a_{t-1}^1] - \omega_0^1 a_t^1 \sigma_z^2}{a_t^1 \sigma_z^2} \]  

(31)
\[ \omega_{t+1} - \omega_t = -\beta \left( \frac{\pi}{\sigma_z^2} \right)^2 \left( 1 - \beta \right) a_t^0 (b - c_t^{0,-1}) z_t + a_t^1 (b - c_t^{0}) \sigma_z^2 \frac{a_t^0}{a_t^1} - (R^f - 1) \pi \frac{a_{t-1}^1}{a_t^0} \]

The second-order term for wealth, \(a_{t+1}^2\), drops out of the expression for \(\omega_t^1\) due to \(z\)'s lack of correlation with all other variables. Relaxing this assumption would therefore require the computation of \(a_t^2\). This is achieved by differentiating the intertemporal Euler equation, Eq(9), a second time with respect to \(\epsilon\) before setting \(\epsilon = 0\). The result is another second-order difference equation of the same form as for \(a_t^0\) and \(a_t^1\). The only difference is that instead of \(\delta_t\), the difference equation is equal to a new random variable, \(\gamma_t\). For our present purposes, we will restrict ourselves to a linear approximation for wealth and will not go into the details about \(\gamma_t\). \(^1\)

### 3.3 Discussion

What do these coefficients mean? Let’s start with the approximation for the portfolio choice. Eq(27) implies that the centering point for the approximation is increasing in the risk premium and decreasing in the variance of the risky return. This is consistent with standard intuition. As consumption or wealth decrease, the portfolio weight increases. This may seem counterintuitive: as the consumer gets poorer he takes larger risks in his portfolio. In fact, the solution of \(\omega_t^0\) is identical to that in Judd and Guu (2001). The difference is that they do not impose any parametric restrictions on the single period utility function as we do here and our model is dynamic rather than static. The portfolio centering point in Judd and Guu is the ratio of the risk premium to the variance divided by the coefficient of relative risk aversion. With quadratic utility, relative risk aversion is increasing.

The first-order coefficient for \(\omega_t\) changes sign depending on the relative

\(^1\)We feel that our choice of a linear approximation to the decision rule for wealth is sound due to the high degree of linearity associated with the policy functions of standard growth models. In these models there is curvature in the production function, but still the policy functions are nearly linear. Here, the production function itself is linear. See Christiano (1990).
magnitudes of $z_t$ and $a^1_t$. As wealth increases, $\omega^1_t$ shrinks in absolute value both due to a decreasing numerator (consumption approaches the bliss point) and an increasing denominator. As each of these occur, the consumer becomes locally more risk averse leading to a smaller allocation to the risky asset.

Turning to the first-order coefficient for wealth, Eq(30), when the return is above average ($z_t > 0$), $a^1_t$ is positive which results in an increase in end of period wealth. After substituting the approximations into the budget constraint, Eq(2), and ignoring terms in $\epsilon^2$ and higher, consumption increases as well.

4 Numerical Implementation

With our approximate decision rules for wealth and the portfolio share we can chose values for the parameters of the model and some initial conditions and then run simulations. Alternatively, given initial conditions, we can compute the optimal choices for $a$ and $\omega$ across the possible realizations for income and the return on the risky asset. The random walk nature of consumption in the unperturbed model can lead simulations into economically uninteresting parts of the state space.\textsuperscript{3} Therefore, we will take the second approach and characterize the approximated decision rules for given initial conditions under several assumptions about the stochastic process for income.

Before proceeding to characterize the decision rules, we should note that simulating a time path for wealth and portfolio share requires resetting the initial conditions for the approximation for each period. At time $t$, $a_{t-1}$ is known exactly. In addition, the first-order coefficient, $a^1_{t-1}$ is set to zero every period. In other words, we treat Eq(30) as:

$$a^1_t = -\beta \delta_t = \beta^2 \frac{\pi}{\sigma^2} z_t (b - c^0_{t-1}). \quad (32)$$

\textsuperscript{2}We will focus on the first term in Eq(31) for reasons that will be clear shortly.

\textsuperscript{3}Negative consumption and consumption in excess of the bliss point are both possibilities in the unperturbed model.
If we were to allow $a_{t-1}$ to vary from period to period we would be introducing a random walk component into the approximation. The problem is not time-dependent so the approximate solution should not be time-dependent either. Note that when we allow for correlation between returns and income the infinite sum in Eq(25) is not affected, only the value of $a_{t-1}$.

Given initial conditions for wealth, $a_0$, and income, $y_0$, we still need a initial portfolio allocation, $\omega_0$. We will use the following procedure to derive an approximately optimal choice of $\omega_0$. The decision rule for wealth from the unperturbed permanent income model is linear in income, lagged wealth, and a constant: $a_t = F x_t$ for some row vector $F$. Recall that one element of the state vector $x_t$ is lagged wealth, $a_{t-1}$. We can use the linear decision rule, $a_0$, and $y_0$ to infer a value for $a_{-1}$. This allows us to construct $x_0$. Using our constructed state vector, we can compute $c_0$ using the decision rule from the unperturbed model. This value is plugged into Eq(27) to compute $\omega_0$.

5 Extensions

It is possible to extend the model discussed above to include serially correlated returns, returns that are correlated with income, the absence of a truly riskless asset, and multiple assets. Our choice of the linear quadratic permanent income model as our special case also allows us to consider non-expected utility preferences and portfolio adjustment costs. In particular, we can specify preferences recursively as

$$U_t = -\frac{1}{2} (c_t - b)^2 - \theta(\omega_t, \omega_{t-1}) + \beta \frac{-2}{\alpha} \log E_t \left[ \exp \left( \frac{-\alpha}{2} U_{t+1} \right) \right],$$

where $\theta$ is an adjustment cost function and $\alpha$ is a parameter that measures risk aversion. Values of $\alpha > 0$ indicate greater risk aversion relative to expected utility. Our structure can also accommodate linear habit formation, durable goods, multiple types of goods, and taxation. The addition of a portfolio adjustment cost is a straightforward addition to the bifurcation methodology.
detailed in Section 3. The addition of non-expected utility is somewhat more challenging since the Euler equations for consumption and portfolio choice will also depend on the level of utility at the optimum. Note, however, that we can follow the same procedure outline above in this case. That is, from Hansen, Sargent, and Tallarini (1999) we have an exact (quadratic) solution for this utility function in the constant-return special case. We then repeatedly differentiate equation Eq(33) as we did in equation Eq(11) to obtain the perturbation terms for values of $\epsilon \neq 0$. As with wealth, the level of utility is likely to be quite smooth, hence it will be well-approximated by a low-order perturbation around the stochastic special case. These extensions are left for future work.

References


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