Disasters Implied by Equity Index Options

David Backus (NYU), Mikhail Chernov (LBS),
and Ian Martin (Stanford)

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Summary

Problem: disasters infrequent $\Rightarrow$ hard to estimate their distribution

Solution: infer from option prices (market prices of bets on disasters)

What we find

- disasters apparent in options data
- more modest than disasters in macro data
Outline

Preliminaries: entropy, AJ bound, cumulants

Disasters in macroeconomic data

Disasters in options data

Extensions
Hans-Otto Georgii (quoted by Hansen and Sargent):

*When Shannon had invented his quantity and consulted von Neumann on what to call it, von Neumann replied: “Call it entropy. It is already in use under that name and, besides, it will give you a great edge in debates because nobody knows what entropy is anyway.”*
Alvarez-Jerman bound

Entropy: for $x > 0$

$$L(x) \equiv \log E x - E \log x \geq 0$$

AJ bound

$$L(m) \geq E (\log r^j - \log r^1)$$
Cumulants

Cumulant generating function

\[ k(s; x) = \log E e^{sx} = \sum_{j=1}^{\infty} \kappa_j(x) s^j / j! \]

Cumulants are almost moments

- mean \(=\) \(\kappa_1\)
- variance \(=\) \(\kappa_2\)
- skewness \(=\) \(\kappa_3 / \kappa_2^{3/2}\)
- (excess) kurtosis \(=\) \(\kappa_4 / \kappa_2^2\)
Entropy and cumulants

Entropy of pricing kernel

\[ L(m) = \log E e^{\log m} - E \log m = \sum_{j=2}^{\infty} \kappa_j (\log m)/j! \]

Zin’s “never a dull moment” conjecture

\[ L(m) = \underbrace{\kappa_2 (\log m)/2!}_{(\log)\text{normal term}} + \underbrace{\kappa_3 (\log m)/3!}_{\text{high-order cumulants (incl disasters)}} + \underbrace{\kappa_4 (\log m)/4!}_{\text{high-order cumulants (incl disasters)}} + \cdots \]
Plan of attack

Modelling assumptions

- iid
- Tight link between consumption growth and equity returns
- Representative agent with power utility [if needed]

Parameter choices

- Match mean and variance of log consumption growth
- Ditto log equity return
- Base “disasters” on Barro’s macroeconomic evidence
- Or on equity index options

Compare macro- and option-based examples
Macro disasters: environment

Consumption growth and “equity” return

\[ g_{t+1} = \frac{c_{t+1}}{c_t} \]
\[ d_t = c_t^\lambda \]
\[ \log r^e_{t+1} = \text{constant} + \lambda \log g_{t+1} \]

Power utility

\[ \log m_{t+1} = \log \beta - \alpha \log g_{t+1} \]

Yaron’s “bazooka”

\[ \kappa_j(\log m)/j! = \kappa_j(\log g)(-\alpha)^j/j! \]
Macro disasters: Poisson-normal mixture

Consumption growth

\[ \log g_{t+1} = w_{t+1} + z_{t+1} \]

\[ w_{t+1} \sim \mathcal{N}(\mu, \sigma^2) \]

\[ z_{t+1} | j \sim \mathcal{N}(j\theta, j\delta^2) \]

\[ j \geq 0 \text{ has probability } e^{-\omega \omega j / j!} \]

Parameter values

- Match mean and variance of log consumption growth
- Jump probability ($\omega = 0.01$), mean ($\theta = -0.3$), and variance ($\delta^2 = 0.15^2$) [similar to Barro, Nakamura, Steinsson, and Ursua]
Macro disasters: entropy

Cumulant generating functions

\[ k(s; w) \equiv \log E e^{sw} = s\mu + (s\sigma)^2/2 \]

\[ k(s; z) \equiv \log E e^{sz} = \omega \left( e^{s\theta} + (s\delta)^2/2 - 1 \right) \]

Entropy

\[ L(m) = (-\alpha \sigma)^2/2 + \omega \left( e^{-\alpha \theta} + (\alpha \delta)^2/2 - 1 \right) + \alpha \omega \theta, \]
Macro disasters: entropy

Graph showing the relationship between entropy of the pricing kernel $L(m)$ and risk aversion $\alpha$. The graph includes the Alvarez–Jermann lower bound and a normal distribution line.
Macro disasters: entropy

![Graph showing entropy of pricing kernel L(m) versus risk aversion α. The graph includes lines for Alvarez–Jermann lower bound, normal, and disasters.]

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Disasters in options
Macro disasters: entropy

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Macro disasters: cumulants

- Cumulants
- Contributions

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Macro disasters: cumulants

<table>
<thead>
<tr>
<th>Model ((\alpha = 10))</th>
<th>Entropy</th>
<th>Variance/2</th>
<th>Odd</th>
<th>Even</th>
</tr>
</thead>
<tbody>
<tr>
<td>Normal</td>
<td>0.0613</td>
<td>0.0613</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Poisson disaster</td>
<td>0.5837</td>
<td>0.0613</td>
<td>0.2786</td>
<td>0.2439</td>
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<tr>
<td>Poisson boom</td>
<td>0.0266</td>
<td>0.0613</td>
<td>-0.2786</td>
<td>0.2439</td>
</tr>
</tbody>
</table>
Macro disasters: equity premium

![Graph showing the relationship between entropy and equity premium with respect to risk aversion (α). The x-axis represents risk aversion, ranging from 0 to 12, and the y-axis represents entropy and equity premium, ranging from -0.5 to 2. The graph includes two curves, one labeled "entropy" and another labeled "sample mean = AJ bound equity premium."
Option disasters: overview

Options an obvious source of information, but ...

- Options on equity, not consumption
- Determine risk-neutral, not true distribution
- True distribution has the usual lack of data problems

Plan of attack

- Estimate risk-neutral distribution from options
- Estimate true distribution two ways
- Compare options implied by macro-based disaster model
Option disasters: overview

Options an obvious source of information, but ...
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Plan of attack
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Risk-neutral probabilities

Notation: states $x$ have (true) probabilities $p(x)$

Risk-neutral probabilities $p^*$

$$p^*(x) = \frac{p(x)m(x)}{q^1}$$
$$m(x) = \frac{q^1 p^*(x)}{p(x)}$$
$$q^1 = Em \ (1\text{-period bond price})$$

Entropy (aka “relative entropy” or “Kullback-Leibler divergence”)

$$L(m) = L(p^*/p) = E \log(p/p^*)$$
Normal log consumption growth

- If $\log g \sim \mathcal{N}(\mu, \sigma^2)$ (true distribution)
- Then risk-neutral distribution also lognormal with
  \[ \mu^* = \mu - \alpha \sigma^2, \sigma^* = \sigma \]

Poisson log consumption growth

- Jumps have probability $\omega$ and distribution $\mathcal{N}(\theta, \delta^2)$
- Risk-neutral distribution has same form with
  \[ \omega^* = \omega \exp[-\alpha \theta + (\alpha \delta)^2/2], \theta^* = \theta - \alpha \delta^2, \delta^* = \delta \]
Option disasters: information in option prices

Put option (bet on low returns)

\[ q_t^p = q_t^1 E_t^* (b - r_{t+1}^e)^+ \]

Strategy

- Estimate \( p^* \) by varying strike price \( b \) (cross section)
- Estimate \( p \) and \( q^1 \) from time series data

Black-Scholes-Merton benchmark

- Quote prices as implied volatilities (high price ⇔ high vol)
- Horizontal line if lognormal
- “Skew” suggests disasters
Option disasters: Merton model

Equity returns iid

\[
\log r_{t+1}^e = \log r^1 + w_{t+1} + z_{t+1}
\]

\[
w_{t+1} \sim \mathcal{N}(\mu, \sigma^2)
\]

\[
z_{t+1} | j \sim \mathcal{N}(j\theta, j\delta^2)
\]

\[
j \geq 0 \text{ has probability } e^{-\omega \omega^j / j!}
\]

Risk-neutral distribution: ditto with *s
Choose \((\mu, \sigma, \omega, \theta, \delta)\) to match distribution of equity returns

- **Jumps:** \(\omega = 1.512, \theta = -0.0259, \delta = 0.0229\)
- **Equity premium:** \(\mu + \omega \theta\)
- **Variance of equity returns:** \(\sigma^2 + \omega (\theta^2 + \delta^2)\)

Set \((\omega^*, \theta^*, \delta^*)\) to match option prices

- **Jumps:** \(\omega^* = \omega, \theta^* = -0.0482, \delta^* = 0.0981\)
- **Set** \(\sigma^* = \sigma\)
- **Set** \(\mu^*\) to satisfy pricing relation \((q^1 E^* r^e = 1)\)

All of this from Broadie, Chernov, and Johannes (JF, 2007)
Option disasters: implied volatility

- Implied Volatility (annual units)
- Estimated Merton model
- Smaller $\theta^*$
- Smaller $\delta^*$
- Smaller $\delta^*$ and positive $\theta^*$

Moneyness: difference of return from zero

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## Option disasters: components of entropy

<table>
<thead>
<tr>
<th>Model</th>
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<th>Variance/2</th>
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<th>Even</th>
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<tbody>
<tr>
<td><strong>Consumption-based models</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Normal ((\alpha = 10))</td>
<td>0.0613</td>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Poisson ((\alpha = 10))</td>
<td>0.5837</td>
<td>0.0613</td>
<td>0.2786</td>
<td>0.2439</td>
</tr>
<tr>
<td>Poisson ((\alpha = 5.38))</td>
<td>0.0449</td>
<td>0.0177</td>
<td>0.0173</td>
<td>0.0099</td>
</tr>
<tr>
<td><strong>Option-based model</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Option model</td>
<td><strong>0.7647</strong></td>
<td><strong>0.4699</strong></td>
<td><strong>0.1130</strong></td>
<td><strong>0.1819</strong></td>
</tr>
</tbody>
</table>
Option disasters: cumulants

Cumulants $j!$

$p$

$p^*$

$m$

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Disasters in options
Comparing macro- and option-based models

Entropy and cumulants of pricing kernel

Consumption growth implied by option prices
- Scale option-based \( p^* \) to consumption
- Find \( p \) using power utility
- Result: more modest skewness and kurtosis, tail probabilities

Option prices implied by consumption growth
- Find macro-based \( p^* \) using power utility
- Scale to equity returns
- Compute option prices
- Result: steeper volatility smile
## Comparing models: consumption implied by options

<table>
<thead>
<tr>
<th></th>
<th>Consumption Process Based on Cons Growth</th>
<th>Option Prices</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$\alpha$</strong></td>
<td>5.38</td>
<td>10.07</td>
</tr>
<tr>
<td><strong>$\omega$</strong></td>
<td>0.0100</td>
<td>1.3864</td>
</tr>
<tr>
<td><strong>$\theta$</strong></td>
<td>-0.3000</td>
<td>-0.0060</td>
</tr>
<tr>
<td><strong>$\delta$</strong></td>
<td>0.1500</td>
<td>0.0229</td>
</tr>
<tr>
<td><strong>Skewness</strong></td>
<td>-11.02</td>
<td>-0.31</td>
</tr>
<tr>
<td><strong>Excess Kurtosis</strong></td>
<td>145.06</td>
<td>0.87</td>
</tr>
<tr>
<td><strong>Tail prob ($\leq -3$ st dev)</strong></td>
<td><strong>0.0090</strong></td>
<td><strong>0.0086</strong></td>
</tr>
<tr>
<td><strong>Tail prob ($\leq -5$ st dev)</strong></td>
<td><strong>0.0079</strong></td>
<td><strong>0.0002</strong></td>
</tr>
</tbody>
</table>
Comparing models: options implied by consumption

Moneyness: difference of return from zero

Implied Volatility (annual units)

consumption model (12 months)

consumption model (3 months)
Comparing models: options implied by consumption

Option disasters

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Risk aversion in the option model

“Risk aversion” implied by arbitrary pricing kernel

$$
RA \equiv -\frac{\partial \log m}{\partial \log g} = -\frac{\partial \log(p^*/p)}{\partial \log r^e} \cdot \frac{\partial \log r^e}{\partial \log g}
$$

Obviously not power utility

- Risk aversion not constant ("state dependent")
- Parameters imply greater aversion to adverse risks
Risk aversion in the option model

![Graph showing risk aversion in the option model]

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Bottom line

Barro, Longstaff & Piazzesi, Rietz

- Disasters contribute to equity premium, entropy
- Evident in macro data

We look at options

- Smile/smirk suggests something like disasters
- But more modest than macro data
- High entropy suggests it’s not enough to match equity premium
Open questions

Sources of apparent risk aversion
- Exotic preferences
- Heterogeneous agents
- Examples: Alvarez, Atkeson, and Kehoe; Bates; Chan and Kogan; Du; Guvenen; Lustig and Van Nieuwerburgh

Consumption and dividends
- Examples: Bansal and Yaron, Gabaix, Longstaff and Piazzesi

Time dependence
- Short rate, predictable returns, stochastic volatility
- Examples: Drechsler and Yaron, Wachter, Shaliastovich
Time dependence: irrelevant?

Cochrane and Hansen

\[
\text{Var}(m) = E\text{Var}_t(m_{t+1}) + \underbrace{\text{Var}[E_t(m_{t+1})]}_{\text{small}}
\]

Analog for entropy

\[
L(m) = E\text{L}_t(m_{t+1}) + \underbrace{\text{L}[E_t(m_{t+1})]}_{\text{ditto}}
\]

Picture something like this

\[
\log m = \underbrace{\text{white noise}}_{\text{big}} + \underbrace{\text{predictable component}}_{\text{small}}
\]
Time dependence: entropy

Alvarez-Jermann bound

\[ L(m) \geq E \left( \log r^j - \log r^1 \right) + L(q^1) \]

Conditional and unconditional entropy

\[ L(m) = EL_t(m_{t+1}) + L(q^1) \]

Mean conditional entropy ("drift irrelevant")

\[ EL_t(m_{t+1}) \geq E \left( \log r^j - \log r^1 \right) \]
Examples with time dependence

Time dependence: examples

Explore mechanisms for magnifying entropy

- Recursive preferences
- Habits
- Heterogeneous preferences [maybe later]

Exploits loglinearity of entropy and (many) asset pricing models

Thanks to Stan Zin for much of this
Recursive preferences: traditional version

Equations (Kreps-Porteus/Epstein-Zin/Weil)

\[ U_t = \left[ (1 - \beta)c_t^\rho + \beta \mu_t(U_{t+1})^\rho \right]^{1/\rho} \]
\[ \mu_t(U_{t+1}) = \left( E_t U_{t+1}^\alpha \right)^{1/\alpha} \]
\[ IES = \frac{1}{1 - \rho} \]
\[ CRRA = 1 - \alpha \]
\[ \alpha = \rho \Rightarrow \text{additive preferences} \]

Note: weakly separable, not additively separable
Recursive preferences: pricing kernel

Scale problem by $c_t$ ($u_t = U_t/c_t$, $g_{t+1} = c_{t+1}/c_t$)

$$u_t = [(1 - \beta) + \beta \mu_t (g_{t+1}u_{t+1})^\rho]^ {1/\rho}$$

Pricing kernel (mrs)

$$m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\rho-1} \left( \frac{U_{t+1}}{\mu_t(U_{t+1})} \right)^{\alpha - \rho}$$

$$= \beta g_{t+1}^{\rho-1} \left( \frac{g_{t+1}u_{t+1}}{\mu_t(g_{t+1}u_{t+1})} \right)^{\alpha - \rho}$$
Examples with time dependence

Recursive preferences: loglinear approximation

Approximation

\[ \log u_t = \rho^{-1} \log \left[ (1 - \beta) + \beta \mu_t (g_{t+1} u_{t+1})^\rho \right] \]

\[ = \rho^{-1} \log \left[ (1 - \beta) + \beta e^{\rho \log \mu_t (g_{t+1} u_{t+1})} \right] \]

\[ \approx b_0 + b_1 \log \mu_t (g_{t+1} u_{t+1}). \]

Exact if \( \rho = 0 \): \( b_0 = 0, \ b_1 = \beta \)

Solve by guess and verify
Example: Bansal-Yaron

Consumption growth

\[
\log g_t = g + \gamma(L) v_{t-1}^{1/2} w_{1t} \\
v_t = v + \nu(L) w_{2t} \\
(w_{1t}, w_{2t}) \sim \text{NID}(0, I)
\]

Guess value function

\[
\log u_t = u + \omega_g(L) v_{t-1}^{1/2} w_{1t} + \omega_v(L) v_t
\]

Solution includes

\[
\omega_{g0} + \gamma_0 = \gamma(b_1) \\
\omega_{v0} = b_1(\alpha/2) \gamma(b_1)^2 \nu(b_1)
\]
Example: Bansal-Yaron (continued)

Pricing kernel

\[
\log m_{t+1} = \log \beta + (\rho - 1)g - (\alpha - \rho)(\alpha/2)\omega^2_v \\
+ (\rho - 1)[\gamma(L)/L] + v_{t-1}^{1/2}w_{1t} - (\alpha - \rho)(\alpha/2)\gamma(b_1)^2v_t \\
+ [(\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1)]v_t^{1/2}w_{1t+1} \\
+ (\alpha - \rho)\omega^2_v w_{2t+1}
\]

Conditional entropy (monthly)

\[
L_t(m_{t+1}) = [(\rho - 1)\gamma_0 + (\alpha - \rho)\gamma(b_1)]^2v_t/2 + (\alpha - \rho)^2\omega^2_v/2 \\
0.0218 = 0.0065 + 0.0153 \\
0.0026 = 0.0026 + 0.0000 \text{ if } \rho = \alpha
Example: Wachter

Consumption growth

\[ \log g_t = g + \sigma w_{1t} + z_t \]
\[ \lambda_t = (1 - \varphi)\lambda + \varphi \lambda_{t-1} + \sigma \lambda w_{2t} \]
\[ (w_{1t}, w_{2t}) \sim \text{NID}(0, I) \]
\[ z_t | j \sim \mathcal{N}(j\theta, j\delta^2) \]
\[ j \geq 0 \text{ has jump intensity } \lambda_{t-1} \]

Guess value function

\[ \log u_t = u + \omega \lambda \lambda_t \]

Solution includes

\[ \omega \lambda = (1 - b_1 \varphi)^{-1} b_1 \left[ e^{\alpha \theta + (\alpha \delta)^2 / 2} - 1 \right] / \alpha \]
Example: Wachter (continued)

Pricing kernel

\[
\log m_{t+1} = \log \beta + (\rho - 1)x - (\alpha - \rho)(\alpha/2)[\sigma^2 + (\omega \lambda \sigma \lambda)^2] \\
- \lambda_t \left( e^{\alpha \theta + (\alpha \delta)^2/2 - 1} - 1 \right)/\alpha \\
+ (\alpha - 1)(\sigma w_{1t+1} + z_{t+1}) + (\alpha - \rho)(\omega \lambda \sigma \lambda)w_{2t+1}
\]

Conditional entropy (monthly) [where’s the bazooka?]

\[
L_t(m_{t+1}) = (\alpha - 1)^2 \sigma^2/2 + (\alpha - \rho)^2(\omega \lambda \sigma \lambda)^2/2 \\
+ \lambda_t \left\{ e^{(\alpha - 1) \theta + (\alpha - 1)^2 \delta^2/2 - 1} - (\alpha - 1)\theta \right\}
\]

\[
0.0100 = 0.0001 + 0.0087 + 0.0012 \\
0.0013 = 0.0001 + 0.0000 + 0.0012 \text{ if } \rho = \alpha
\]
Example: Abel/Chan-Kogan external habit

Additive preferences with

\[ v(c_t, x_t) = \left( \frac{c_t}{x_t} \right)^\alpha / \alpha \]

\[ x_{t+1} = x + \chi(L)y_t \quad \text{("predetermined")} \]

Pricing kernel

\[ m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha-1} \left( \frac{x_{t+1}}{x_t} \right)^\alpha \]

Entropy: habit irrelevant to conditional entropy [!?]
Example: Abel/Chan-Kogan external habit

Additive preferences with

\[ v(c_t, x_t) = \left( \frac{c_t}{x_t} \right)^\alpha / \alpha \]

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Entropy: habit irrelevant to conditional entropy [!?]
Example: recursive Abel/Chan-Kogan

Consumption growth

\[
\log g_t = g + \gamma_0 w_t, \quad \{w_t\} \sim \text{NID}(0, 1)
\]

Pricing kernel

\[
\log m_{t+1} = \text{constants} + \text{things dated } t \text{ and before} \\
+ \{(\rho - 1) + (\alpha - \rho)[1 - b_1 \chi(b_1)]\} \gamma_0 w_{t+1}
\]

Note: habit introduces persistent component
Example: Campbell-Cochrane external habit

Additive preferences with

\[ \nu(c_t, x_t) = \frac{(c_t - x_t)^\alpha}{\alpha} \]
\[ \log g_t = g + w_{t+1}, \quad \{w_t\} \sim \text{NID}(0, \sigma^2) \]

Approximation

\[ s_t = \frac{(c_t - x_t)}{c_t} \text{ ("surplus")} \]
\[ \log s_{t+1} = (1 - \varphi) \log s + \varphi \log s_t + \lambda(\log s_t)w_{t+1} \]
\[ (1 + \lambda)^2 = \frac{[1 - 2(\log s_t - \log s)][1 - \varphi - b/(1 - \alpha)]}{(1 - \alpha)\sigma^2} \]
Example: Campbell-Cochrane external habit (continued)

Pricing kernel

\[ m_{t+1} = \beta \left( \frac{c_{t+1}}{c_t} \right)^{\alpha - 1} \left( \frac{s_{t+1}}{s_t} \right)^{\alpha - 1} \]

\[ \log m_{t+1} = \text{constants} + \text{things dated } t \text{ and before} \]
\[ - (1 - \alpha)[1 + \lambda (\log s_t)]w_{t+1} \]

Conditional entropy (monthly)

\[ EL_t(m_{t+1}) = \frac{[(1 - \alpha)(1 - \varphi) - b]}{2} \]

Campbell-Cochrane 0.0100
Wachter 0.0082
Verdelhan 0.0052
Time dependence: summary

Little time-dependence in pricing kernel

But: modest dynamics + recursive preferences can magnify entropy

Habits, too

Options: same devices can magnify the impact of disasters
(Benzoni-Collin-Dufresne-Goldstein, Drechsler-Yaron, Eraker-Shaliastovich, & Campbell-Cochrane with jumps)