Optimality Results in Inventory-Pricing Control:  
An Alternate Approach

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Abstract

We study a stationary, single-stage inventory system, under periodic review, with fixed ordering costs and multiple sales levers (such as pricing, advertising, etc.). We show the optimality of \((s, S)\)-type policies in these settings under both the backordering and lost-sales assumptions. Our proof is novel, more direct and easier to understand than earlier ones.
1 Introduction

In this paper, we establish the optimal inventory control policies for a single-stage system with fixed ordering costs and stationary stochastic demand. In each period, a manager makes inventory decisions as well as decisions that influence demand, for example, the price choice or the advertisement budget. We refer to these decision variables as sales levers.

For a single stage system with inventory and pricing control and a fixed ordering cost, the optimality of the \((s, S)\)-type inventory structure has been established in recent papers such as Chen and Simchi-Levi (2004a,b) and Chen et al. (2003). The only sales lever in these papers is the price. The proofs in these papers rely heavily on induction arguments and dynamic programming formulations, which are pervasive in the area of stochastic inventory theory. A majority of their results require joint-concavity of the single-period expected profit function with respect to inventory and price. We extend the optimality of the \((s, S)\)-type structure for stationary systems by allowing a multi-dimensional sales lever, and a less-restrictive single-period expected profit function (permitting, for example, quasi-concavity).

Our proofs, completely different from the earlier proofs, are based on a “unifying condition” which we identify as being key to proving the \((s, S)\) structure. We believe that our proofs are important in their own right because they are, in our opinion, more direct and easier to understand than traditional proofs. We exploit some basic characteristics of stochastic inventory systems which have been hidden in the existing body of knowledge in this area, perhaps, due to the pervasion of proofs based on dynamic programming and induction arguments. Our ideas are closest in spirit to some arguments used by Veinott (1966).

\[\footnote{We use the terminology “\((s, S)\)-type policies”, which are the obvious generalization of \((s, S, p)\) policies, when we discuss our results.}\]
1.1 Problem Definition

Consider the following periodic review system with a planning horizon of $T$ periods ($T$ can be finite or infinite), and a discount factor $\gamma$. All parameters of the system are assumed to be stationary. Periods are indexed forwards. At the beginning of period $t$ ($t \leq T$), we have $x$ units of inventory. At this instant, an order can be placed to raise the inventory to some level $y$ instantaneously (that is, there is no lead time). There is a fixed cost or a set-up cost, $K$, associated with ordering any strictly positive quantity.\(^2\) This ordering opportunity is a control on the supply or inflow process. Similarly, we have a set of levers to control the demand or outflow process. Examples of these levers are prices, sales-force incentives and advertisements. We model the sales lever control by a vector $d$ within a compact, convex set $D \subseteq \mathbb{R}^n$; the first component could denote the price discount and the second component could denote the advertisement expense and so on.\(^3\)

After $y$ and $d$ are chosen, the demand in period $t$ is realized next. It is a random variable $D(d, \epsilon)$, where $\epsilon$ is an exogenous random variable. (That is, the distribution of demand depends on the sales lever control, and $\epsilon$ is the source of randomness.) The net inventory at the end of the period is $y - D(d, \epsilon)$, and holding and shortage costs is charged based on this quantity. We let $\pi(y, d)$ denote the expected profit in this period, excluding the set-up cost (that is, the total expected profit is $\pi(y, d) - K \cdot 1[y > x]$); $\pi$ includes sales revenue, the cost of choosing the sales lever and the holding and shortage costs. The inventory level at the beginning of the next period is given by $\psi(y - D(d, \epsilon))$, where $\psi$ is one of the following two operators: (a) an identity map if excess demand is completely backordered, or, (b) the

\(^2\)A purchase cost, linear in the order size $y - x$, could also be present. However, a simple assumption about salvaging inventory left at the end of $T$ periods can be used to transform the system into one in which this proportional cost is zero and the other cost parameters are suitably modified. Please see Appendix A.1 for a discussion on salvage values. Consequently, we will not consider this linear cost in our analysis.

\(^3\)We also allow the possibility of $D$ containing a single element; this represents traditional inventory control without any sales levers.
positive part operator if excess demand is lost. The objective is to find a pair \((y, d)\) for every \(x\) and \(t\), that maximizes the expected discounted-profits in periods \(t, t + 1, \ldots, T\).

Before proceeding to define the finite-horizon and infinite-horizon problems mathematically, we introduce some basic assumptions. Let \((y^*, d^*)\) be the maximizer of \(\pi(y, d)\), and let \(\pi^*\) be the maximum value. We remark that \((y^*, d^*)\) would be the solution chosen in a single-period problem in which there is no fixed cost and the starting inventory is lower than \(y^*\). We make the following assumption throughout the paper.

**Assumption 1.**

(a) Either \(\psi(x) = x\) (complete backlogging), or \(\psi(x) = (x)^+\) (lost sales).

(b) \(\pi^*\) is finite.

(c) The demand model is stationary, that is, the sequence of \(\epsilon\)'s in time periods \(\{1, 2, \ldots, T\}\) is independent and identically distributed.

The dynamic programming formulation for the finite-horizon problem is given by

\[
U_t(x) = \max_{\{y: y \geq x\}} \left[ V_t(y) - K \cdot 1[y > x] \right]
\]

where

\[
V_t(y) = \max_{\{d: d \in D\}} [W_t(y, d)], \quad \text{and}
\]

\[
W_t(y, d) = \begin{cases} 
\pi(y, d) + \gamma \cdot E \epsilon U_{t+1}(\psi(y - D(d, \epsilon))), & \text{if } t < T; \\
\pi(y, d), & \text{if } t = T.
\end{cases}
\]

(The subscript \(t\) in the above formulation denotes the period index.)

An optimal policy for this finite-horizon problem specifies a feasible pair \((y, d)\), for every \(x\) and \(t\), that maximizes \(W_t(y, d) - K \cdot 1[y > x]\). The infinite-horizon optimal policy specifies a feasible pair \((y, d)\), for every \(x\), that maximizes \(W(y, d) - K \cdot 1[y > x]\), where \(W(\cdot, \cdot)\) is the
point-wise limit of $W_1(\cdot, \cdot)$ as $T$ approaches infinity.\footnote{Theorem 4.2.3 and Lemma 4.2.8 of Hernandez-Lerma and Lassere (1996) give conditions for the existence of an optimal stationary policy and for the convergence of the finite-horizon dynamic program to the infinite-horizon dynamic program. The models studied in this paper satisfy these conditions. Note that in a modified problem in which $\pi^*$ is subtracted from the profit of every period, the expected profit in every period is at most 0. This property is useful in verifying the conditions.} In this paper, we study finite-horizon problems and infinite-horizon discounted-profit problems.

The following definitions are based on the dynamic programming formulation and become useful later:

- $y_t^* := \arg \max V_t(y)$, and $y^* := \arg \max V(y)$, where $V(y) := \max \{W(y, d) : d \in D\}$;
- $Q(y) := \max \{\pi(y, d) \mid d \in D\}$.

Next, we list the three kinds of demand models we use to capture the dependence of $D(d, \epsilon)$ on $d$.

- **Additive Demand Model:** For any $d \in D$, where $D$ is a closed real interval, $D(d, \epsilon)$ is nonnegative for almost every $\epsilon$, and can be expressed as $D(d, \epsilon) = d + \epsilon$, where $E[\epsilon] = 0$. (In this model, we use $d$ instead of $d$ since $D \subseteq \mathbb{R}^1$.)
- **Linear Demand Model:** For any $d \in D$, $D(d, \epsilon)$ is nonnegative for almost every $\epsilon$, and can be expressed as $D(d, \epsilon) = \alpha \cdot d + \beta$, where $\epsilon = (\alpha, \beta)$, and $\alpha \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$.
- **Concave Demand Model:** For any $d \in D$, $D(d, \epsilon)$ is nonnegative, monotonic and concave in $d$ for almost every $\epsilon$.

Among these models, the strongest structural results hold under the Additive Demand Model. For example, Chen and Simchi-Levi (2004a) show the optimality of $(s, S, p)$ policies for finite horizon problems with backordering and the Additive Demand Model; however, they show that the result does not always hold under the Linear Demand Model. Similarly,
Chen et al. (2003) require the Additive Demand Model in order to prove the optimality of \((s, S, p)\) policies with the lost sales assumption. Under the backordering assumption, the infinite horizon results in Chen and Simchi-Levi (2004b) are proved using the Linear Demand Model. Notice that the Linear Demand Model is a generalization of the Additive Demand Model, and the Concave Demand Model is a generalization of the Linear Demand Model. Federgruen and Heching (1999) introduced the Concave Demand Model when there is no fixed cost for ordering. To our knowledge, no results have been established so far under the Concave Demand Model when there is a fixed cost of ordering.

1.2 Literature Review and Summary of Contributions

The primary focus of this paper is to provide an alternate, more direct proof technique for important results in simultaneous control of inventory and pricing. In this section, we first review early classical work on inventory models with fixed ordering costs without any pricing or sales-lever decisions. Then, we review recent papers on joint inventory-pricing control, followed by an explanation of our contribution to the literature.

1.2.1 Classical Results without Pricing Decisions

Scarf (1960) and Veinott (1966), both seminal papers in Inventory Theory, establish the optimality of \((s, S)\) policies for inventory systems with fixed ordering costs. (No sales lever decisions are considered.)

Scarf shows that the cost function in the finite horizon dynamic program possesses a property, he named as \(K\)-convexity. A key step in the proof is demonstrating \textit{by induction} that this property holds for an \((n+1)\)-period problem if it holds for the \(n\) period problem. The proof, although originally presented for stationary models, can be extended to non-stationary environments as noted by Veinott (1966). Scarf’s main assumption is that the single-period
expected holding and shortage cost function is convex with respect to the inventory level after ordering.

Veinott presents another proof for the \((s, S)\) optimality result under different assumptions. He generalizes the assumption on the expected holding and shortage cost function to include quasi-convex functions, but requires the sequence of minimizers of these functions to be weakly increasing. His proof does not use induction based on the dynamic program. While his proof is algebraic, he sketches another proof that is constructive. Our work is inspired by this constructive approach.

1.2.2 Joint Inventory-Pricing Control

We now discuss the literature on joint inventory-pricing control. (We refer the reader to these papers for an exhaustive list of references.)

With complete backlogging, no fixed ordering costs, and the Concave Demand Model, Federgruen and Heching (1999) show the optimality of the base-stock list-price policy for the non-stationary finite-horizon model as well as the stationary infinite-horizon model. (A base-stock list-price policy is defined as follows: if the starting inventory level is less than some level \(y^*_t\), then order up to \(y^*_t\), and charge a fixed list-price in period \(t\); otherwise, do not order and offer a price discount.) They assume that the single-period expected profit function \(\pi(y, d)\) is jointly concave in inventory (after ordering) and price.\(^5\)

With complete backlogging, positive fixed cost, and the Additive Demand Model, Chen and Simchi-Levi (2004a) show that the \((s, S, p)\) policy is optimal in the finite horizon. (An \((s, S, p)\) policy is defined as follows: order nothing if inventory exceeds \(s\); order up to \(S\) otherwise. The price chosen depends on \(y\), the inventory level after ordering, through a specified function \(p(y)\).) They also present an example indicating that the \((s, S, p)\) policy may not be optimal in the finite-horizon problem when demand is not additive. With the Linear

\(^5\)As Chen and Simchi-Levi (2004a) point out, the only sufficient condition provided by Federgruen and Heching (1999) for this assumption is the linearity of the demand model.
Demand Model, Chen and Simchi-Levi (2004b) show the optimality of the \((s, S, p)\) policy in infinite-horizon models both with the discounted-profit and the average-profit criteria. They require joint concavity of \(\pi(y, d)\) for their results on the finite-horizon problem as well as the infinite-horizon discounted-profit problem. They develop the notion of symmetric \(K\)-concavity, a generalization of \(K\)-concavity, and show that the profit function in the finite horizon dynamic program possesses this property. This is the key step in their proof, which uses inductive arguments and is quite involved.

With the lost sales assumption, Chen et al. (2003) study a finite-horizon problem with the Additive Demand Model. They introduce some restrictions on the function relating expected demand and price as well as additional restrictions on the distribution of \(\epsilon\). With these assumptions, they demonstrate the optimality of the \((s, S, p)\) policy.

### 1.2.3 Contributions

The papers mentioned in Section 1.2.2 use several inductive arguments for finite-horizon results and rather involved convergence arguments to establish the infinite-horizon results. The contrast between the simplicity of the structure in the optimal policies (\((s, S, p)\) policies and base-stock list-price policies) and the complexity of the optimality proofs is quite striking. As mentioned earlier, the main contribution of this paper is to provide constructive proofs, which are simple and intuitive, for these types of results.

In the process of developing this proof technique, we identify a “Unifying Assumption” which is sufficient to guarantee the optimality of \((s, S)\)-type policies. This Unifying Assumption can be verified by simply studying a single-period problem; meanwhile, the existing results in the literature are proved by studying the value function from the dynamic program, which is typically more difficult to analyze. This is our second contribution.

Finally, our results complement the existing literature in some ways. In particular, we relax the usual assumptions on the demand model and profit functions; for example, we replace the joint concavity of \(\pi(y, d)\) with joint quasi-concavity. Instead, we require the
stationarity assumption. Table 1 summarizes the recent work on single-stage systems with sales levers along with our results. In addition, we improve existing bounds on the optimal $s$ and $S$ values.

For joint inventory-pricing problems, there has been a remarkably long time gap between Thomas (1974)’s conjecture of the optimality of $(s, S)$-type policies and Chen and Simchi-Levi’s proof. Even through their proof is inspired by Scarf’s work, their work has rightfully gained a lot of attention\footnote{In less than three years from the first version of their work, there have been at least ten citations, excluding self-citations, of their papers.} because the generalization is significant both in terms of the result and the methodology. Our work is a similar advancement of the approach initiated by Veinott. More specifically, proving optimal policy structure with inventory and price as decision variables is much more involved than merely appending the price variable in the proofs of classical papers in which prices are exogenous.

1.3 Organization

We present our analysis in the following sequence. In Section 2, we present the Unifying Assumption. In Section 3, we present our proof for the optimality of $(s, S)$-type policies based on the Unifying Assumption. The validity of the Unifying Assumption is shown in Section 4 for the backordering case and in Section 5 for the lost sales case. We conclude in Section 6.

2 A Unifying Assumption

In this section, we present a set of assumptions on the expected single-period profit $\pi$ that lies at the core of our proofs. Although the following condition appears technical, we show in Sections 4 and 5 that common modeling assumptions found in the literature satisfy this condition.
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<td>Concave Demand Model, jointly concave (\pi): Base-stock list-price policy is optimal.(^b)</td>
<td>Concave Demand Model: Base-stock list-price policy is optimal with weaker assumptions, e.g., quasi-concave (\pi).</td>
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<td>Positive</td>
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<td>Additive Demand Model, jointly concave (\pi): ((s, S, p)) is optimal. Linear Demand Model, jointly concave (\pi): ((s, S, p)) may not be optimal, but ((s, S, A, p)) is optimal.(^c)</td>
<td>Additive Demand Model: ((s, S, p)) is optimal with weaker assumptions, e.g., (\pi(r + d, d) = \phi^R(d) - \phi^H(r)) where (\phi^R) is quasi-concave and (\phi^H) is quasi-convex.</td>
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<td>Lost Sales Models (Finite Horizon)</td>
<td>Additive Demand Model: ((s, S, p)) is optimal with technical assumptions.(^e)</td>
<td>Additive Demand Model: ((s, S, p)) is optimal with the same assumptions.</td>
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<td></td>
<td>Lost Sales Models (Infinite Horizon)</td>
<td>none</td>
<td>Additive Demand Model: ((s, S, p)) is optimal for the discounted-profit problems with the same assumptions as the finite-horizon case.</td>
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\(^a\)All our results here assume stationarity of all parameters.

\(^b\)Federgruen and Heching (1999). See Footnote 5.

\(^c\)Chen and Simchi-Levi (2004a).

\(^d\)Chen and Simchi-Levi (2004b).

\(^e\)Chen et al. (2003). They assume stationarity.

Table 1: Summary of \((s, S)\)-Type Results with Stationary Parameters
Assumption 2 (Unifying Assumption).

(a) $Q(y) = \max_{d \in D} \pi(y, d)$ is quasi-concave\footnote{A function $f: \mathbb{R}^n \to \mathbb{R}$ is quasi-convex if the level set $\{w : f(w) \leq l\}$ of $f$ is convex for any $l \in \mathbb{R}$. A convex function is quasi-convex. We say $f$ is quasi-concave if $-f$ is quasi-convex.}, and

(b) for any $y^1$ and $y^2$ satisfying $y^* \leq y^1 < y^2$ and $d^2$, there exists

\[ d^1 \in \{d \mid \pi(y^1, d) \geq \pi(y^2, d^2)\} \]

such that for any $\epsilon$,

\[ \psi(y^1 - D(d^1, \epsilon)) \leq \max\{\psi(y^2 - D(d^2, \epsilon)), y^*\}. \]

Recall that $(y^*, d^*)$ maximizes $\pi(y, d)$. Thus, $y^*$ maximizes $Q(y)$. It follows from part (a) that the set in (1) is nonempty.

The above assumption ensures that the problem has sufficient structure. An intuitive explanation of the condition follows. Part (a) indicates that the closer the starting inventory level (after ordering) is to $y^*$, greater the single-period profit the system can generate. As a result, in a single-period problem without the fixed cost, it is optimal to order up to $y^*$ if $y < y^*$ and to order nothing if $y \geq y^*$. By part (b), if $y^* \leq y^1 < y^2$, the system starting from $y^1$ is capable of ensuring that it will be at a “better” inventory level in the immediate following period than the system starting from $y^2$; in the next period, the starting inventory of the $y^1$-system is closer to $y^*$ than that of the $y^2$-system, or ordering up to $y^*$ is possible since the starting inventory of the $y^1$-system is below $y^*$.

3 Optimality of $(s, S)$-Type Policies

This section contains the proof of the optimality of $(s, S)$-type policies for both backordering and lost sales models. All the results in this section are consequences of the Unifying
Assumption. Preliminary results are established in Section 3.1. Section 3.2 considers the special case of zero fixed ordering cost. One of the main results of this paper is in Section 3.3, which shows the sufficiency of the Unifying Assumption for proving the optimality of \((s, S)\)-type policies in the infinite-horizon discounted profit model. The finite-horizon case is discussed in Section 3.4, which shows the same result under an additional condition. We provide bounds on \(s\) and \(S\) values in Section 3.5.

### 3.1 Preliminary Results

In this section, we establish that in period \(t\), if the starting inventory level before ordering is below \(y^o_t\), and an order is placed, then the optimal order-up-to level is \(y^o_t\) (Proposition 3.1). Furthermore, if the starting inventory level is above \(y^o_t\), then it is optimal not to order (Corollary 3.3).

Throughout the paper, we use \(x_t\) to denote the starting inventory level before ordering, whereas \(y_t\) is the inventory level after ordering.

**Proposition 3.1.** If \(x_t \leq y^o_t\) and the fixed cost \(K\) is waived in period \(t\), then the optimal quantity to order is \(y^o_t - x_t\), i.e., order up to \(y^o_t\).

**Proof.** It follows from the optimality of \(y^o_t\) and the zero variable ordering cost assumption. \(\Box\)

**Proposition 3.2.** Suppose the Unifying Assumption holds. For any \(y^1_t\) and \(y^2_t\) satisfying \(y^* \leq y^1_t < y^2_t\) or \(y^2_t < y^1_t \leq y^*\), we have

\[
V_t(y^1_t) + \gamma K \geq V_t(y^2_t).
\]

**Proof.** We prove the result for the finite-horizon case. The infinite-horizon discounted-profit case follows directly. If \(t = T\), Assumption 2 (a) implies \(V_t(y^1_t) \geq V_t(y^2_t)\), and the required result holds. We proceed by assuming \(t < T\).

We compare two systems starting with \(y^1_t\) and \(y^2_t\), and use the superscript 1 and 2 to denote each of them. Suppose the \(y^2\)-system follows the optimal decision to attain \(V_t(y^2_t)\).
Let $d_t^2$ be the sales lever decision of the $y^2$-system. We claim that there exists $d_t^1$ such that $\pi(y_t^1, d_t^1) \geq \pi(y_t^2, d_t^2)$ and $x_{t+1}^1 \leq \max\{x_{t+1}^2, y^*\} \leq \max\{y_{t+1}^2, y^*\}$. If $y_t^2 < y_t^1 \leq y^*$, Assumption 2 (a) implies the existence of $d_t^1$ such that $\pi(y_t^1, d_t^1) \geq \pi(y_t^2, d_t^2)$, and we know $x_{t+1}^1 \leq y^*$ since $y_t^1 \leq y^*$. Otherwise, we have $y^* \leq y_t^1 < y_t^2$, and Assumption 2 (b) is applicable. In either case, the claim is true, and either $x_{t+1}^1 \leq y_{t+1}^2$ or $y_{t+1}^2 < x_{t+1}^1 \leq y^*$ holds. Refer to Figure 1.

- **Case (a):** $x_{t+1}^1 \leq y_{t+1}^2$: In the next period $t+1$, set the ordering quantity of the $y^1$-system to $y_{t+1}^2 - x_{t+1}^1$. Thus, $y_{t+1}^1 = y_{t+1}^2$. From period $t+2$ onwards, let the $y^1$-system mimic the $y^2$-system.

- **Case (b):** $y_{t+1}^2 < x_{t+1}^1 \leq y^*$: In period $t+1$, the $y^1$-system does not order, i.e., set $y_{t+1}^1 := x_{t+1}^1$. By Assumption 2 (a), we choose $d_{t+1}^1$ such that $\pi(y_{t+1}^1, d_{t+1}^1) \geq \pi(y_{t+1}^2, d_{t+1}^2)$. We continue choosing the sales lever in the $y^1$-system in this way until we come across the first period in which Case (a) is encountered.

Therefore, the $y^1$-system generates as much profit $\pi$ as the $y^2$-system in each period after period $t$. Furthermore, the $y^1$-system does not place an order in periods in which the $y^2$-system does not order, with possibly one exception (where the first case is applied). Thus, the ordering cost of the $y^1$-system is at most $\gamma K$ more than the $y^1$-system. Hence, the discounted-profit of the $y^1$-system is at worst $\gamma K$ less than the discounted profit in the $y^2$-system.

A corollary of this proposition is the optimality of not placing any order when the starting inventory level $x_t$ is at least $y^*$ or $y_t^2$.

**Corollary 3.3.** Under the Unifying Assumption, we have the following. If $x_t \geq \min\{y^*, y_t^2\}$, then it is optimal not to order in period $t$, i.e., $y_t = x_t$.

**Proof.** Suppose that the starting inventory before ordering is $x_t$, and we order up to $y_t$, where $y_t > x_t$. The ordering cost $K$ is incurred in period $t$. One of the following cases occurs.
(a) Case $x^1_{t+1} \leq y^2_{t+1}$

(b) Case $y^2_{t+1} < x^1_{t+1} \leq y^*$

Figure 1: Proof of Proposition 3.2
• Case $x_t \geq y^*$: Since $y_t > x_t \geq y^*$, we apply Proposition 3.2 to get

$$V_t(x_t) \geq V_t(y_t) - \gamma K \geq V_t(y_t) - K.$$  

• Case $y^o_t < x_t < y^*$: By Proposition 3.2 and the choice of $y^o_t$, we have

$$V_t(x_t) \geq V_t(y^o_t) - \gamma K \geq V_t(y_t) - \gamma K \geq V_t(y_t) - K.$$  

It follows that when the starting inventory level is greater than $\min(y^*, y^o_t)$, ordering a positive quantity does not increase the discounted profit by more than the fixed cost $K$ of ordering.

3.2 A Special Case: Zero Fixed Cost

In this section, we present a simple elementary proof for the optimality of the base-stock list-price policy in the absence of the fixed ordering cost. This result is similar to Federgruen and Heching (1999).

**Theorem 3.4.** If the Unifying Assumption holds and $K = 0$, then, the optimal ordering policy for the finite horizon problem or the infinite horizon discounted profit problem has the following structure. If $x_t < y^*$, then order up to $y^*$, and set the sales lever to $d^*$. Otherwise, do not order.

**Proof.** By Theorem 6.1 of Porteus (2002), it is easy to verify that $(y^*, d^*)$ is the optimal action pair if $x_t < y^*$. If $x_t \geq y^*$, it is optimal not to order by Corollary 3.3.

**Remark.** We make the following two remarks on Theorem 3.4: (a) If $x_t \leq y^*$, then the myopic policy is optimal; (b) Theorem 3.4 does not specify the optimal sales lever $d_t$ in period $t$ for $x_t > y^*$. This computation requires solving a dynamic program. In Appendix A.2, we show how to use the optimal structure to improve the computational efficiency of this dynamic program.
We point out the following important differences between the results in this section and the corresponding results in Federgruen and Heching (1999) under the backordering assumption. They require the concavity of the expected single-period profit whereas we require Assumption 2, which, for example, is satisfied if $\pi$ is quasi-concave (see Section 4.1). Their sales lever is single dimensional whereas ours is multi-dimensional. However, our simplification relies upon the stationarity of all the parameters of the system, whereas their finite-horizon results do not. For the lost sales case, Chen et al. (2003) prove this result under some assumptions which imply the Unifying Assumption, as we will show later (in Section 5). It should be noted that subject to the validity of the Unifying Assumption, our result holds for both the backordering and the lost sales cases.

### 3.3 Infinite-Horizon Discounted Profit Model with Positive $K$

In this section, we establish the optimality of the $(s, S)$-type policy in the infinite-horizon discounted-profit model, in which $T = \infty$ and $\gamma < 1$.

**Theorem 3.5.** Suppose the Unifying Assumption holds. In the infinite-horizon discounted-profit model, there exist $s$ and $S$ such that if $x_t \geq s$, it is optimal to not order, and otherwise to order up to $S$. That is, an $(s, S)$ policy is optimal.\(^8\)

**Proof.** We first prove the following claim: for $x^2 < x^1 \leq y^*$, if it is optimal to place an order when the beginning inventory level is $x^1$, then, it is also optimal to order when the starting inventory level is $x^2$.

For the infinite-horizon discounted-profit model, the profit function and the optimal policy are both stationary. (See the comments following Assumption 1.) Thus, we assume the current period $t$ is 0, and drop the subscript $t$ when $t = 0$. By Corollary 3.3, we proceed by assuming $x^1 < y_0^* = y^*$, since it is optimal to order at $x^1$.

\(^8\)Since we do not make any claims about the optimal sales lever to be chosen, we prefer referring to this as an $(s, S)$ policy rather than an $(s, S, d)$ policy.
If we order when the starting inventory level is either $x^1$ or $x^2$, then the order-up-to level is $y^o$ by Proposition 3.1, and the maximum profit from period $t$ onward is $v^o := V(y^o) - K$. Suppose the starting inventory level is $x^1$. By deferring order placement to the next period, we obtain a present value of profit equal to $Q(x^1) + \gamma v^o$. Since it is optimal to order in the current period, we must have $v^o \geq Q(x^1) + \gamma v^o$. Thus,

$$\left(1 - \gamma\right)v^o \geq Q(x^1) \geq Q(x)$$

for all $x \leq x^1$, where the second inequality follows from Assumption 2 (a).

Now, suppose the starting inventory level is $x^2$. Let $J$ denote the period in which the next order is placed according to the optimal solution. Thus, $x^2 = x_0^2 \geq x_1^2 \geq x_2^2 \geq \cdots \geq x_{J-1}^2$. From $x^2 < x^1$ and (3), an upper bound on the present value of the maximum profit between periods 0 and period $J - 1$ is given by

$$(1 + \gamma + \cdots + \gamma^{J-1})Q(x^2) \leq (1 + \gamma + \cdots + \gamma^{J-1})Q(x^1) \leq (1 + \gamma + \cdots + \gamma^{J-1}) \cdot (1 - \gamma)v^o = (1 - \gamma^J)v^o.$$ 

In period $J$, an order is placed. Thus, the present value of the maximum expected profit from $J$ onwards is $\gamma^J v^o$. In summary, the present value of the expected profit in all periods is bounded above by $(1 - \gamma^J)v^o + \gamma^J v^o = v^o$, which is attained if we order in the current period. Thus, we complete the proof of the claim.

Let $s = \sup \{ x : \text{it is optimal to order when the inventory level is } x \}$. Corollary 3.3 implies $s \leq y^o$. Thus, by Proposition 3.1, the optimal order-up-to level is $S = y^o$ for all $x \leq s$.

\[\square\]

### 3.4 Finite-Horizon Model with Positive $K$

In this section, we prove the optimality of an $\langle s, S \rangle$-type policy for the finite horizon problem. For proving this result, we need the following condition in addition to the Unifying Assumption. We will validate this condition in Sections 4 and 5.
Assumption 3. $V_t(y_t)$ is nondecreasing in the interval $(-\infty, \min\{y^*, y^0_t\}]$ for each $t$.

Theorem 3.6. Suppose the Unifying Assumption and Assumption 3 hold. Then, an $(s_t, S_t)$ policy is optimal in period $t$.

Proof. Proposition 3.1 and Corollary 3.3 state that it is optimal to order only if $x_t < \min(y^*, y^0_t)$, and the order-up-to level is $S_t = y^0_t$. Thus, if an order is placed in period $t$, the optimal profit from the remaining periods is independent of the starting inventory level $x_t$. For $x_t < \min(y^*, y^0_t)$, it is optimal to order if $V_t(x_t) \leq V_t(y^0_t) - K$, and not to order if $V_t(x_t) \geq V_t(y^0_t) - K$. Consequently, the monotonicity of $V_t$ establishes the existence of $s_t$. \qed

3.5 Bounds on $s$ and $S$

In Sections 3.3 and 3.4, we have shown the optimality of $(s, S)$-type policies. In this sections, we establish bounds on the values of $s$ and $S$ in the case of the infinite-horizon discounted profit case, and on $s_t$ and $S_t$ in the case of the finite-horizon case. Our bounds improve those existing in the literature.

Let

\[
\begin{align*}
m^* &= \inf \{ y \mid Q(y^*) - Q(y') > \gamma K \text{ for each } y' \geq y \} \\
m &= \sup \{ y \mid Q(y^*) - Q(y') > K \text{ for each } y' \leq y \} \\
M &= \sup \{ y \mid Q(y^*) - Q(y') > (1 + \gamma) K \text{ for each } y' \leq y \}.
\end{align*}
\]

Theorem 3.7. The following statements are true:

(a) In Theorems 3.5, $s$ and $S$ satisfy $m \leq s \leq S \leq m^*$.

(b) In Theorems 3.6, $s_t$ and $S_t$ satisfy $M \leq s_t \leq S_t \leq m^*$.

Proof. See Appendix A.3. \qed

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Chen and Simchi-Levi (2004b) have first established bounds on $s$ and $S$ for the infinite-horizon discounted profit model with backordering. We note that our bounds are tighter than their bounds. In particular, as $K \to 0^+$, our bounds converge to $y^*$ whereas their bounds do not. Furthermore, our bounds are applicable to the finite-horizon case and the lost sales case.

4 Application to the Backordering Model

In this section, we apply the results of the previous section to the case where excess demand is backordered. We first present (in Section 4.1) two sets of sufficient conditions for the Unifying Assumption to hold. Then, we formally establish (in Section 4.2) the optimality of $(s, S)$-type policies for finite and infinite horizon models under the backordering assumption.

4.1 Sufficient Conditions for the Unifying Assumption

The specification of the Unifying Assumption in Section 2 is quite technical. However, we will now show that it is a generalization of common modeling assumptions found in the literature.

Before proceeding to establish sufficient conditions, we note that if $f$ is a quasi-concave function and $w^*$ is its maximizer, then $f(w^* + \lambda v)$ is nonincreasing in $\lambda \geq 0$ for any $v$.

Condition 1: Joint Quasi-Concavity of $\pi$

It is shown in the following proposition that the Concave Demand Model and the quasi-concavity of the expected single-period profit $\pi$ are a sufficient condition for the Unifying Assumption. These modeling assumptions are more general than those used in the literature. Federgruen and Heching (1999) assume both the concavity of $D(p, \epsilon)$ in $p$, and the joint concavity of the single-period profit $\pi(y, p)$. For the infinite-horizon discounted-profit
criterion, Chen and Simchi-Levi (2004b) use a Linear Demand Model and assume that the single-period profit function is concave. Thus, the following proposition implies that these models also satisfy the Unifying Assumption.

**Proposition 4.1.** *With the Concave Demand Model, the joint quasi-concavity of $\pi$ implies the Unifying Assumption.*

**Proof.** Since $\pi$ is jointly quasi-concave, $Q(y) = \max_d \pi(y, d)$ is quasi-concave, and part (a) of the Unifying Assumption is satisfied.

Suppose $y^1$ and $y^2$ satisfy $y^* \leq y^1 < y^2$. Let $\lambda := (y^2 - y^1)/(y^2 - y^*)$. For any $d^2$, let $d^1$ be the following convex combination of $d^*$ and $d^2$: $d^1 = \lambda d^* + (1 - \lambda) d^2$. Since $\pi$ is quasi-concave,

$$\pi(y^1, d^1) \geq \pi(y^2, d^2).$$

The concavity of $D(d, \epsilon)$ in $d$ implies $D(d^1, \epsilon) \geq \lambda D(d^*, \epsilon) + (1 - \lambda) D(d^2, \epsilon)$. Thus,

$$y^1 - D(d^1, \epsilon) \leq \lambda [y^* - D(d^*, \epsilon)] + (1 - \lambda)[y^2 - D(d^2, \epsilon)]$$

$$\leq \max\{y^* - D(d^*, \epsilon), y^2 - D(d^2, \epsilon)\}$$

$$\leq \max\{y^*, y^2 - D(d^2, \epsilon)\}$$

It follows $\psi(y^1 - D(d^1, \epsilon)) \leq \max\{y^*, \psi(y^2 - D(d^2, \epsilon))\}$, satisfying part (b) of Assumption 2.

**Condition 2: Additive Demand Model and the “Separability” of $\pi$**

We now show that when the Additive Demand Model is used, a separability-like condition on $\pi$ is sufficient to guarantee the Unifying Assumption.

**Assumption 4.** *There exists quasi-concave $\phi^R$ and quasi-convex $\phi^H$ such that $\pi(r + d, d) = \phi^R(d) - \phi^H(r)$.***
This assumption holds, for example, under the Additive Demand Model with backordering in which revenues are received based on total demand and not on demand satisfied. In this case, \( \phi^R \) denotes the revenue function and \( \phi^H \) denotes the holding and shortage cost function. Notice that this assumption does not imply the joint quasi-concavity of \( \pi \). Assumption 4 is more general than the Additive Demand Model of Chen and Simchi-Levi (2004a) which assumes concave \( \phi^R \) and convex \( \phi^H \).

Let \( d^* := \arg \max \phi^R(d) \), and \( r^* := \arg \min \phi^H(r) \). Clearly, \( y^* = r^* + d^* \). The following proposition shows that under the Additive Demand Model, Assumption 4 implies the Unifying Assumption.

**Proposition 4.2.** Under the Additive Demand Model, if Assumption 4 holds, then we have a stronger version of the Unifying Assumption in which \((2)\) is replaced with

\[
\psi(y^1 - D(d^1, \epsilon)) \leq \psi(y^2 - D(d^2, \epsilon)).
\]

**Proof.** Suppose \( y^1 \) and \( y^2 \) satisfy \( y^* \leq y^1 < y^2 \). For any fixed \( d^2 \), set \( r^2 := y^2 - d^2 \). Set \( r^1 := \min\{r^2, y^1 - d^*\} \), and \( d^1 := y^1 - r^1 \). (The existence of such a \( d^1 \) in \( \mathcal{D} \) can be shown by demonstrating that \( y^1 - r^1 \) is sandwiched between \( d^* \) and \( d^2 \) since set \( \mathcal{D} \) is convex.) Clearly, by \( r^1 \leq r^2 \) and the additivity of demand, \((4)\) is satisfied.

Now refer to Figure 2 for the following two cases.

- **Case** \( r^1 = r^2 \): Clearly, \( \phi^H(r^1) = \phi^H(r^2) \). Since \( r^1 \leq y^1 - d^* \), we get

\[
d^* \leq y^1 - r^1 < y^2 - r^1 = y^2 - r^2 = d^2.
\]

Since \( d^1 := y^1 - r^1 \), we have \( d^* \leq d^1 < d^2 \). It follows from the quasi-concavity of \( \phi^R \) that \( \phi^R(d^1) \geq \phi^R(d^2) \).

- **Case** \( r^1 = y^1 - d^* \): It follows \( d^1 = d^* \), implying \( \phi^R(d^1) = \phi^R(d^*) \geq \phi^R(d^2) \). We have

\[
r^2 \geq r^1 = y^1 - d^1 \geq y^* - d^1 = y^* - d^* = r^*.
\]
(a) Case $r^1 = r^2$

(b) Case $r^1 = y^1 - d^*$

Figure 2: Proof of Proposition 4.2
By the quasi-concavity of $\phi^H$, it follows that $\phi^H(r^2) \geq \phi^H(r^1)$.

Therefore, for every $d^2$, there exists $d^1$ such that $\pi(y^1, d^1) = \phi(r^1, d^1) \geq \phi(r^2, d^2) = \pi(y^2, d^2)$. We thus verify Assumption 2 (b). Furthermore, we obtain that $Q(y)$ is nonincreasing for $y \geq y^*$. If $y^1$ and $y^2$ satisfy $y^* \geq y^1 > y^2$, let $r^1 := \max\{r^2, y^1 - d^*\}$ instead. A similar argument shows that $Q(y)$ is nondecreasing for $y \leq y^*$. Thus, we have established the quasi-concavity of $Q$.

4.2 Main Results

We are now ready to show the infinite horizon optimality result holds under the Concave Demand Model if $\pi$ is jointly quasi-concave.

**Theorem 4.3.** Assume $\pi$ is jointly quasi-concave. For the infinite-horizon discounted-profit model, an $(s, S)$ policy is optimal.

**Proof.** By Proposition 4.1, we know that the Unifying Assumption is satisfied if $\pi$ is jointly quasi-concave. The result now follows from Theorem 3.5.

The above theorem shows the optimality of an $(s, S)$-type policy in the infinite-horizon discounted-profit model when excess demand is backordered. We point out the following differences between our results and those contained in Chen and Simchi-Levi (2004b). They assume the concavity of the expected single-period profit function whereas quasi-concavity is found to be sufficient for our results. While they use the Linear Demand Model, we use the more general Concave Demand Model. In addition, our proof holds with multi-dimensional sales levers. However, they are able to show the optimality of $(s, S, p)$ policies even in the infinite-horizon average-profit case, which we have not shown.

Next, for the Additive Demand Model, we establish that Assumption 4 implies the optimality of $(s, S)$-type policies for finite and infinite horizon problems.
Theorem 4.4. Consider the Additive Demand Model. Suppose Assumption 4 holds. In the finite-horizon model, an \((s_t, S_t)\) policy is optimal in period \(t\). For the infinite horizon case, an \((s, S)\) policy is optimal.

Proof. Recall that Assumption 4 satisfies the Unifying Assumption by Proposition 4.2. Thus, the infinite horizon optimality result follows from Theorem 3.5. To establish the finite horizon result, it suffices, by Theorem 3.6, to prove that Assumption 3 holds, i.e.,

\[ V_t(y_t) \text{ is nondecreasing in the interval } (-\infty, \min\{y^*, y_t^\circ\}]. \]

Suppose \(y_t^1\) and \(y_t^2\) satisfy \(y_t^2 < y_t^1 \leq \min(y^*, y_t^\circ)\). We want to show \(V_t(y_t^1) \geq V_t(y_t^2)\). Suppose the \(y^2\)-system with the starting inventory level \(y_t^2\) follows an optimal policy and attains \(V_t(y_t^2)\). Below, we construct a policy for the \(y^1\)-system such that

- the expected profit of the \(y^1\)-system before accounting for the ordering cost in every period is at least that of the corresponding quantity in the \(y^2\)-system, and
- the \(y^1\)-system places an order in a period only if the \(y^2\)-system places an order in that period.

Let \(d^2\) be the optimal sales lever of the \(y^2\)-system in period \(t\). By the proof of Proposition 4.2, there exists \(d^1\) such that \(\pi(y^1, d^1) \geq \pi(y^2, d^2)\) and \(x^1_{t+1} \geq x^2_{t+1}\) for any realization of \(\epsilon\) in the demand model. Thus, if \(x^1_{t+1} < y^2_{t+1}\) occurs, an order must have been placed in the \(y^2\)-system. In that case, let the \(y^1\)-system place an order such that \(y^1_{t+1} = y^2_{t+1}\), and mimic the \(y^2\)-system for the remaining periods. Otherwise, we repeat the above process until both the systems have the same ending inventory level. This concludes the proof of the claim.

Chen and Simchi-Levi (2004a) also show the finite horizon optimality result. Our result is more general in that, for instance, the single-period profit function does not even need to be quasi-concave, whereas they assume concavity of this profit function. (Also see comments following Assumption 4.)
5 Application to the Lost Sales Model

Chen et al. (2003) study a finite-horizon model in which any unsatisfied demand is lost. They introduce some technical assumptions that facilitate their dynamic programming, induction-based proof of the \((s, S)\) optimality result. In this section, we will demonstrate that their assumptions are sufficient to prove the result for both the finite and the infinite horizon discounted profit models using our proof technique. We apply results from Section 3.

They use the Additive Demand Model in which the sales lever is the per-unit selling price \(p\) per unit\(^9\), i.e.,

\[ D(p, \epsilon) = d(p) + \epsilon \quad \text{(5)} \]

where \(d(p)\) is the deterministic part of demand. Let \(f\) and \(F\) denote the probability density and cumulative distribution functions of \(\epsilon\), respectively. Let \(P := [0, P^u]\) be the domain of \(p\), any \(p \in P\) satisfy \(d(p) \geq 0\). They impose the following additional technical assumptions.

**Assumption 5** (Chen et al. (2003)). Demand is additive and is given by \((5)\), where \(d(p)\) is a deterministic function, and \(\epsilon\) is a nonnegative, continuous random variable defined on a closed interval \([0, B]\). The probability density of \(\epsilon\) is strictly positive on \((0, B)\). We have

(a) \(d(p)\) is strictly decreasing, concave, and \(3d'' + pd'' \leq 0\) on \(P\), and

(b) The failure rate function \(r(u) := f(u)/(1 - F(u))\) of \(\epsilon\) satisfies \(r'(u) + 2[r(u)]^2 > 0\) for any \(u \in (0, B]\).

(c) The expected single-period profit function is given by

\[ \pi(y, p) = p \cdot E[min(y, D(p, \epsilon))] - h \cdot E[(y - D(p, \epsilon))^+] - b \cdot E[(D(p, \epsilon) - y)^+], \]

where \(h\) is a holding cost per unit and \(b\) is a penalty cost per unit.

\(^9\)In this section, we use \(p\) in place of \(d\) because in this model, the only sales lever is the price, \(p\).
This assumption is satisfied by a wide range of demand functions and distribution functions of \(\epsilon\); see Chen et al. (2003) for a discussion. Our proof makes use of some intermediate results they derived using the technical assumption above.

**Proposition 5.1.** Assumption 5 implies the Unifying Assumption.

*Proof.* See Appendix A.4.1.

**Proposition 5.2.** In a finite-horizon model, Assumption 5 implies Assumption 3.

*Proof.* See Appendix A.4.2.

We are now ready to demonstrate the application of our proof technique to the optimality of \((s, S, p)\) policies for both the finite and the infinite horizon discounted profit models with lost sales.

**Theorem 5.3.** Suppose Assumption 5 holds. In the finite-horizon model, an \((s_t, S_t)\) policy is optimal in period \(t\). For the infinite horizon case, an \((s, S)\) policy is optimal.

*Proof.* For the infinite-horizon case, we know from Proposition 5.1 that the Unifying Assumption is satisfied; the result now follows from Theorem 3.5. Moreover, for the finite-horizon case, Proposition 5.1 implies Assumption 3 holds. Therefore, the finite horizon result now follows from Theorem 3.6.

6 Extension and Conclusion

6.1 The Case of Stochastically Increasing, Additive Demands

All the results we have shown in this paper so far assume that all cost parameters and the demand distributions are stationary. In this section, we extend our results to a special class of non-stationary problems. We use the Additive Demand Model in this section. Due to non-stationarity, we will use the time index as a subscript for all parameters.
Theorem 6.1. In the finite-horizon model, suppose the following conditions hold:

(a) Demand is additive.

(b) \( \{y_t^*\} \) is an increasing sequence.\(^{10}\)

(c) \( K_t \geq \gamma \cdot K_{t+1} \) holds for every \( t \).

(d) Either (i) excess demand is backordered and Assumption 4 holds for every \( t \); or, (ii) excess demand is lost and Assumption 5 holds for every \( t \).\(^{11}\)

Then, an \((s_t, S_t)\) policy is optimal in each period \( t \).

The proof of this theorem can be obtained by adapting arguments used in Theorems 4.4 and 5.3 for the non-stationary case.

6.2 Concluding Remarks

In this paper, we have developed an alternate proof technique for establishing the optimality of \((s, S)\) type policies when inventory replenishment decisions are taken jointly with sales-lever or demand-influencing decisions. Our proof technique is more direct and easy to understand as compared to existing proofs in the literature. The center-piece of the technique is our development of a Unifying Assumption which we show is sufficient for these optimality results. This assumption can be verified entirely by studying a single period problem. This stands in contrast to the \( K \)-convexity or symmetric \( K \)-concavity properties established earlier which have to be verified for the value function in a dynamic program; this verification is usually difficult and algebraic. We have also developed new bounds on the \((s, S)\) parameters of the optimal policy using the same proof technique. It is hoped that

\(^{10}\)In models with stationary cost parameters, this assumption usually holds when \( \{\epsilon_t\} \) is a sequence of stochastically increasing random variables.

\(^{11}\)In (i), Assumption 4 now involves \( \pi_t, \phi_t^R \) and \( \phi_t^H \). In (ii), Assumption 5 now involves \( d_t(p), \epsilon_t, B_t, f_t, F_t, \tau_t, \pi_t, h_t \) and \( b_t \).
this proof technique will be useful for proving structural results for other models in stochastic inventory theory. Recently, we have successfully used this technique and the Unifying Assumption, in particular, for proving the optimality of \((s, S)\) results for models in which sales are conducted through auctions (Huh and Janakiraman (2005)).

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**References**


A Appendix

A.1 Salvage Costs

Throughout this paper, we assume that the salvage value of a unit of inventory remaining at the end of a finite planning horizon is equal to the unit purchase cost. Our proofs for the finite horizon results depend on this assumption. Although it is useful for notational simplicity, our infinite horizon results and proofs do not require this assumption. Veinott (1966) makes a similar observation (see page 1072).

A.2 Dynamic Program for Section 3.2

In this appendix, we demonstrate how Theorem 3.4 can be used to improve the computational efficiency of the dynamic program for computing $d_t$ when $x_t > y^*$. Suppose the horizon $T$ is finite or the profit is discounted, i.e., $\gamma < 1$. Federgruen and Heching (1999) give the
following dynamic programming formulation. Recall $U_t(x_t)$ is the optimal value function as a function of the beginning inventory $x_t$ in period $t$, i.e., $U_t(x_t) = \max_{y_t \geq x_t} V_t(y_t)$. Then,

$$U_t(x_t) = \max_{y_t \geq x_t} \pi(y_t, d_t) + \gamma E[U_{t+1}(\psi(y_t - D_t(d_t, \epsilon)))$$

for $t \leq T$, and $U_{T+1}(\cdot) = 0$. This dynamic program is well-defined when $T$ is finite.

Once the starting inventory level $x_t$ falls below $y^*$, the expected profit in each subsequent period is $Q(y^*)$. Thus, the expected profit (discounted to period $t$) from period $t$ to $T$ is

$$Q(y^*)(1 + \gamma + \gamma^2 + \cdots + \gamma^{T-t}) = Q(y^*) \cdot (1 - \gamma^{T-t+1})/(1 - \gamma).$$

As a result, we simplify the dynamic programming recursion as follows: for any $t \leq T$,

$$U_t(x_t) = \begin{cases} 
\max_{d_t} \pi(x_t, d_t) + \gamma E[U_{t+1}(\psi(x_t - D_t(d_t, \epsilon)))], & \text{for } x_t > y^*, \\
Q(y^*) \cdot (1 - \gamma^{T-t+1})/(1 - \gamma), & \text{for } x_t \leq y^*, 
\end{cases}$$

and $U_{T+1}(\cdot) = 0$. It is well-defined when $T$ is finite. For the infinite-horizon discounted-profit problem, we let $T \to \infty$ and substitute $U_t$ with its pointwise limiting function.

### A.3 Proof of Theorem 3.7

In this section, we prove Theorem 3.7 which contains the following results: (i) the order-up-to levels ($S$ and $S_t$) in Theorems 3.5 and 3.6 are bounded above by $m$; and (ii) the reorder points ($s$ and $s_t$) in Theorems 3.5 and 3.6 are bounded below by $m$ and $M$, respectively.

By the quasi-concavity of $Q$ (Assumption 2), we have $M \leq m \leq y^* \leq \overline{m}$. First, we show that $\overline{m}$ is an upper bound on the value of $S$ or $S_t$. By Proposition 3.1, the order-up-to level in both Theorem 3.5 and Theorem 3.6 is $y_t^\circ$, which maximizes $V_t(\cdot)$.

**Proposition A.1.** $y_t^\circ \leq \overline{m}$ for each $t$.

**Proof.** Suppose, by way of contradiction, $y_t^\circ > \overline{m}$ for some $t$. We compare two systems starting with $y_t^1 = y^*$ and $y_t^2 = y_t^\circ$, respectively. Thus, $y_t^1 < y_t^2$. 

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Suppose the \( y^2 \)-system follows the optimal decision to attain \( V_t(y^2_t) \), and let \( d^2_t \) be its sales lever decision in period \( t \). Let \( d^1_t \) be such that \( Q(y^1_t) = \pi(y^1_t, d^1_t) \). Then, from \( y^2_t > m \), the expected profits of two systems satisfy

\[
\pi(y^1_t, d^1_t) = Q(y^*) > Q(y^2_t) + \gamma K \geq \pi(y^2_t, d^2_t) + \gamma K .
\] (6)

Consider the next period \( t + 1 \). We have either (i) \( x^1_{t+1} < y^2_{t+1} \) or (ii) \( y^2_{t+1} \leq x^1_{t+1} \). In the first case, the \( y^1 \)-system can incur the fixed cost of \( K \), and order up to \( y^2_{t+1} \), i.e., \( y^1_{t+1} = y^2_{t+1} \). Thus, in period \( t + 1 \),

\[
U_{t+1}(x^1_{t+1}) \geq V_{t+1}(y^2_{t+1}) - K .
\] (7)

In the second case, let the \( y^1 \)-system place no order, i.e., \( y^1_{t+1} = x^1_{t+1} \). It follows \( y^2_{t+1} \leq y^1_{t+1} \leq y^* \). By Proposition 3.2, we obtain

\[
V_{t+1}(y^1_{t+1}) \geq V_{t+1}(y^2_{t+1}) - K .
\] (8)

In either case, combining (7) or (8) with inequality (6), we obtain \( V_t(y^1_t) > V_t(y^2_t) \), contradicting the choice of \( y^1_t = y^2_t \). \(\square\)

We now provide lower bounds for the value of the reorder point. In Proposition A.2, we show \( \underline{M} \) is a lower bound on \( s \) or \( s_t \).

**Proposition A.2.** Reorder points \( s \) in Theorem 3.5 and \( s_t \) in Theorem 3.6 are both bounded below by \( \underline{M} \).

**Proof.** It suffices to show that for any \( y^2_t < \underline{M} \),

\[
V_t(y^*) > V_t(y^2_t) + K
\] (9)

holds. Then, when the starting inventory is \( x^2_t = y^2_t \), it is better to order up to \( y^* \) than not to order. Let \( d^2_t \) be any sales lever decision of the \( y^2 \)-system.
Let $y^1_t = y^*$, and let $d^1_t$ be such that $Q(y^1_t) = \pi(y^1_t, d^1_t)$. Then, $y^2_t < y^1_t = y^*$ implies (8) by Proposition 3.2. Furthermore, from $y^2_t < M$, we have
\[
\pi(y^1_t, d^1_t) = Q(y^*) > Q(y^2_t) + (1 + \gamma)K \geq \pi(y^2_t, d^2_t) + (1 + \gamma)K ,
\]
which is an analogous statement to (6). These two results yield (9) by a similar argument used in the proof of Proposition A.1. \(\square\)

The following result is used in establishing Proposition A.4.

**Proposition A.3.** In Theorem 3.5, $S \geq m$ holds.

**Proof.** Suppose, by the way of contradiction, $S = y^o$ is strictly less than $m$. We compare two systems starting with $y^1_1 = y^*$ and $y^2_1 = y^o < m$. Let $y^1$-system follow order-up-to $y^*$ and choose $d^*$ in each period. The expected single-period profit of the $y^1$-system is $Q(y^*) - K$ in each period, except the first period in which it is $Q(y^*)$. Let $y^2$-system follow the optimal decision to attain $V(y^*)$. By Theorem 3.5, $y^1_t$ never exceeds $m$ in each period $t$, and the $y^1$-system’s expected single-period profit $\pi(y^1_t, d^1_t)$ is bounded $Q(y^*) - K$. Thus, we obtain $V(y^o) < V(y^*)$, a contradiction. \(\square\)

We recall that Theorem 3.5 considers the infinite-horizon discounted-profit case. In this case, we are able to prove a stronger lower bound on $s$.

**Proposition A.4.** The reorder point $s$ in Theorem 3.5 is bounded below by $m$.

**Proof.** By Proposition A.3, $m$ is a lower bound on $y^o$, the optimal order-up-to level. It suffices to show that for each $t$, $y^2_t < m$ implies
\[
V(y^o) > V(y^2_t) + K .
\]
Without loss of generality, we proceed by assuming $t = 1$.

Let the $y^2$-system follow the optimal policy to achieve $V(y^2_1)$. Let $\tau = \min\{t \mid y^2_t > x^2_t\}$ be the first period in which the $y^2$-system places an order (and incurs the fixed cost of $K$).
By Theorem 3.5, \( y_\tau^2 = y^* \). For each \( t < \tau \), we have \( x_t^2 \leq y_t^2 < m \), and thus the expected single-period profit in period \( t \) is at most \( Q(y_t^2) < Q(y^*) - K \). It follows
\[
V(y_1^2) < \sum_{t=1}^{\tau-1} \gamma^{t-1}[Q(y^*) - K] + \gamma^{\tau-1}[V(y^*) - K]. \tag{10}
\]

We make the following claim:
\[
\frac{Q(y^*) - K}{1 - \gamma} \leq V(y^*) - K.
\]

Consider a system which starts at \( y^* \) and orders up to \( y^* \) in every period. The infinite-horizon discounted profit of this system is given by
\[
Q(y^*) + \sum_{t=2}^{\infty} \gamma^{t-1}(Q(y^*) - K) = K + \sum_{t=2}^{\infty} \gamma^{t-1}(Q(y^*) - K) = K + \frac{Q(y^*) - K}{1 - \gamma}.
\]

However, this cost is bounded above by \( V(y^*) \) by the choice of \( y^* \). Thus we complete the proof of the above claim.

Now from (10) and the claim, we obtain
\[
V(y_1^2) \leq \sum_{t=1}^{\tau-1} \gamma^{t-1}(1 - \gamma)[V(y^*) - K] + \gamma^{\tau-1}[V(y^*) - K]
\]
\[
= \sum_{t=1}^{\tau-1} \gamma^{t-1}(1 - \gamma)[V(y^*) - K] + \sum_{t=\tau}^{\infty} \gamma^{t-1}(1 - \gamma)[V(y^*) - K]
\]
\[
= V(y^*) - K.
\]

This completes the proof.

\[ \square \]

**A.4 The Chen et al. (2003) Model**

(We use their July 30, 2003 version.)

We introduce the following notation and definitions used in Chen et al. (2003). Let \( z := y - d(p) \) be the “riskless” leftover inventory at the end of a period. Let \( G(z, p) \) be the expected single-period profit without considering the fixed ordering cost, i.e., \( G(z, p) = \)
Let $P(y)$ be the optimal price when the starting inventory level after ordering is $y$, i.e., $P(y) := \arg \max_p \pi(y, p)$. Thus, $\pi(y, P(y)) = Q(y)$. Let $Z(y) := y - d(P(y))$. It follows that

$$G(Z(y), P(y)) = Q(y) = \pi(y, P(y)). \quad (11)$$

Also, let $p(z) := \arg \max_p G(z, p)$. We denote the maximizer of $G(z, p(z))$ by $Z$. Then, the single-period optimal stocking quantity satisfies $y^* = Z + d(p(Z))$.

Chen et al. (2003) give the following list of properties based on Assumption 5. They are related to the expected single-period profit function.

**Fact A.5 (Equation (6)).**

$$\frac{\partial^2 G(z, p)}{\partial p^2} < 0.$$  

**Fact A.6 (Corollary of Lemma 1).** $p(z)$ is continuous on $[0, +\infty)$.

**Fact A.7 (Lemma 3).** $Z(y)$ is nonnegative and increasing on $[0, +\infty)$.

**Fact A.8 (Theorem 1).** $P(y) \leq p(z)$ for $y > y^*$, and $P(y) \geq p(z)$ for $y < y^*$ where $y = z + d(p(z))$. Furthermore, $G(Z(y), P(y))$ is unimodal on $[0, +\infty)$ and $y^*$ is its maximizer.

### A.4.1 Proof of Proposition 5.1

Fact A.8 shows the quasi-concavity of $Q(y) = G(Z(y), P(y))$, implying Assumption 2 (a). Suppose that a pair of $y^1$ and $y^2$ satisfy $y^* \leq y^1 < y^2$, and $p^2$ is given. Let $z^2 := y^2 - d(p^2)$. By (11), Fact A.8, and the definition of $P(y)$,

$$\pi(y^1, P(y^1)) = G(Z(y^1), P(y^1)) \geq G(Z(y^2), P(y^2)) = \pi(y^2, P(y^2)) \geq \pi(y^2, p^2).$$

Thus, if $Z(y^1) \leq \max\{y^*, z^2\}$ holds, $p^1 = P(y^1)$ satisfies Assumption 2 (b). As a result, we proceed by supposing

$$Z(y^1) > y^* \text{ and } Z(y^1) > z^2. \quad (12)$$

\(^{12}\)In this section, $p$ corresponds to the sales lever $d$ used in earlier sections.
(a) Case $z^2 + d(p(z^2)) \leq y^1$

(b) Case $z^2 + d(p(z^2)) > y^1$

Figure 3: Proof of Proposition 5.1
We consider the following two cases: $z^2 + d(p(z^2)) \leq y^1$ and $z^2 + d(p(z^2)) > y^1$. Refer to Figure 3.

Case $z^2 + d(p(z^2)) \leq y^1$: Choose $p^3$ such that $d(p^3) = y^1 - z^2$. (This is possible because $d(\cdot)$ is continuous and $y^1 - z^2$ is bounded below by $d(p(z^2))$ and bounded above by $y^2 - z^2$, which is the same as $d(p^2)$.) Since $z^2 + d(p^3) = y^1$ and $z^2 + d(p^2) = y^2$, we obtain that

$$z^2 + d(p(z^2)) \leq y^1 < y^2$$

is equivalent to

$$d(p(z^2)) \leq d(p^3) < d(p^2).$$

Therefore we obtain $p(z^2) \geq p^3 > p^2$. Now, the definition of $p(z^2) = \arg \max_p G(z^2, p)$ and Fact A.5 imply that $G(z^2, p)$ is increasing in $p$ for $p \leq p(z^2)$. Thus,

$$\pi(y^1, p^3) = G(z^2, p^3) \geq G(z^2, p^2) = \pi(y^2, p^2).$$

Thus, Assumption 2 (b) is satisfied with $p^3$.

Case $z^2 + d(p(z^2)) > y^1$: By the continuity of $d(\cdot)$ and Fact A.6, $d(p(\cdot))$ is continuous. Thus, there must exist $z'$ in the closed interval defined by $Z$ and $z^2$ such that $z' + d(p(z')) = y^1$. Recalling $Z + d(p(Z)) = y^*$, we get $Z \leq y^*$ and consequently,

$$z' \leq \max\{Z, z^2\} \leq \max\{y^*, z^2\} < Z(y^1),$$

where the last inequality follows from (12). Since

$$Z(y^1) + d(P(y^1))) = y^1 = z' + d(p(z')),$$

we get $d(fp(z')) > d(P(y^1))$, and thus $p(z') < P(y^1)$. However, this contradicts Fact A.8 since $y(z') = y^1 > y^* = y(Z)$. Thus, this case cannot happen.
(a) Assuming $y' < y^1$

(b) Assuming $y' \geq y^1$

Figure 4: Proof of Proposition 5.1
A.4.2 Proof of Proposition 5.2

Consider a pair \( y_1 \) and \( y_2 \) such that \( y_2 < y_1 < y^* \). For any \( p^2 \), let \( z^2 := y^2 - d(p^2) \). As in the proof of Theorem 4.4, it suffices to show that there exist \( p^1 \) such that (i) \( z^1 := y^1 - d(p^1) \geq z^2 \), and (ii) \( G(z^1, p^1) \geq G(z^2, p^2) \). Recall \( Z(y^1) = y^1 - d(P(y^1)) \). If \( z^2 \leq Z(y^1) \), then \( p^1 = P(y^1) \) satisfies (i). Furthermore, the quasi-concavity of \( Q \) implies

\[
\pi(y^1, P(y^1)) = Q(y^1) \geq Q(y^2) = \pi(y^2, P(y^2)) \geq \pi(y^2, p^2),
\]

which shows (ii). Thus, we proceed by assuming \( Z(y^1) < z^2 \).

\[ \begin{equation}
Z(y^1) < z^2. 
\end{equation} \] (13)

Let \( y' := z^2 + d(p(z^2)) \). Refer to Figure 4. We claim \( y' \geq y^1 \). By the way of contradiction, suppose \( y' < y^1 \). Thus,

\[
z^2 + d(p(z^2)) = y' < y^1 < y^* = Z + d(p(Z)).
\]

It follows from the continuity of \( p \) (Fact A.6) and \( d \) that there must exist \( z' \leq y^1 \) in the open interval defined by \( z^2 \) and \( Z \) such that \( z' + d(p(z')) = y^1 \). Thus,

\[
d(p(z')) = y^1 - z' < y^1 - \min\{z^2, Z\} \leq y^1 - Z(y^1) = d(P(y^1)).
\]

(The first inequality follows from the choice of \( z' \). Since Fact A.7 implies \( Z = Z(y^*) \geq Z(y^1) \), the second inequality follows from (13).) Therefore, \( p(z') > P(y^1) \). However, it contradicts Fact A.8, and we prove the claim that \( y' \geq y^1 \).

We choose \( p^1 \) such that \( y^1 = z^2 + d(p^1) \). Thus, \( z^2 = z^1 = y^1 - d(p^1) \) shows (i). (Note that such a \( p^1 \) can be chosen because \( d(p(\cdot)) \) is a continuous function, and \( y^1 - z^2 \) is bounded below by \( d(p^2) = y^2 - z^2 \) and above by \( d(P(y^1)) = y^1 - Z(y^1) \).) Note that

\[
y^2 = z^2 + d(p^2) < y^1 = z^2 + d(p^1) \leq y' = z^2 + d(p(z^2))
\]

implies \( p^2 > p^1 \geq p(z^2) \), and \( p = p(z^2) \) maximizes \( G(z^2, p) \). Thus, Fact A.5 shows \( G(z^1, p^1) = G(z^2, p^1) \geq G(z^2, p^2) \), verifying (ii).