A Comparison of the Optimal Costs of Two Canonical Inventory Systems

Ganesh Janakiraman\textsuperscript{1}, Sridhar Seshadri\textsuperscript{2}, J. George Shanthikumar \textsuperscript{3}

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Abstract

We compare two inventory systems, one in which excess demand is lost and the other in which excess demand is backordered. Both systems are reviewed periodically. They experience the same sequence of identically and independently distributed random demands. Holding and shortage costs are considered. The holding cost parameter is identical, however the cost of a lost sale could be different from the per period cost of backlogging a unit sale. When these costs are equal, we prove that the optimal expected cost for managing the system with lost sales is lower. When the cost of a lost sale is greater, we establish a relationship between these parameters that ensures that the reverse inequality is true. These results are useful for designing inventory systems. We also introduce a new stochastic comparison technique in this paper.

\textsuperscript{1}IOMS-OM Group, Stern School of Business, New York University, 44 W. 4th Street, Room 8-71, New York, NY 10012-1126.
\textsuperscript{2}IOMS-OM Group, Stern School of Business, New York University, 44 W. 4th Street, Room 8-73, New York, NY 10012-1126.
\textsuperscript{3}4129 Etcheverry Hall, Department of Industrial Engineering and Operations Research, University of California, Berkeley, CA 94720.
1 Introduction and Motivation

The literature in the area of inventory theory almost entirely assumes that there are only two consequences for stocking out of inventory when a customer demand arrives. The customer either leaves this inventory system taking her demand elsewhere, causing the system to lose the profit on this sale and also suffer loss of goodwill, or, she waits until inventory is available and then completes the transaction thus delaying the realization of revenue, causing a loss due to delayed revenue, as well as, the loss of goodwill. The former event is called a “lost sale” whereas the latter is called a “backorder”. Furthermore, a typical inventory model assumes either that every customer who does not find inventory available upon arrival causes a lost sale or every such customer causes a backorder. These two assumptions form the basis for two canonical inventory systems, which we will refer to as the “lost sales system” (L) and the “backorder system” (B) respectively.

In this paper, we compare the cost performances of the two systems. In particular, we confirm the common belief that a lost sales system has a lower optimal cost than a backorder system when the two systems have the same holding cost and the lost sales penalty cost per unit in L is equal to the backorder penalty cost per unit per period in B. This belief is based on the intuition that a unit of demand incurs a shortage penalty at most once in L, whereas it could incur the penalty in multiple periods in B (because it might take several periods for a backorder to be cleared). We also derive an upper bound on the penalty cost per unit per period in B relative to the unit penalty cost in L below which the optimal cost in B is lower than the optimal cost in L. Our results are the first of their kind in terms of providing comparative results between these two inventory systems that are of great theoretical and practical significance. From a managerial perspective, our results are the first in the direction of answering the following important question: in an environment in which a firm usually loses a sale when a customer order cannot be delivered or satisfied from stock immediately, how much of a discount can the firm afford to give the customer to induce her to backorder her demand without adversely affecting the firm’s profits? (For literature
on consumer response to stock-outs, please see Bhargava et al. (2005) and the references therein.) From a technical perspective, we introduce a new stochastic comparison technique.

1.1 Systems $B$ and $L$

Both $B$ and $L$ have the following common characteristics. Both are single stage inventory systems with a replenishment lead time of $\tau$ periods. The number of periods in the planning horizon is $N$. Demands in different periods are independent and identically distributed. $D$ and $D_t$ are used to denote the random demand in a generic period and a specific period $t$, respectively. Furthermore, we assume that, in every period, the same demands are realized in the two systems and that $D$ has a continuous distribution function. The discount factor is $\alpha$.

In general, we use $x_t$ and $q_t$ to denote the inventory at the beginning of period $t$ and the order placed in period $t$, respectively, in either system. When necessary, we will use the superscript $B$ or $L$ to denote which system is being discussed. At the beginning of period 0, both systems are endowed with the following state vector, $x_0$, where

$$x_0 = (q_{-1}, \ldots, q_{-\tau+2}, q_{-\tau+1}, x_0).$$

The components of $x_0$ are given, thus unaffected by the ordering policies used in $B$ and $L$.

In any period $t$, the sequence of events is as follows: The replenishment order placed in period $t - \tau$ arrives, and, the inventory available and the vector of outstanding orders are observed. Next, the order quantity $q_t$ is chosen. The demand, $D_t$, is realized. Finally, in $B$ ($L$), a cost, $g^B(x^B_t - D_t) (g^L(x^L_t - D_t))$, is charged capturing the imbalance between supply and demand. The evolution of the two systems is captured by the following two recursions.

\begin{align}
  x_{t+1}^B &= (x^B_t - D_t) + q_{t+1-\tau}^B, \quad (1.1) \\
  x_{t+1}^L &= (x^L_t - D_t)^+ + q_{t+1-\tau}^L. \quad (1.2)
\end{align}

We assume the following about the cost functions throughout the paper.
Assumption 1.1. The function, \( g^B (g^L) \), is non-negative and convex. Furthermore, \( g^B(x) \) (\( g^L(x) \)) is decreasing if \( x < 0 \).

The standard cost models with linear holding and shortage cost functions satisfy this assumption. For the sake of specificity later, we present such a linear cost model below.

Assumption 1.2. \( g^B(x) = h^B \cdot (x)^+ + b^B \cdot (x)^- \) and \( g^L(x) = h^L \cdot (x)^+ + b^L \cdot (x)^- \) for some strictly positive parameters \( h^B \), \( b^B \), \( h^L \) and \( b^L \).

Let \( C^*_B \) (\( C^*_L \)) be the optimal expected discounted cost in \( B \) (\( L \)) over a horizon of \( N \) periods for a given \( x_0 \) and \( \alpha \), that is, the minimum of this cost over the class of all non-anticipatory policies.

The main results of this paper are the following: When the functions \( g^B \) and \( g^L \) are identical, \( C^*_L \leq C^*_B \) under a reasonable assumption about \( x_0 \), the starting state vector. We provide an example to show that this result is not true for all starting states. Finally, we show that the result holds without any assumption on the starting state for the infinite horizon, average cost performance measure case. In words, a lost-sales system incurs lower cost than a backorder system when they both experience the same demands and have the same cost structure. We will explain later why this is an intuitive result; however, we will argue why this “intuition” does not lead to a proof; our proof, by itself, is an interesting and challenging exercise that uses the concept of convex ordering of random variables and a construction of a sequence of systems connecting \( L \) and \( B \). This is the subject of Section 2. In Section 3, we work with the cost model of Assumption 1.2. We present a condition comparing the shortage cost parameters in \( B \) and \( L \) that ensures that \( C^*_B \leq C^*_L \) when the holding costs are identical. This result holds for all \( x_0 \).

1.2 Brief Literature Review

Single and multi-stage inventory models with stochastic demands and replenishment lead times have been extensively studied with the assumption that excess demands are backo-
ordered. It is well known that order-up-to or base-stock policies and echelon order-up-to or echelon base-stock policies are optimal in the classical versions of these models. Furthermore, when demands are independently and identically distributed across periods, these optimal policies can be determined simply by solving an appropriate static optimization or myopic problem. See, for example, Veinott (1965).

The corresponding model with the assumption that excess demand is lost has also been recognized to be an important problem (see Karlin and Scarf (1958)); however, very little is known about the optimal inventory policy in this setting. Morton (1969), in which elementary properties of the optimal ordering function are proved and bounds on this function are derived, is a notable contribution to this area; Zipkin (2006b) generalizes some of Morton’s results and provides elegant proofs using properties related to sub-modularity and diagonal dominance. Levi et al. (2005) have developed an approximation algorithm that is guaranteed to have an expected cost that is at most two times that attained by an optimal policy for independent demand processes and certain types of correlated demand processes. Zipkin (2006a) presents structural results that reduce the effort required to evaluate the performance of a given ordering policy; using these results, he performs a computational investigation of several heuristic policies. Our comparative analysis is one more step in the direction of understanding this problem.

2 Cost Comparison under Identical, Convex Cost Functions

In this section, we will assume that the cost functions in $\mathcal{L}$ and $\mathcal{B}$ are the same.

**Assumption 2.1.** $g^B(x) = g^C(x) = g(x) \forall x$.

Before getting into the technical details about our results or the proofs, notice that in $\mathcal{B}$, there is a chance that some customers are backordered for more than one period; however,
in a lost sales model, any such penalty is incurred at most once for each customer. So, it is natural to conjecture that $C^L \leq C^B$ always. It turns out that this intuition is not always correct as shown in Lemma 2.4.1. The problem with using this intuitive argument is that the difference in the holding costs incurred by the two systems can sometimes make $L$ more expensive to manage than $B$. Therefore, instead of pursuing this way of justifying the result, we use the observation that the stochastic process representing inventory at the beginning of a period is more variable in the backorder system than the lost sales system, i.e., the random variable $(x - D)$ has higher variance than the random variable $(x - D)^+$. We are able to exploit this phenomenon to derive our results.

2.1 Convex Ordering and Cost Comparison

**Definition 2.1** (2.A.1 of Shaked and Shanthikumar (1994)). Let $X$ and $Y$ be two random variables such that

$$E[\phi(X)] \leq E[\phi(Y)]$$

for all convex functions $\phi : \mathbb{R} \to \mathbb{R}$, provided the expectations exist. Then, $X$ is said to be smaller than $Y$ in the convex order (denoted by $X \leq_{cx} Y$).

Next, we state a sufficient condition for the relation $X \leq_{cx} Y$ to be true.

**Lemma 2.1.1** (from Theorem 2.A.17 of Shaked and Shanthikumar (1994)). Assume (i) $E[X] = E[Y]$, (ii) $\exists a$ such that $P(X \leq u) \leq P(Y \leq u)$ $\forall u < a$ and $P(X \leq u) \geq P(Y \leq u)$ $\forall u \geq a$. Then, $X \leq_{cx} Y$.

In other words, if two random variables have the same means and their distribution functions cross each other exactly once (“single crossing property”), then they can be compared using the convex order.

We are now ready to relate this concept to the systems we study.
Lemma 2.1.2. Assume $X$ and $D$ are two random variables, and, $q^B$ and $q^L$ are two constants such that

\[ E[(X - D)^+ + q^L] = E[(X - D) + q^B]. \]  

Then,

\[ E[\phi((X - D)^+ + q^L)] \leq E[\phi((X - D) + q^B)] \forall \text{ convex } \phi(\cdot). \]  

Proof: It is easy to see that $q^L \leq q^B$ for the equality $E[(X - D)^+ + q^L] = E[(X - D) + q^B]$ to hold. Next, observe that

\[ P((X - D)^+ + q^L \leq u) = 0 \leq P((X - D) + q^B \leq u) \forall u < q^L \] and

\[ P((X - D)^+ + q^L \leq u) \geq P((X - D) + q^B \leq u) \forall u \geq q^L, \]

where the last inequality holds because $q^L \leq q^B$. So, the conditions of Lemma 2.1.1 hold. This implies the result. □

The consequence of the lemma is the following. Consider the function $\phi(y) = E[g(y - D)]$. Then, $\phi$ is clearly a convex function since $g$ is convex. Let $X$ be the same (random) inventory at the beginning of period $t$ in $B$ as well as in $L$. Let $q^L$ and $q^B$ denote the replenishment quantities received in $L$ and $B$, in period $t + 1$. For a given $q^B$, if $q^L$ is chosen such that the expectation of the inventory in period $t + 1$ in $L$ equals the corresponding expectation in $B$, then, $L$ dominates $B$ in period $t + 1$.  

This “expectation matching” idea is at the crux of our analysis. If this idea can be used repeatedly in every period, we can establish the dominance of $L$ over $B$ over the horizon $[0, N]$. Unfortunately, this cannot be done repeatedly because, beyond period 0, the inventory variables in the two systems are not the same. This leads us to our next idea which involves the construction of a sequence of systems $\{\Gamma_n\}$ that has $B$ as its first element and $L$ as its last element.

\footnote{We use “dominates” to mean “has an expected cost smaller than or equal to” throughout.}
2.2 Analysis Outline

We develop an inductive analysis for the finite horizon model as follows. We consider a sequence of systems \( \{ \Gamma_n : n = 0, 1, \ldots, N+1 \} \). For all \( n \), in \( \Gamma_n \), excess demand is lost in all periods up to \( n-1 \); in subsequent periods, excess demand is backordered. All systems start with the same initial conditions. Observe that \( \Gamma_0 \) is the same as \( \mathcal{B} \) and \( \Gamma_{N+1} \) is the same as \( \mathcal{L} \). The main reason for this construction is that \( \Gamma_n \) and \( \Gamma_{n+1} \) differ in their dynamics in only one period, namely period \( n \). This makes it much easier to use the expectation matching idea and Lemma 2.1.2 in a dynamic way in the interval \([n+1, N]\). We outline these arguments below.

Let \( \pi^*_n \) be the optimal policy in \( \Gamma_n \) and let \( C^*_n(t) \) be the cost incurred in period \( t \) in \( \Gamma_n \) under \( \pi^*_n \). Let \( C^*_n \) be the total cost over \([0, N]\) in \( \Gamma_n \) under \( \pi^*_n \). We shall construct a (sub-optimal) policy \( \tilde{\pi}_{n+1} \) for \( \Gamma_{n+1} \) using \( \pi^*_n \), with corresponding costs \( \tilde{C}^*_n(t) \) and \( \tilde{C}^*_n \), such that 
\[
E[\tilde{C}^*_n+F_0] \leq E[C^*_n+F_0] \text{ for } n = 0, 1, \ldots, N.
\]

Let \( D[t_1, t_2] = D_{t_1} + D_{t_1+1} + \ldots + D_{t_2} \) and \( q[t_1, t_2] = q_{t_1} + q_{t_1+1} + \ldots + q_{t_2} \). When \( t_1 > t_2 \), \([t_1, t_2]\) refers to the null set and \( q[t_1, t_2] = D[t_1, t_2] = 0 \). Let 
\[
S^B_m = \max \{ \arg \min_y E[g(y-D[1, m+1])] \}.
\]

The quantity \( S^B_m \) plays a pivotal role in backorder systems as explained in the lemma below. 

**Inventory position**, a term used in the Lemma, refers to the total inventory in the system including outstanding orders; that is, the inventory position at the beginning of period \( t \) before (after) ordering is \( x_t + \sum_{t-\tau+1}^{t} q_u \) (\( x_t + \sum_{t-\tau+1}^{t} q_u \)).

**Lemma 2.2.1.** (i) The optimal policy in \( \mathcal{B} \) is to order up to \( S^B_{\tau} \), that is, raise the inventory position to the target level \( S^B_{\tau} \) in every period. (ii) \( S^B_m \geq 0 \) \( \forall m \) and \( \{S^B_m\} \) is an increasing sequence.
**Proof of (i):** This result is well known in inventory theory. The reader is referred to the discussion on “myopic optimal policies” in Veinott (1965) or in Chapter 3 of Heyman and Sobel (1984) for details.

**Proof of (ii):** The quantity $S_m^B$ is clearly non-negative because $D[1, m + 1] \geq 0$ and $g(z)$ is a decreasing function when $z < 0$ by Assumption 1.1. Also, $S_m^B$ satisfies the first order condition $E[g'(S_m^B - D[1, m + 1])] = 0$. Since $g'$ is an increasing function (by the convexity of $g$) and $D[1, m + 2] \geq D[1, m + 1]$, it is clear that $S_{m+1}^B \geq S_m^B$. □

We know that $\pi_n^*$ is an order-up-to-$S_\tau^B$ policy from period $n$ onwards since all excess demand is backordered from period $n$ onwards in $\Gamma_n$.

We use $\tilde{x}_t$ to denote the net inventory (= inventory on hand - backorders) at the beginning of period $t$ in $\Gamma_{n+1}$ under policy $\tilde{\pi}_{n+1}$. Let $\tilde{q}_t$ denote the order quantity placed in period $t$ in system $\Gamma_{n+1}$ under policy $\tilde{\pi}_{n+1}$. Similarly, $x_t^*$ and $q_t^*$ are the corresponding quantities for $\Gamma_n$ under policy $\pi_n^*$. We use $x_n$ to denote $\tilde{x}_n$, as well as, $x_n^*$ since they are identical, as shown below. We use $\mathcal{F}_t$ to denote the information about the history of demands in either system and the history of orders in $\Gamma_{n+1}$ and $\Gamma_n$ up to the beginning of period $t$.

**Definition 2.2.** : Construction of $\tilde{\pi}_{n+1}$

1. $\tilde{q}_t = q_t^* \forall t \in [0, n - \tau]$.

2. $\tilde{q}_t = (q^*_{n+1-\tau, t} - \tilde{q}_{n+1-\tau, t-1} - E[(D_n - x_n)^+ | \mathcal{F}_t])^+ \forall t \in [\max(0, n+1-\tau), n]$.

3. $\tilde{q}_t = (S_{\tau}^B - \tilde{x}_t - \tilde{q}_{t+1-\tau, t-1})^+ \forall t \in [n + 1, N]$.

The policy $\tilde{\pi}_{n+1}$ which we construct for $\Gamma_{n+1}$ is also an order-up-to-$S_\tau^B$ policy from period $n + 1$ onwards. The order quantities it prescribes are identical to the order quantities prescribed by $\pi_n^*$ in periods \{0, 1, \ldots, (n - \tau)\} when $n \geq \tau$. Moreover, orders placed before period zero are identical in both systems. Consider any realization of $(D_0, D_1, \ldots, D_{n-1})$. 

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Since the receipts of inventory, the demands, and the starting states in the two systems are the same, the inventory on hand in $\Gamma_n$ under $\pi^*_n$ equals the corresponding quantity in $\Gamma_{n+1}$ under $\tilde{\pi}_{n+1}$ in periods \{0, 1, \ldots, n\}. We now provide intuition for our choice of $\tilde{q}_m$ for $m > n-\tau$.

Since $\Gamma_n$ backorders excess demand in period $n$ whereas $\Gamma_{n+1}$ loses it, we have

$$\tilde{x}_{n+1} = (x_n - D_n)^+ + \tilde{q}_{n+1-\tau} \quad \text{and} \quad x^*_{n+1} = (x_n - D_n) + q^*_{n+1-\tau}.$$ 

Observe that these relations are of the same form as those given in Lemma 2.1.2. We develop similar relations for subsequent periods. For any $t \in [n + 1 - \tau, n]$, define

$$\tilde{y}_{t+\tau} = (x_n - D_n)^+ + \tilde{q}[n + 1 - \tau, t] \quad \text{and} \quad (2.6)$$

$$y^*_{t+\tau} = (x_n - D_n) + q^*[n + 1 - \tau, t]. \quad (2.7)$$

We now have the more general relations,

$$\tilde{x}_{t+\tau} = \tilde{y}_{t+\tau} - D[n + 1, t + \tau - 1] \quad \text{and} \quad (2.8)$$

$$x^*_{t+\tau} = y^*_{t+\tau} - D[n + 1, t + \tau - 1] \quad \forall \ t \in [n + 1 - \tau, n]. \quad (2.9)$$

In each period $t$, where $t \in [n + 1 - \tau, n]$, $\tilde{q}_t$ will be chosen such that $E[\tilde{y}_{t+\tau}|F_t] = E[y^*_{t+\tau}|F_t]$, if possible. Then, Lemma 2.1.2 can be invoked using the convex function $G(y) = E[g(y - D[n + 1, t + \tau])]$ to show that $\Gamma_{n+1}$ dominates $\Gamma_n$ in period $t + \tau$. This gives a cost comparison for the interval $[n + 1, n + \tau]$. However, there might be periods when such an expectation matching is impossible for all non-negative choices of $\tilde{q}_t$. For such periods, a more elaborate analysis is necessary. Finally, we will demonstrate that the expected costs incurred in $[n + \tau + 1, N]$ in $\Gamma_n$ and $\Gamma_{n+1}$ can be captured by an identical function that is convex in the inventory position at the beginning of $n + 1$. We will exploit a relation between the inventory positions in $n + 1$ in the two systems and use the convexity to show the dominance of $\Gamma_{n+1}$ over $\Gamma_n$ in $[n + \tau + 1, N]$. We now present our results and our analysis under Assumption 1.1 in detail.
2.3 Analysis

There are two cases to consider, \( n \geq \tau - 1 \) and \( n < \tau - 1 \). In the first case, we will be able to construct \( \tilde{\pi}_{n+1} \) for \( \Gamma_{n+1} \) such that \( \Gamma_{n+1} \) dominates \( \Gamma_n \) over \([0,N]\). In the second case, we will show the same result under some assumptions regarding the starting state \( x_0 \).

2.3.1 Case 1: \( n \geq \tau - 1 \): Periods Beyond the Lead Time

We start our analysis by deriving a useful property of \( \pi_n^* \), the optimal policy for \( \Gamma_n \).

**Lemma 2.3.1.** For every \( t \in \{n + 1 - \tau, n\} \), the inequality \( q_t^*[n + 1 - \tau, t] \leq S_{t+\tau-n}^B \) holds.

**Proof:** We prove the result first for \( t = n + 1 - \tau \) by contradiction. Assume \( q_{n+1-\tau}^* > S_{1}^B \) (to be more rigorous, \( q_{n+1-\tau}^* \) is the smallest optimal order quantity in that period). We now construct a policy for which the order quantities and inventory levels will be represented using \( q' \) and \( x' \) with the appropriate period subscripts. From now, we will refer to the two policies as the \( ' \) policy and \( ^* \) policy, respectively. Let \( C'(m) \) denote the cost incurred by \( ' \) in period \( m \). We will show that \( E[\sum_{m=0}^{m=N} C'(m) | \mathcal{F}_t] < E[\sum_{m=0}^{m=N} C_n^*(m) | \mathcal{F}_t] \), which contradicts the assumption of optimality of the \( ^* \) policy.

Let \( q_m' = q_m^* \forall m \in [0, n - \tau] \). So, the inventory levels and consequently the costs of both the policies are identical in \([0, n]\), implying that \( \sum_{m=0}^{m=n} C_n^*(m) = \sum_{m=0}^{m=n} C'(m) \forall \mathcal{F}_N \). Let \( q_{n+1-\tau}' = S_{1}^B ; q_{n+2-\tau}' = q_{n+2-\tau}^* + (q_{n+1-\tau}^* - q_{n+1-\tau}') \) (which is strictly positive by assumption) and \( q_m' = q_m^* \forall m \in [n + 3 - \tau, N] \).

From (2.7) and (2.9), it should be clear that \( x_m^* = x_m' \forall m \in [n + 2, N] \) and so, \( \sum_{m=n+2}^{m=N} C_n^*(m) = \sum_{m=n+2}^{m=N} C'(m) \forall \mathcal{F}_N \). The only costs that we are yet to compare are those incurred in period \( n + 1 \).

Consider the costs \( E[g(x_{n+1}^* - D_{n+1}) | \mathcal{F}_{n+1-\tau}] \) and \( E[g(x_{n+1}' - D_{n+1}) | \mathcal{F}_{n+1-\tau}] \). We know that the random variables \( x_n^* \) and \( x_n' \) are the same, by construction of the new policy; so, we
refer to both as $x_n$. We now have

$$x_n + q^*_{n+1-\tau} > x_n + S^B_1 = x_n + q'_{n+1-\tau} \geq S^B_1,$$

for any realization of $(D_{n+1-\tau}, \ldots, D_{n-1})$; the last inequality holds because $x_n$ is a non-negative random variable since excess demand is lost in period $n-1$ in $\Gamma_n$. Now, by definition, $S^B_1$ is the largest minimizer of $E[g(y - D_n - D_{n+1})]$, which is a convex function of $y$. So, by convexity, we get

$$E[g(x_n + q^*_{n+1-\tau} - D_n - D_{n+1}) | \mathcal{F}_{n+1-\tau}] > E[g(x_n + q'_{n+1-\tau} - D_n - D_{n+1}) | \mathcal{F}_{n+1-\tau}].$$

Rewriting this in terms of $x^*_n$ and $x'_{n+1}$, we get

$$E[g(x^*_n - D_{n+1}) | \mathcal{F}_{n+1-\tau}] > E[g(x'_{n+1} - D_{n+1}) | \mathcal{F}_{n+1-\tau}] \quad \forall \mathcal{F}_{n+1-\tau},$$

that is,

$$E[C^*_n(n+1) | \mathcal{F}_{n+1-\tau}] > E[C'(n+1) | \mathcal{F}_{n+1-\tau}].$$

So, we now have

$$E[\sum_{m=0}^{m=N} C^*_n(m) | \mathcal{F}_{n+1-\tau}] > E[\sum_{m=0}^{m=N} C'(m) | \mathcal{F}_{n+1-\tau}].$$

This contradicts the assumption that $*$ constitutes an optimal policy. So, we have proved our result for $t = n + 1 - \tau$.

Next, we assume the result is true for some $t-1 \in [n+1-\tau, n-1]$ and inductively prove the result for $t$, again by contradiction. We now assume that $q^*[n+1-\tau, t] > S^B_{t+\tau-n} \geq S^B_{t+\tau-n-1} \geq q^*[n+1-\tau, t-1]$. The first inequality is the assumption that we will contradict, the second is a property of the myopic solutions $\{S^B_m\}$ and the third follows from the result for $t-1$. We will construct an alternate policy $'$ that is identical to $*$ in $[0, t-1]$ and in $[t+2, N]$. Let $q'_t = S^B_{t+\tau-n} - q'[n+1-\tau, t-1]$, which is a non-negative quantity. So, $q'[n+1-\tau, t] = S^B_{t+\tau-n}$. Let $q'_{t+1} = q^*_{t+1} + (q'^*_t - q'_t)$. The specification of $'$ is now complete. As before, the costs and the inventory levels attained by the two policies are identical in all periods except period $t+\tau$. 

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We now only need to show that \( E[C'(t + \tau)|\mathcal{F}_t] < E[C^*_n(t + \tau)|\mathcal{F}_t] \), thereby contradicting the assumption about \( \ast \). We have

\[
x_n + q^*[n + 1 - \tau, t] > x_n + q'[n + 1 - \tau, t] \geq S^B_{t+\tau-n} \quad \forall (D_t, \ldots, D_{n-1}) .
\]

But, \( S^B_{t+\tau-n} \) minimizes \( E[g(y - D[n, t + \tau])] \), which again is a convex function of \( y \). So, we get

\[
E[g(x_n + q^*[n + 1 - \tau, t] - D[n, t + \tau])|\mathcal{F}_n] > E[g(x_n + q'[n + 1 - \tau, t] - D[n, t + \tau])|\mathcal{F}_n] \quad \forall \mathcal{F}_n ,
\]

which, along with (2.7) and (2.9), implies that

\[
E[g(x^*_t + \tau - D_t)|\mathcal{F}_t] > E[g(x^*_t + \tau - D_t)|\mathcal{F}_t] .
\]

Therefore,

\[
E[\sum_{m=0}^{m=N} C^*_n(m)|\mathcal{F}_t] > E[\sum_{m=0}^{m=N} C'(m)|\mathcal{F}_t] ,
\]

thus contradicting the optimality of \( \ast \). \( \square \)

Next, we show that \( \Gamma_{n+1} \) under \( \tilde{\pi}_{n+1} \) (defined in Section 2.2) dominates \( \Gamma_n \) under \( \pi^*_n \). Recall that

(i) \( \tilde{q}_t = q^*_t \forall t \in [0, n - \tau] \).

(ii) \( \tilde{q}_t = (q^*[n + 1 - \tau, t] - \tilde{q}[n + 1 - \tau, t - 1] - E[(D_n - x_n)^+|\mathcal{F}_t])^+ \forall t \in [n + 1 - \tau, n] \).

(iii) \( \tilde{q}_t = (S^B_t - \bar{x}_t - \tilde{q}[t + 1 - \tau, t - 1])^+ \forall t \in [n + 1, N] \).

In words, \( \tilde{\pi}_{n+1} \) mimics \( \pi^*_n \) in \([0, n - \tau]\). Also, it orders up to \( S^B_t \) from period \( n + 1 \) onwards. During the interval \([n + 1 - \tau, n]\), the policy has been carefully constructed to have the expectation matching property a lead time ahead; we will prove this and another useful property of this policy next.

**Lemma 2.3.2.** The following two properties hold for all \( t \in [n + 1 - \tau, n] \) and any \( \mathcal{F}_t \).

(i) Either \( \tilde{q}_t = 0 \) or \( E[\tilde{y}_{t+\tau}|\mathcal{F}_t] = E[y^*_t|\mathcal{F}_t] \). In either case, \( E[\tilde{y}_{t+\tau}|\mathcal{F}_t] \geq E[y^*_t|\mathcal{F}_t] \).

(ii) \( \tilde{q}[n + 1 - \tau, t] \leq q^*[n + 1 - \tau, t] \leq S^B_{t+\tau-n} \).
Proof: (i) Assume $\hat{q}_t \neq 0$. This implies that

$$\hat{q}_t = (q^*[n+1-\tau,t] - \bar{q}[n+1-\tau,t-1] - E[(D_n-x_n)^+|\mathcal{F}_t]) .$$

(2.10)

Using the identity $z = z^+ - (-z)^+$, this relation can be rewritten as

$$E[(x_n-D_n)^+|\mathcal{F}_t] + \bar{q}[n+1-\tau,t] = E[(x_n-D_n)|\mathcal{F}_t] + q^*[n+1-\tau,t] .$$

Recall from (2.6) and (2.7) that the left and right sides of the equality above are $E[y_{t+\tau}|\mathcal{F}_t]$ and $E[y^*_{t+\tau}|\mathcal{F}_t]$, respectively.

When $\hat{q}_t = 0$, we know

$$q^*[n+1-\tau,t] \leq \bar{q}[n+1-\tau,t-1] + E[(D_n-x_n)^+|\mathcal{F}_t] .$$

Adding $E[(x_n-D_n)|\mathcal{F}_t]$ to both sides, we see that $E[y_{t+\tau}|\mathcal{F}_t] \geq E[y^*_{t+\tau}|\mathcal{F}_t]$.

(ii) The second inequality has already been shown in Lemma 2.3.1. We show the first inequality now. Let us begin with $t = n + 1 - \tau$. If $\hat{q}_t = 0$, the result is trivial from the non-negativity of $q^*_t$. If $\hat{q}_t > 0$, then (2.10) and the non-negativity of $E[(D_n-x_n)^+|\mathcal{F}_t]$ imply the result.

Let us now assume the result is true for $t-1$ and use it to prove the result for $t$. If $\hat{q}_t = 0$, then

$$\bar{q}[n+1-\tau,t] = \bar{q}[n+1-\tau,t-1] \leq q^*[n+1-\tau,t-1] \leq q^*[n+1-\tau,t] .$$

If $\hat{q}_t > 0$, again (2.10) and the non-negativity of $E[(D_n-x_n)^+|\mathcal{F}_t]$ imply the result.

We will use the lemma above to establish the dominance of $\Gamma_{n+1}$ under $\tilde{\pi}_{n+1}$ over $\Gamma_n$ under $\pi^*_n$ in $[n+1+\tau,N]$.

Lemma 2.3.3. Let $i^*_{n+1} = x^*_{n+1} + q^*[n+2-\tau,n+1]$ denote the inventory position in $\Gamma_n$ after ordering in period $n+1$, using $\pi^*_n$. Let $\hat{i}_{n+1}$ be similarly defined for $\Gamma_{n+1}$ under $\tilde{\pi}_{n+1}$.

(i) The inequality

$$S^B_{\tau} \leq \hat{i}_{n+1} \leq i^*_{n+1} \text{ holds } \forall \mathcal{F}_{n+1} .$$
(ii) Furthermore, there exists a convex function $V$ such that, for any $\mathcal{F}_{n+1}$, the expected discounted sums of the costs incurred by $\Gamma_n$ and $\Gamma_{n+1}$ in $[n+1, \tau, N]$, given $\mathcal{F}_{n+1}$, are $V(i^*_n + 1)$ and $V(\tilde{i}_n + 1)$, respectively. Also, $V$ is minimized at $S^B_\tau$. (iii) Therefore, given any $\mathcal{F}_{n+1}$, $E[\sum_{t=n+1+\tau}^{N} \tilde{C}_{n+1}(t)|\mathcal{F}_{n+1}] \leq E[\sum_{t=n+1+\tau}^{N} C^*_n(t)|\mathcal{F}_{n+1}]$. (iv) This implies that, $E[\sum_{t=n+1+\tau}^{N} \tilde{C}_{n+1}(t)|\mathcal{F}_0] \leq E[\sum_{t=n+1+\tau}^{N} C^*_n(t)|\mathcal{F}_0]$.

**Proof:** Using Lemma 2.3.2 (ii) for $t = n$, we get

$$ \tilde{q}[n+1-\tau, n] \leq q^*[n+1-\tau, n] \leq S^B_\tau. \quad (2.11) $$

Note that

$$ i^*_n + 1 = (x_n - D_n) + q^*[n+1-\tau, n] + q^*_n + 1 \quad \text{and} \quad (2.12) $$

$$ \tilde{i}_n + 1 = (x_n - D_n)^+ + \tilde{q}[n+1-\tau, n] + \tilde{q}_n. \quad (2.13) $$

Furthermore,

$$ q^*_n + 1 = (S^B_\tau - (x_n - D_n) - q^*[n+1-\tau, n])^+ \quad \text{and} \quad (2.14) $$

$$ \tilde{q}_n + 1 = (S^B_\tau - (x_n - D_n)^+ - \tilde{q}[n+1-\tau, n])^+. \quad (2.15) $$

The first equality above is due to the previously discussed optimality of the order-up-to-$S^B_\tau$ policy in systems with backorders and a lead time of $\tau$ periods, and, the second equality is due to the construction of $\tilde{\pi}_{n+1}$.

When $D_n \geq x_n$, (2.11) and equations (2.12)-(2.15) together imply that $\tilde{i}_n + 1 = i^*_n + 1 = S^B_\tau$. This establishes (i) for sample paths in which $x_n \leq D_n$. When $x_n > D_n$, (2.11) and equations (2.12)-(2.15) together imply $S^B_\tau \leq \tilde{i}_n + 1 \leq i^*_n + 1$. We have now proved (i) for all sample paths of demand.

Statement (ii) is a well known result in inventory theory for systems with backordering when an optimal policy is followed. Since excess demands are backordered in $[n+1, N]$ in both $\Gamma_n$ and $\Gamma_{n+1}$, the order-up-to-$S^B_\tau$ policy is optimal in both from $n + 1$ onwards. By definition, $\pi^*_n$ is the optimal policy in $\Gamma_n$ and by construction, $\tilde{\pi}_{n+1}$ is the order-up-to-$S^B_\tau$ policy from $n + 1$ onwards. So, the result on the existence of a convex function $V$
that represents the expected discounted costs over \([n + 1 + \tau, N]\) and the property that \(S^B_{t}\) minimizes \(V\) apply here.

Statement (iii) is a corollary of the convexity of \(V\) and statement (i). Statement (iv) follows from (iii) by taking the expectation over all possible realizations of \(\mathcal{F}_{n+1}\). □

We know that \(\tilde{C}_{n+1}(t) = C^*_n(t) \forall t \in [0, n]\) for every sample path and that
\[
E[\sum_{t=n+1+\tau}^{n} \tilde{C}_{n+1}(t) | \mathcal{F}_0] \leq E[\sum_{t=n+1+\tau}^{n} C^*_n(t) | \mathcal{F}_0].
\]
It remains to show
\[
E[\sum_{t=n+1}^{n+\tau} \tilde{C}_{n+1}(t) | \mathcal{F}_0] \leq E[\sum_{t=n+1}^{n+\tau} C^*_n(t) | \mathcal{F}_0].
\]

Let us consider some period \(t \in [n + 1 - \tau, n]\). By Lemma 2.3.2 (i), we know that either \(\tilde{q}_t = 0\) or \(E[\tilde{y}_{t+\tau}|\mathcal{F}_t] = E[y^*_t|\mathcal{F}_t]\). In the following two lemmas, we prove the dominance of \(\Gamma_{n+1}\) over \(\Gamma_n\) in period \(t + \tau\) for the latter and former cases, respectively.

Lemma 2.3.4. Let \(E[\tilde{y}_{t+\tau}|\mathcal{F}_t] = E[y^*_t|\mathcal{F}_t]\) for some \(t \in [n + 1 - \tau, n]\) and \(\mathcal{F}_t\). (i) Then, \((\tilde{y}_{t+\tau}|\mathcal{F}_t) \leq_{cx} (y^*_t|\mathcal{F}_t)\). (ii) Consequently, \(E[g(\tilde{x}_{t+\tau} - D_{t+\tau})|\mathcal{F}_t] \leq E[g(x^*_t - D_{t+\tau})|\mathcal{F}_t]\).

Proof: Let \(X = (x_n|\mathcal{F}_t), D = D_n, q^B = q^*[n + 1 - \tau, t]\) and \(q^C = \tilde{q}[n + 1 - \tau, t]\). Let \(\phi(y) = E[g(y - D[n + 1, t + \tau])].\) Using the relations established in (2.6) and (2.7), the results follow from Lemma 2.1.2. □

Lemma 2.3.5. Let \(\tilde{q}_t = 0\) for some \(t \in [n + 1 - \tau, n]\) and \(\mathcal{F}_t\). (i) Then, \(\exists \Delta^* \leq S^B_{t+\tau-n-1}\) such that \(E[\max(y^*_t, \Delta^*)|\mathcal{F}_t] = E[\tilde{y}_{t+\tau}|\mathcal{F}_t]\) and \((\tilde{y}_{t+\tau}|\mathcal{F}_t) \leq_{cx} (\max(y^*_t, \Delta^*)|\mathcal{F}_t)\). (ii) Also,
\[
E[g(\max(y^*_t, \Delta^*) - D[n + 1, t + \tau])|\mathcal{F}_t] \leq E[g(y^*_t - D[n + 1, t + \tau])|\mathcal{F}_t].
\]
Consequently, \(E[g(\tilde{x}_{t+\tau} - D_{t+\tau})|\mathcal{F}_t] \leq E[g(x^*_t - D_{t+\tau})|\mathcal{F}_t]\).

Proof: Assume \(E[\tilde{y}_{t+\tau}|\mathcal{F}_t] > E[y^*_t|\mathcal{F}_t]\). (Otherwise, \(E[\tilde{y}_{t+\tau}|\mathcal{F}_t] = E[y^*_t|\mathcal{F}_t]\) and the trivial choice of \(\Delta = -\infty\) can be used.) So, \(E[\max(y^*_t, \Delta)|\mathcal{F}_t] < E[\tilde{y}_{t+\tau}|\mathcal{F}_t]\) when \(\Delta = -\infty\).

Next, consider \(\Delta = \tilde{q}[n + 1 - \tau, t]\); that is, \(\Delta = \tilde{q}[n + 1 - \tau, t - 1]\) because \(\tilde{q}_t = 0\). Now, observe the following. For all \(\mathcal{F}_{n+1}\) such that \(D_n \leq x_n, y^*_t \geq \tilde{y}_{t+\tau}\) from (2.6), (2.7) and Lemma 2.3.2 (ii); this implies that \(\max(y^*_t, \Delta) \geq \tilde{y}_{t+\tau}\) for such \(\mathcal{F}_{n+1}\). For all \(\mathcal{F}_{n+1}\) such that \(D_n > x_n, \max(y^*_t, \Delta) \geq \tilde{q}[n + 1 - \tau, t - 1] = \tilde{y}_{t+\tau}\). Combining these
two observations and taking the expectation over all realizations of \( F_{n+1} \), given \( F_t \), we get
\[
E[\max(y_{t+\tau}^*, \Delta)|F_t] \geq E[\tilde{y}_{t+\tau}|F_t].
\]

Furthermore, \( E[\max(y_{t+\tau}^*, \Delta)|F_t] \) is a continuous and increasing function of \( \Delta \). The inequalities involving \( E[\max(y_{t+\tau}^*, \Delta)|F_t] \) and \( E[\tilde{y}_{t+\tau}|F_t] \) switch directions when the choice of \( \Delta \) changes from \(-\infty\) to \( \tilde{q}[n+1-\tau, t-1] \). So, there is a \( \Delta^* \leq \tilde{q}[n+1-\tau, t-1] \) when the expectations become equal. \( \Delta^* \) is bounded above by \( S_{t+\tau-n-1}^g \) because \( \tilde{q}[n+1-\tau, t-1] \leq S_{t+\tau-n-1}^g \) by Lemma 2.3.2.

Also, observe that (a) \( P(\max(y_{t+\tau}^*, \Delta^*) \leq u|F_t) \geq 0 = P(\tilde{y}_{t+\tau} \leq u|F_t) \forall u < \tilde{q}[n+1-\tau, t] \) and (b) \( P(\max(y_{t+\tau}^*, \Delta^*) \leq u|F_t) \leq P(\tilde{y}_{t+\tau} \leq u|F_t) \forall u \geq \tilde{q}[n+1-\tau, t] \). This single crossing property, the equality of the expectations and Lemma 2.1.1 imply that \( (\tilde{y}_{t+\tau}|F_t) \leq cx (\max(y_{t+\tau}^*, \Delta^*)|F_t) \).

Next,
\[
E[g(\tilde{y}_{t+\tau} - D[n+1, t+\tau])|F_t] \leq E[g(\max(y_{t+\tau}^*, \Delta^*) - D[n+1, t+\tau])|F_t] \quad \text{and}
E[g(\max(y_{t+\tau}^*, \Delta^*) - D[n+1, t+\tau])|F_t] \leq E[g(y_{t+\tau}^* - D[n+1, t+\tau])|F_t]
\]
follow from the convexity of the function \( E[g(y - D[n+1, t+\tau])|F_t] \), the facts that \( S_{t+\tau-n-1}^g \) minimizes this function and \( \Delta^* \leq S_{t+\tau-n-1}^g \). Using (2.8) and (2.9) and these inequalities, we obtain \( E[g(\tilde{x}_{t+\tau} - D_{t+\tau})|F_t] \leq E[g(x_{t+\tau}^* - D_{t+\tau})|F_t] \). \( \square \)

**Corollary 2.3.6.** For all \( F_0 \), \( E[\sum_{t=n+1}^{t=n+\tau} C_{n+1}(t)|F_0] \leq E[\sum_{t=n+1}^{t=n+\tau} C_n(t)|F_0] \).

**Proof:** Lemmas 2.3.2 (i), 2.3.4 and 2.3.5 together imply that
\[
E[\tilde{C}_{n+1}(t+\tau)|F_t] \leq E[C_n(t+\tau)|F_t] \quad \text{for any } t \in [n+1-\tau, n] \text{ and any } F_t.
\]
Taking expectation over all \( F_t \) and summing over all \( t \) in \([n+1-\tau, n]\), we get the desired result. \( \square \).

**Note:** The policy \( \tilde{x}_{n+1} \) uses the information \( D_t \) in period \( t+1 \) even if excess demand is lost in period \( t \). This is more information than a system that loses sales in a period would normally have; that is, only sales can be observed not demand. However, since demands are assumed to be i.i.d. and these systems are Markovian, we know that the optimal policy
for $\Gamma_{n+1}$ does not use the information $D_t$ in period $t+1$, when sales are lost in period $t$, if $t \leq n$. Therefore, the optimal expected cost of managing $\Gamma_{n+1}$ is not more than the cost of managing $\Gamma_{n+1}$ using $\tilde{\pi}_{n+1}$.

We state the conclusion of the analysis of the subcase $n \geq \tau - 1$ formally as a theorem.

**Theorem 2.1.** Assume $n \geq \tau - 1$. For any $(x_0, \alpha, N)$ and for any $n \in \{0, 1, \ldots, N\}$, $E[C^*_n|F_0] \leq E[C^*_n|F_0]$.

### 2.3.2 Case 2: $n < \tau - 1$: Periods within the Lead Time

In this section, we shall show that $\Gamma_{n+1}$ dominates $\Gamma_{n}$ under some assumptions about the starting state $x_0$. These assumptions are motivated by the optimal starting state for $B$, the system in which excess demand is backordered in all periods.

Let $S_0 = x_0$ and $S_t = (x_0 + q_{-\tau+1} + \ldots + q_{-\tau+t}) \forall t \in \{1, 2, \ldots, \tau - 1\}$. That is, $S_t$ represents the cumulative supply of inventory for the interval $[0, t]$. We now present a result on the optimal $x_0$ for system $B$. Recall that

$$S_m^B = \arg \min_y E[g(y - D[0, m])] .$$

In the next lemma, we use the notation $C^*_B(x_0)$ to explicitly show the dependence of $C^*_B$ on $x_0$.

**Lemma 2.3.7.** Let $x_0^B = (S_{\tau-1}^B - S_{\tau-2}^B, \ldots, S_1^B - S_0^B, S_0^B)$. Then, $x_0^B$ minimizes $C^*_B(x_0)$ for all $(\alpha, N)$.

**Proof:** If $x_0 = x_0^B$, then $x_0 = S_0^B$ and $q_{-\tau+t} = S_t^B - S_{t-1}^B \forall t \in \{1, \ldots, \tau - 1\}$. Therefore, $S_t = S_t^B \forall t \in \{0, 1, \ldots, \tau - 1\}$. Next, observe that the ordering policy from period 0 onwards can affect the expected costs only from period $\tau$ onwards. By Lemma 2.2.1, we know that the optimal ordering policy from period 0 onwards is the order-up-to-$S_\tau^B$ policy. Since $S_{\tau-1} = S_{\tau-1}^B$ and $S_{\tau-1}^B \leq S_\tau^B$ (by Lemma 2.2.1), we know that ordering up to $S_\tau^B$ is also feasible. This shows that by using the order-up-to-$S_\tau^B$ policy from period 0 and state
$x_0^B$, the expected total discounted cost in $[\tau, N]$ is the minimum achievable by any policy from any state.

Furthermore, the expected cost incurred in period $t$ ($t \in \{0,1,\ldots,\tau-1\}$) is $E[g(S_t - D[0,t])]$. Since $S_t = S_t^B \forall t \in \{1,2,\ldots,\tau-1\}$, the state $x_0$ minimizes the expected cost incurred in each period in $\{0,1,\ldots,\tau-1\}$.

Combining the properties discussed above, we see that $x_0^B$ minimizes the total expected discounted cost over $[0,N]$. $\square$

Let $x_0^B$ represent the optimal starting state derived in Lemma 2.3.7. We now state an assumption on $x_0$, the starting state vector, under which $\Gamma_{n+1}$ dominates $\Gamma_n$.

**Assumption 2.2.** $S_{m+n} - S_m \leq S_{n-1}^B \forall (m,n)$ such that $m \in \{0,\ldots,\tau-1\}$, $n \geq 1$ and $(m+n) \in \{1,\ldots,\tau-1\}$.

In words, we assume that the cumulative supply over the interval $[m+1,m+n]$ does not exceed the optimal order-up-to level for a system with an $(n-1)$-period lead time.

We now proceed to prove that $\Gamma_{n+1}$ dominates $\Gamma_n$ when Assumption 2.2 is satisfied. Recall that our construction of $\tilde{\pi}_{n+1}$ is as follows:

(i) $\tilde{q}_t = (q^*[0,t] - \tilde{q}[0,t-1] - E[(D_n - x_n)^+ | F_t])^+ \forall t \in [0,n]$, and

(ii) $\tilde{q}_t = (S^B_t - \tilde{x}_t - \tilde{q}[t+1-\tau,t-1])^+ \forall t \in [n+1,N]$.

We will now compare the costs in $\Gamma_n$ and $\Gamma_{n+1}$ over the horizon of $N$ periods in four parts: (i) $[0,n]$, (ii) $[n+1,\tau-1]$, (iii) $[\tau,n+\tau]$, and (iv) $[n+\tau+1,N]$.

The costs in $\Gamma_n$ and $\Gamma_{n+1}$ are the same in $[0,n]$. Next, we show the dominance result for the interval $[n+1,\tau-1]$.
Lemma 2.3.8. Consider any $t$ in $[n + 1, \tau - 1]$. Under Assumption 2.2, the following statements hold.

(i) For any $\mathcal{F}_{n+1}$, either

$$y_t^* \leq \tilde{y}_t \leq S_{t-(n+1)}^B \text{ or } y_t^* = \tilde{y}_t.$$ 

(ii) For any $\mathcal{F}_{n+1}$,

$$E[g(\tilde{x}_t - D_t)|\mathcal{F}_{n+1}] \leq E[g(x_t^* - D_t)|\mathcal{F}_{n+1}] .$$

(iii)

$$E[g(\tilde{x}_t - D_t)|\mathcal{F}_0] \leq E[g(x_t^* - D_t)|\mathcal{F}_0] .$$

Proof: For all $\mathcal{F}_{n+1}$ such that $D_n \leq x_n$, we can observe that $y_t^* = \tilde{y}_t$ from (2.6), (2.7) and the fact that $\tilde{q}[n + 1 - \tau, t - \tau] = q^*[n + 1 - \tau, t - \tau] = q[n + 1 - \tau, t - \tau]$.

Let us now consider some $\mathcal{F}_{n+1}$ such that $D_n > x_n$. Then, $\tilde{y}_t = q[n + 1 - \tau, t - \tau] = S_t - S_n \leq S_{t-(n+1)}^B$ by Assumption 2.2; also,

$$y_t^* = (x_n - D_n) + q[n + 1 - \tau, t - \tau] < q[n + 1 - \tau, t - \tau] = \tilde{y}_t.$$

So, we obtain the inequalities $y_t^* \leq \tilde{y}_t \leq S_{t-(n+1)}^B$.

Statement (ii) is simply a consequence of the first statement, the relations in (2.6-2.9) and the convexity of the function $E[g(y - D[n+1, t])|\mathcal{F}_{n+1}]$. Statement (iii) follows from (ii) by taking the expectation over all possible realizations of $\mathcal{F}_{n+1}$. □

To show the dominance over the intervals $[\tau, n + \tau]$ and $[n + 1 + \tau, N]$, we use results that are identical to Lemmas 2.3.2-2.3.5 and Corollary 2.3.6. We summarize these results next.

Lemma 2.3.9. Under Assumption 2.2, the following statements hold.

(i) For every $t \in [n + 1 - \tau, n]$, the inequality $q^*[n + 1 - \tau, t] \leq S_{t+\tau-n}^B$ holds.

(ii) For all $t \in [0, n]$ and any $\mathcal{F}_t$, either $\tilde{q}_t = 0$ or $E[\tilde{y}_{t+\tau}|\mathcal{F}_t] = E[y_{t+\tau}^*|\mathcal{F}_t]$. In either case, $E[\tilde{y}_{t+\tau}|\mathcal{F}_t] \geq E[y_{t+\tau}^*|\mathcal{F}_t]$.

(iii) For all $t \in [n + 1 - \tau, n]$ and any $\mathcal{F}_t$, $\tilde{q}[n + 1 - \tau, t] \leq q^*[n + 1 - \tau, t] \leq S_{t+\tau-n}^B$. 

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(iv) Let $i_{n+1}^*, \tilde{i}_{n+1}$ and $V$ be as defined in the statement of Lemma 2.3.3. Then, for any $\mathcal{F}_{n+1}$, $S^B_{\tau} \leq \tilde{i}_{n+1} \leq i_{n+1}^*$. Given any $\mathcal{F}_{n+1}$, $E[\sum_{t=n+1+\tau}^{n} \tilde{C}_{n+1}(t)|\mathcal{F}_{n+1}] \leq E[\sum_{t=n+1+\tau}^{n} C^*_n(t)|\mathcal{F}_{n+1}]$. Therefore, $E[\sum_{t=n+1+\tau}^{n} \tilde{C}_{n+1}(t)|\mathcal{F}_0] \leq E[\sum_{t=n+1+\tau}^{n} C^*_n(t)|\mathcal{F}_0]$.

(v) Let $E[\tilde{y}_{t+\tau}|\mathcal{F}_t] = E[y^*_n|\mathcal{F}_t]$ for some $t \in [0, n]$ and $\mathcal{F}_t$. Then, $(\tilde{y}_{t+\tau}|\mathcal{F}_t) \leq E_{\mathcal{F}_t}$ (iv) Let $\tilde{q}_t = 0$ for some $t \in [0, n]$ and $\mathcal{F}_t$. Then, $\exists \Delta^* \leq S^B_{t+\tau-n}$ such that $E[\max(y^*_t, \Delta^*)]|\mathcal{F}_t] = E[\tilde{y}_{t+\tau}|\mathcal{F}_t]$ and $(\tilde{y}_{t+\tau}|\mathcal{F}_t) \leq E_{\mathcal{F}_t}$ (iv) Let $\tilde{x}_t = 0$ for some $t < \tau$. For $0 < t < n$, the proof of Lemma 2.3.1 applies.

Proof: (i) For $t < 0$, $q^*[n+1+\tau, t] = q[n+1-\tau, t] = S_t - S_{n-\tau} \leq S^B_{t+\tau-n}$ by Assumption 2.2. Therefore, by Lemma 2.2.1 (ii), we get $q^*[n+1-\tau, t] \leq S^B_{t+\tau-n}$. For $0 \leq t \leq n$, the proof of Lemma 2.3.2 (i) applies.

(ii) For all $t < 0$,

$q^*[n+1-\tau, t] = \tilde{q}[n+1-\tau, t] = q[n+1-\tau, t] = S_t - S_{n-\tau} \leq S^B_{t+\tau-n}$

For all $t \geq 0$, the proof of Lemma 2.3.2 (ii) applies.

(iv) The proof of Lemma 2.3.3 applies.

(v) The proof of Lemma 2.3.4 applies.

(vi) The proof of Lemma 2.3.5 applies.

(vii) Statements (v) and (vi) imply this. \(\Box\)

We conclude the analysis of the case in which $n < \tau - 1$ formally in the next theorem.

**Theorem 2.2.** Assume $n < \tau - 1$. For any $x_0$ satisfying Assumption 2.2, for any $(\alpha, N)$ and for any $n \in \{0, 1, \ldots, N\}$, $E[C^*_n|\mathcal{F}_0] \leq E[C^*_n|\mathcal{F}_0]$.  

"
We conclude this section by stating two other assumptions that jointly imply Assumption 2.2. The new assumptions are more directly connected to the basic parameters of the systems under consideration.

**Assumption 2.3.** $x_0 \leq x_0^B$ component-wise. That is, $x_0 \leq S_0^B$ and $q_{-\tau+t} \leq S_t^B - S_{t-1}^B \forall t \in \{1, \ldots, \tau - 1\}$.

**Assumption 2.4.** $S_{m+n}^B - S_m^B \leq S_{n-1}^B \forall (m,n)$ such that $m \in \{0, \ldots, \tau - 1\}$, $n \geq 1$ and $(m+n) \in \{1, \ldots, \tau - 1\}$.

**Lemma 2.3.10.** Assumptions 2.3 and 2.4 imply Assumption 2.2.

**Proof:**

\[
S_{m+n}^B - S_m^B = \sum_{t=m+1}^{n+m} q_{-\tau+t} \leq S_{m+n}^B - S_m^B \leq S_{n-1}^B .
\]

\[\square\]

Let us now take a closer look at Assumptions 2.3 and 2.4. Assumption 2.3 is simply an assumption on the starting state. Assumption 2.4, however, is an assumption on the other parameters of the system, namely the demand distribution, $g(\cdot)$ and $\tau$. Next, we show that Assumption 2.4 is satisfied when $D$ is normally distributed and the holding and shortage costs are linear.

**Lemma 2.3.11.** Assume (a) $D_n$ has a normal distribution and (b) $G(x) = h \cdot E(x - D)^+ + b \cdot E(D - x)^+$, where $0 \leq h \leq b$. Then, Assumption 2.4 is satisfied.

**Proof:** For the normal distribution, it is easy to derive the following expression for $S_t^B$ using the newsvendor formula.

\[
S_t^B = \mu \cdot (t + 1) + k \cdot \sigma \cdot \sqrt{t+1},
\]

where $k$ is a positive constant. Observe that

\[\sqrt{m+n+1} - \sqrt{m+1} \leq \sqrt{n+1} .\]

Assumption 2.4 can now be verified easily. \[\square\]
2.4 Results

We are now ready to state our main results comparing the costs in $L$ and $B$.

**Theorem 2.3.** If $x_0$ satisfies Assumption 2.2, then, for any $(\alpha, N)$, $C^*_{L} \leq C^*_{B}$.

**Proof:** $B$ is the same as $\Gamma_0$ and $L$ is the same as $\Gamma_{N+1}$. Now, Theorem 2.1 applied sequentially from $n = 0$ to $n = \tau - 2$ and Theorem 2.2 applied sequentially from $n = \tau - 1$ to $n = N$ imply the result. \(\square\)

Next, we present a counterexample to show that Theorem 2.3 can fail if Assumption 2.2 does not hold.

**Lemma 2.4.1.** Without any assumptions on $x_0$, the inequality $C^*_{L}(x_0, \alpha, N) \leq C^*_{B}(x_0, \alpha, N)$ does not always hold.

**Proof:** Let $\tau = 2$ and $N \geq 1$. Let $x_0 = (100, 0)$. [0 units are on hand at the beginning of period 0 and 100 units are scheduled to arrive at the beginning of period 1]. Demand is deterministic and equals 50 units every period. Let $g(x) = h \cdot (x)^+ + b \cdot (x)^-$ for some $h \geq 0$ and $b \geq 0$. Now, the optimal cost in the backorder system is lower than the optimal cost in the lost sales system by $\alpha \cdot h \cdot 50$. \(\square\)

*Note: The demand distribution in the example above was concentrated at 50. However, it should be obvious to the reader that similar examples can be constructed for continuous demand distributions using more elaborate computations.*

**Theorem 2.4.** For the infinite horizon, average cost model, the lost sales model dominates the backorder model for any starting state.

**Proof:** It is easy to observe that the average cost in the backorder model when the optimal policy is used is $E[g(S^B_\tau - D[0, \tau])]$, which is independent of the starting state. Schäl (1993) shows that the minimum infinite horizon average cost for a Markov decision model is the limit of $(1 - \alpha) \cdot (\text{the minimum infinite horizon discounted cost with discount factor } \alpha)$ as $\alpha$.
approaches 1 and that this limit is independent of the starting state, when certain conditions are satisfied. Huh and Janakiraman (2005) show that these conditions are satisfied for the lost sales system. Therefore, the dominance of the lost sales system over the backorder system is a direct consequence of this limiting result and Theorem 2.3. □

3 Cost Comparison under a Specific Linear Cost Structure

In this section, we work with the linear holding and shortage cost model of Assumption 1.2. We will provide a cost comparison result between $\mathcal{B}$ and $\mathcal{L}$ in the opposite direction of Theorem 2.3 and Theorem 2.4 for a specific choice of the shortage cost parameters.

**Theorem 3.1.** Let $h$, $b^B$ and $b^L$ be strictly positive constants such that $b^B \leq b^L / (\tau + 1)$. Let $g^B(x) = h \cdot (x)^+ + b^B \cdot (x)^-$ and $g^L(x) = h \cdot (x)^+ + b^L \cdot (x)^-$. (i) Then, for any $(x_0, \alpha, N)$,

$$C^*_{\mathcal{B}} \leq C^*_{\mathcal{L}}$$

holds.

(ii) The inequality holds for the average cost model also.

**Proof:** Let $\mathcal{L}$ use the optimal policy, say $\pi^*_{\mathcal{L}}$. Consider the following policy $\pi^*_{\mathcal{B}}$ for $\mathcal{B}$. In any period $t$, $\pi^*_{\mathcal{B}}$ orders the quantity ordered by $\pi^*_{\mathcal{L}}$ in that period $t$ plus the demand $\mathcal{L}$ lost in period $t - 1$. Moreover, $\pi^*_{\mathcal{B}}$ does not serve customers on a first-come-first-serve basis, which is the optimal service mechanism; if a customer who arrives in period $t - 1$ is backordered, a unit is ordered for that customer in period $t$ and is received in period $t + \tau$ and is given to that customer. Notice that there might be periods between $t$ and $t + \tau$ in which this customer stays backordered by $\pi^*_{\mathcal{B}}$ although units might have been available; in other words, $\mathcal{B}$ could have inventory on hand and backorders at the same time. The reason for this construction is simple. It ensures the following: (a) The inventory on hand in $\mathcal{B}$ equals that in $\mathcal{L}$ in every period. (b) The number of customers that arrive in $t$ and are backordered in $\mathcal{B}$ equals the number of customers that arrive in $t$ and leave unsatisfied in $\mathcal{L}$. (c) Therefore, each such
customer in $\mathcal{B}$ incrementally adds $(\tau + 1) \cdot b^B$ when $\alpha = 1$ (when $\alpha < 1$, the discounted backordering cost is even smaller than this) to the shortage cost since each backordered customer waits for $\tau + 1$ periods to receive a unit. Now, (a) implies that the holding costs in $\mathcal{B}$ and $\mathcal{L}$ are the same in every period. The assumption that $b^B \leq b^L/\tau$ and (c) imply that the shortage costs incurred by $\mathcal{B}$ in $[0, N]$ are lower than those incurred by $\mathcal{L}$ in $[0, N]$. So, $\mathcal{B}$ incurs a lower cost than $\mathcal{L}$ in $[0, N]$ for every $F_{N+1}$. This proves (i). Statement (ii) again follows from the limiting results referred to in the proof of Theorem 2.4. □

4 Remarks

1. Let us consider the linear cost model of Assumption 1.2 and assume that $\mathcal{B}$ and $\mathcal{L}$ have the same holding cost parameter. Our results imply that when $b^B \geq b^L$, $\mathcal{L}$ incurs a lower expected long run average cost, optimally, than $\mathcal{B}$ whereas when $b^B \leq b^L/(\tau + 1)$, the reverse is true. Both these results hold for any demand distribution. The following are interesting open questions: (a) Can the former result be shown over an interval of $b^B$ that is larger than $[b^L, \infty)$? (b) Can the latter result be shown over an interval of $b^B$ that is larger than $[0, b^L/(\tau + 1)]$? (c) Do the answers to (a) and (b) depend on the demand distribution?

2. There has been some interest recently on models where backordered customers are offered rebates or incentives (see for example DeCroix and Arreola-Risa (1998) or Bhardwaj et al. (2005)). An important topic in that area that deserves attention is the question of how much of a rebate should we give backordered customers instead of losing their sales. The larger the rebate, the greater the chance of retaining a customer but lower the profit margin per customer. We hope that the results and analysis we have developed in this paper will be useful in addressing such questions, at least partially.

3. As mentioned in the introductory section, lost sales inventory models have been notoriously intractable when lead times are present. The questions we have posed in this
paper are simple to state and have intuitive answers. However, we demonstrate that the answers could be counter-intuitive sometimes and even when they are intuitive, the analysis can be quite non-trivial. It is hoped that the method of analysis we have employed here can be applied in other inventory models and, in particular, can be used to further our knowledge on lost sales inventory models.

4. The main results of Section 2 hold for independent and stochastically increasing demand models. This can be seen by modifying the definition of $S_m^B$ to include time-dependence:

$$S_{m,t}^B = \max \{ \arg \min_y E[g(y - D[t, m + t])] \}.$$ 

Theorems 2.3 and 2.4 and their proofs (with the obvious modifications to the notation, wherever $S_m^B$ appears) continue to hold.

5. The main result of Section 3, namely Theorem 3.1, holds under any arbitrary demand structure including correlations and non-stationarity. This is because the proof shows the dominance of $B$ over $L$ for every sample path.

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**References**


