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European Journal of Operational Research 159 (2004) 296–317
EUROPEAN JOURNAL OF OPERATIONAL RESEARCH
www.elsevier.com/locate/dsw

# Dynamic balancing of inventory in supply chains

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Available online 23 October 2003

**Abstract**

We consider the dynamic version of the classic problem of allocation of inventories to a set of retailers to rectify the imbalance of inventories amongst them. While most research is focussed on analyzing different allocation strategies with a predetermined time of shipment (static policy), we investigate the benefit of using real time demand (inventory) information to schedule rebalancing shipments in a retail network. We model the dynamic rebalancing problem that has two decisions, the timing of the balancing shipments and determination of the new stocking levels at the retailers, as a dynamic program (DP). We obtain structural properties for the optimal allocation, rebalancing and timing strategies. We also present conditions under which a greedy heuristic to decide how much to ship from one retailer to another is optimal. The DP for determining the optimal timing and quantity of shipments has a very large state space. We present an algorithm to solve this DP efficiently. We also provide a heuristic solution procedure to the dynamic problem that performs very close to optimal. Numerical results show that dynamic allocation policies can lead to substantial benefits over the static policy especially in systems in which the starting inventories at the retailers are balanced or when high service levels are required.


*Keywords:* Supply chain; Allocation strategies; Retailers; Balancing; Schur-convexity; Dynamic program; Majorization ordering

## 1. Introduction

We consider a supply chain in which a vendor sells a single good to several stores. The customer demand at stores is independent. Unmet sales are lost. At the beginning of the period, the inventories at the stores are known and the supplier has a given stock of the good. The supplier's decisions are: (i) To determine the quantities to ship at the beginning of the period to the stores, and (ii) depending on the sales at the stores to rebalance the inventories held by the stores at an appropriate time but before the end of the period. Our main objective in this paper is to investigate the value of inventory information in risk pooling and its impact on allocation decisions for multiple retailers with one supply point.

This problem is different from the ones previously studied because it combines the timing decision with the rebalancing decision. The problem is relevant because often due to the proximity of stores, suppliers

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0377-2217/$ - see front matter 
doi:10.1016/j.ejor.2003.08.017

find it economical to combine shipments to stores into a single large shipment. For example, to supply milk from Philadelphia to a cluster of stores in the New York Metropolitan area it is cost effective to send a large refrigerated truck to supply the cluster of stores rather than to send a small truck to each store. In such cases, the timing of the shipment is obviously an important decision. However, the same shipment can be used to rebalance inventories. Thus, it leads to the question whether it is sufficiently beneficial to consider the two decisions jointly in a dynamic framework when compared to choosing (and fixing) the timing of rebalance initially and rebalancing appropriately at that fixed time, in other words, optimizing in a static framework. This problem is of particular interest in the fashion goods industry where the manufacturer has to decide when to rebalance the inventory during the selling season. Due to the availability of real time data, such rebalancing decisions can be effected within a few days of the beginning of a selling period for goods such as music CDs, new releases of books, cosmetics, etc.

The algorithmic aspects of this problem are also of interest because the dynamic program (DP) that has to be solved to determine the timing and the rebalance quantities has an extremely large state space. Thus, evaluation of heuristics for solving the problem is of theoretical as well as practical value. Finally, classification of the factors that determine the value of dynamic scheduling in such situations, for example, knowing whether savings increase with volatility, is bound to be of value to practitioners.

The last decade has seen tremendous investments in supply chain execution systems that use (nearly) real time information. In addition, we have seen a huge growth in the number of third party logistics providers, i.e., firms that provide less than truckload and local courier services. As a result of these changes, the lead-time from central distribution centers (DCs) or suppliers has gone down resulting in the reduced importance of stocking regional DCs to support retailers. Thus, rebalancing of retail inventories (only if and when it is needed as in a dynamic policy) has become more attractive.

The contributions in this paper are as follows: We present the general version of the dynamic scheduling problem in Section 3. In Section 3.1 we derive several structural properties of dynamic allocation policies that have two decisions, the timing of the balancing shipments and the new stocking levels at the retailers. We present conditions under which the optimal rebalancing strategy is to maximize the minimum inventory. We also present conditions under which a greedy heuristic to decide the shipments from one retailer to another is optimal. In Section 3.2 we formulate the problem as a DP and present several properties of the optimal solution. This DP has a very large state space. We present a simulation technique for solving the DP in Section 3.3. We also provide a heuristic solution to the DP that gives results that are very close to optimal. In Section 4 we exhibit numerical results to demonstrate that dynamic rebalancing policies can lead to substantial benefits compared to a static rebalancing policy especially in systems in which starting inventories at the retailers are balanced or when high service levels are required.

## 2. Related work

### 2.1. Risk pooling

Our work is related to the literature on risk pooling at the DC as exemplified by the work of Jonsson and Silver (1987a,b), Jackson (1988), Jackson and Muckstadt (1989), Schwarz (1989), Axsäter (1990), McGavin et al. (1993), Kumar et al. (1995), and Graves (1996). Recent studies and additional references on the problem of how much stock to hold at a central warehouse, how to set transshipment intervals, and how to allocate stocks from the warehouse to satisfy competing demands, can be found in Mercer and Tao (1996), van der Heijden (1999), and Verrijdt and de Kok (1996).

Of these studies, our model is the closest to that of McGavin et al. (1993). McGavin et al., model an $N$ retailer system, in which demand is generated by a stationary process with independent increments. The demand process is assumed to be independent and identically distributed (i.i.d.) across retailers.

McGavin et al's analysis is restricted to one review period of the DC (though it can be extended to multiple periods) and their aim is to study risk pooling. The DC receives stock at the beginning of its review period, i.e., the DC follows a base stock policy and enjoys instantaneous replenishment from an external supplier. The review period is divided into two intervals. In the first interval, part of the replenished stock is allocated to the retailers. Unmet demand during this interval is lost. At the beginning of the second interval, another allocation is made. Excess demand is lost in this interval too. The lead-time from the DC to the retailers is assumed to be zero. They do not model holding or set up costs but instead focus on minimizing the expected lost sales. The contributions of their paper are: (i) the optimal allocation maximizes the minimum retailer inventory, and that (ii) two well-chosen (not necessarily equal) intervals work equally well when compared to four balanced intervals for replenishing stocks.

McGavin et al. (1997) extend the single period problem to $T$ successive time intervals. In this multi-period problem, the objective is to determine the optimal redistribution of the inventory in *each* period (therefore the problem is not dynamic). They show that when the retailer cost is convex and the retailers are identical, the optimal solution is to balance the retailer inventories in each period. In general, balancing is shown to be not optimal when retailers are non-identical. Our work differs from this stream of work in that it develops the theme of dynamic determination of when to rebalance instead of depending on a prefixed time to rebalance.

### 2.2. Allocation of safety stock

Models dealing with the allocation of safety stocks between DC and retail outlets, have been considered by Jonsson and Silver (1987b), Deuermeyer and Schwarz (1981), Schwarz et al. (1985), and Graves (1996). The numerical results reported in the majority of these papers indicate that the optimal solution is for the DC to hold little or no stocks. This as well as the other factors discussed in Section 1 motivates our focusing upon the rebalancing decision and relegating the option of the DC holding back stock to redistribute to stores some time before the end of the selling season to future work.

### 2.3. Incentive issues

If the DC and retail outlets were not under single ownership, retail outlets will need an incentive to carry most of the safety stocks. The same situation will result, if the stocks at the retailers have to be readjusted to balance the ending stocks, as modeled by Jackson (1988) and Jackson and Muckstadt (1989). In either case, the analysis and resolution of the coordination issues can be based on the cost-sharing scheme proposed in Moses and Seshadri (2000). We come back to this issue in the next section when we discuss the optimal allocation of stocks.

## 3. Problem setting

We consider a two-echelon supply chain in which a vendor supplies a single good to $M$ stores from a DC. The time frame for decision making is a single period, that is further sub-divided into $N$ inventory reviews. Thus, the single period can be considered to comprise of $N$ sub-periods. We assume that the time to transport the good from the warehouse to the stores or between stores is negligible. There is adequate space to store inventories at the warehouse as well as the stores. In the general problem setting, the vendor (centralized decision maker) has complete knowledge of inventory and sales at all stores, Based on this information s/he must decide when and how to restock and/or rebalance the store inventories (jointly termed the rebalancing decision). We assume that the vendor has only one opportunity for making the rebalancing decision. We also assume that sales that are not met are lost, the inventory holding costs at different locations can be expressed as functions of the average sub-period opening and ending inventories, the cost of transportation between locations depends on the quantities shipped, and there is no discounting of costs that are incurred at different instants but within the same period. After formulating the general problem in this section we analyze a special case and provide numerical results pertaining to the special case in Section 4. In order to formally state the scheduling problem define the following quantities:

| | |
|---|---|
| $M$ | number of retailers indexed by $i$. The warehouse is given the index 0 |
| $N$ | number of sub-periods (inventory reviews) indexed by $j$ |
| $x_i$ | starting inventory at retailer $i$ |
| $D_{ij}$ | random demand at retailer $i$ during sub-period $j$, i.e., random demand at retailer $i$ between the $j$th and the $j+1$st review. We assume that $D_{ij}$ is a continuous random variable |
| $u_{ik}(x)$ | cost of transporting a quantity $x$ of the good from store $i$ to store $k$ |
| $\tau$ | sub-period in which the rebalancing is done |
| $h_i(x)$ | cost of holding a quantity $x$ of average inventory of the good at store $i$ |
| $Y_{ik}$ | the quantity of the good transported at $\tau$ from store $i$ to store $k$ |
| $g_i(x)$ | cost of losing a quantity $x$ of sales of the good at store $i$ |
| $I_i$ | average inventory at store $i$ |
| $L_i$ | lost sale at store $i$ |
| $x_{i\tau-}$, $x_{i\tau+}$ | inventory before and after the rebalancing at store $i$ |
| $A^+$ | stands for the positive part of $A$, i.e., $\max\{0, A\}$ |
| $A^-$ | stands for the negative part of $A$, i.e., $\max\{0, -A\}$ |

We can express the average inventory ($I_i$) and the lost sales ($L_i$) as shown below:

$$x_{i\tau-} = [x_i - D_{i1} - D_{i2} - \cdots - D_{i\tau}]^+,$$
$$x_{i\tau+} = [x_i - D_{i1} - D_{i2} - \cdots - D_{i\tau}]^+ + \sum_k Y_{ki} - \sum_i Y_{ik},$$
$$I_i = \tau(x_i + x_{i\tau-})/2 + (N-\tau)(x_{i\tau+} + [x_{i\tau+} - D_{i(\tau+1)} - \cdots - D_{iN}]^+)/2, \text{ and}$$
$$L_i = [x_i - D_{i1} - D_{i2} - \cdots - D_{i\tau}]^- + [x_{i\tau+} - D_{i(\tau+1)} - \cdots - D_{iN}]^-.$$

Let $E[A]$ stand for the expectation of a random variable $A$. Then, the optimization problem can be stated as follows:

$$\min_{\tau, Y_{ik}} \quad E\left(\sum_i \left(h_i(I_i) + g_i(L_i) + \sum_k u_{ik}(Y_{ik})\right)\right)$$
$$\text{subject to:} \quad \sum_k Y_{ik} \leqslant [x_i - D_{i1} - D_{i2} - \cdots - D_{i\tau}]^+, \quad Y_{ik} \geqslant 0.$$

### 3.1. Identical stores and minimizing lost sales

In this subsection, we study a version of the general problem to obtain structural properties of the optimal solution. Numerical results are presented in the next section that illustrate the value of dynamic scheduling. In this subsection we assume (unless stated otherwise) that the objective is to minimize the *expected cost of lost sales* at stores and that the demand at the stores are identically distributed in each sub-period for every store and across stores. Notice that we do not require the demands to be independent across stores.

*Majorization order*: Many of our results are based on inequalities that are governed by the majorization order. Therefore, we briefly define this order relation below and its connection to the usual order relation on the real line via the class of Schur Convex functions. Let the $i$th largest component of any vector $(a_1, a_2, \ldots, a_M)$ be denoted as $a_{[i]}$. Then, given two real-valued $M$-dimensional vectors $(x_1, x_2, \ldots, x_M)$ and $(y_1, y_2, \ldots, y_M)$, the vector $(x_1, x_2, \ldots, x_M)$ is said to *majorize* the vector $(y_1, y_2, \ldots, y_M)$, written as $\mathbf{x} \geqslant_m \mathbf{y}$, if:

$$\begin{aligned} & x_{[1]} \geqslant y_{[1]} \\ & x_{[1]} + x_{[2]} \geqslant y_{[1]} + y_{[2]} \\ & \ldots \\ & x_{[1]} + x_{[2]} + \cdots + x_{[i]} \geqslant y_{[1]} + y_{[2]} + \cdots + y_{[i]} \\ & \ldots \\ & x_{[1]} + x_{[2]} + \cdots + x_{[M]} = y_{[1]} + y_{[2]} + \cdots + y_{[M]}. \end{aligned} \tag{1}$$

*Schur-convex function:* A function, $g$, whose domain is a sub-set $\mathscr{A}$ of the $M$-dimensional Euclidean space, is said to be Schur-convex if given $\mathbf{x}$ and $\mathbf{y}$ that belong to $\mathscr{A}$ and that $\mathbf{x} \geqslant_m \mathbf{y}$ ($\mathbf{x}$ majorizes $\mathbf{y}$) then $g(\boldsymbol{x}) \geqslant g(\boldsymbol{y})$ (definition A.1, Chapter 3, p. 54, Marshall and Olkin, 1979).

**Lemma 3.1.** *If the cost of lost sales is given by*

$$\begin{aligned} & g_1([x_1 - D_{11} - D_{12} - \cdots - D_{1\tau}]^-, \ldots, [x_M - D_{M1} - D_{M2} - \cdots - D_{M\tau}]^-) \\ & \quad + g_2([x_{1\tau+} - D_{1(\tau+1)} - D_{1(\tau+2)} - \cdots - D_{1N}]^-, \ldots, [x_{M\tau+} - D_{M(\tau+1)} - D_{M(\tau+2)} - \cdots - D_{MN}]^-) \end{aligned}$$

*where $g_2$ is a Schur convex function then the optimal solution (given $\tau$) is to set*

$$x_{i\tau+} = \left( \sum_i [x_i - D_{i1} - D_{i2} - \cdots - D_{i\tau}]^+ \right) \Big/ M.$$

**Proof.** Follows from the definition of a Schur convex function and the fact that the completely balanced inventory vector is the smallest of all rebalanced inventory vectors in the majorization order. □

**Remark**

(i) This lemma extends McGavin et al. (1993) result that balancing is optimal when the objective is to minimize the sum of expected lost sales. This is because minimizing the expected total lost sales is a special case of this lemma.

(ii) The lemma will continue to hold even if the function $g_2$ were to depend on the history of the demands and the past actions of the decision maker.

(iii) Balancing is not optimal if it is only specified that the cost functions are identical and convex, i.e., if the total cost were written as the sum: $E(\sum_i g(L_i))$, where $g(.)$ is a convex function. In particular, if the stores are not under common ownership then further incentives might be required to achieve coordination. To see this notice that if any single store has suffered extremely high levels of lost sales and if $g(.)$ were convex then it is optimal to ship a larger quantity to this store to prevent further loss of sales. On the other hand, if the cost function could be written as $E[g(\sum_i L_i)]$ (for example, this could be the objective criterion when the stores and the supplier are under single ownership or could be the criterion of the supplier who does not wish to lose sales) then balancing is optimal if $g(.)$ is convex.

It is straightforward to determine the optimal solution when the cost of transporting the good is linear and identical between all locations—an assumption that is reasonable given the types of contracts that shippers such as Federal Express and UPS have entered into with retailers. Assume that $g_2$ is a strictly convex function. For notational convenience, let

$$L_i(a) = [a - D_{i(\tau+1)} - D_{i(\tau+2)} - \cdots - D_{i(N)}]^-.$$

Let the current inventories at stores $i$ and $j$ be $a$ and $b$ with $a$ greater than $b$. Let the transportation cost be $u$ per unit. Clearly, it is optimal to transport one unit of the good from location $i$ to location $j$ if

$$E[g_2(L_i(a)) + g_2(L_j(b))] \geqslant E[g_2(L_i(a-1)) + g_2(L_j(b+1))] + u. \quad (2)$$

Given a vector $\mathbf{x}$ of inventories, let $S_1(\mathbf{x})$ and $S_2(\mathbf{x})$ denote the set of stores with the largest and the smallest inventory.

**Lemma 3.2.** *Starting with the current inventory vector at time $\tau$ determine the sets $S_1(\mathbf{x})$ and $S_2(\mathbf{x})$, and choose one store in $S_1$ and one store in $S_2$. Apply (2) to determine whether one unit of the good should be shipped from the chosen store in $S_1$ to the store in $S_2$. Make the shipment of the single unit if it reduces the cost. Repeatedly apply this procedure until no further improvement can be made. This algorithm yields the lowest combined expected cost of lost sales and transportation given that the rebalancing is done at time $\tau$.*

**Proof.** Notice that the objective function is convex in the allocation made to each store. If applying condition (2) shows that no cost reduction is possible, then the point reached is a local optimum therefore it is also a global optimum. Moreover, there is never any backtracking in the algorithm because once a shipment is made it is final.

If we assume that the stores are ordered such that store $i$ initially has the $i$th largest inventory. When choosing a store from set $S_1$ choose the one with the largest index. Similarly, when choosing a store from $S_2$ choose the one with the smallest index. Thus, the $i$th largest inventory is always at the $i$th store. In this manner we also get a unique optimal solution. The reader is referred to Fox (1966) for other situations in which a greedy allocation is optimal. □

**Corollary 3.2.** *Let the cost of transporting the good be linear and identical, say $u$, between all locations. Assume that demands at different retailers are independent but not identically distributed. Define a greedy algorithm, GA, that ships one unit at a time from the store that experiences the smallest increase in the expected cost of lost sales due to the unit decrease in inventory; to the store that gains the most due to the transfer and stops when the benefit of such a transfer is less than $u$. Applying GA leads to the optimal solution of rebalancing the combined cost of transportation and lost sales.*

**Proof.** Similar to the proof of Lemma 3.2. □

**Lemma 3.3.** *If the cost of lost sales is given by $g(L_1, L_2, \ldots, L_M)$ where $g$ is a Schur convex function then the minimum inventory should be maximized after rebalancing in the optimal solution.*

**Proof.** Follows from the definition of a Schur convex function and the fact that the smallest of all rebalanced inventory vectors in the majorization order will have the largest minimum inventory, see (1). □

**Remark.** This extends the result due to McGavin et al., that the minimum inventory should be maximized when the function $g$ is convex.

### *3.2. Dynamic programming formulation and its structural properties*

We shall assume that the condition of Lemma 3.1 holds in this section unless stated otherwise. With this result in hand, we now proceed to develop a Dynamic Programming Formulation for solving the problem. We shall discuss the algorithmic extension required to cover other cases after presenting the theoretical analysis. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_M)$ denote the inventory vector at the beginning of sub-period $j$ and $\mathbf{D}_j = (D_{1j}, D_{2j}, \ldots, D_{Mj})$ stand for the demand vector in sub-period $j$. Let $[\mathbf{x} - \mathbf{D}_j]^+$ stand for the vector $([x_1 - D_{1j}]^+, [x_2 - D_{2j}]^+, \ldots, [x_M - D_{Mj}]^+)$. Let $V_j(\mathbf{x})$ be the optimal expected cost to go at the beginning of sub-period $j$ given that the current inventory vector is $\boldsymbol{x}$ and that rebalancing has not yet been done. Thus,

$$V_j(\mathbf{x}) = \min\{A_j(\mathbf{x}), \Lambda_j(\mathbf{x}) + E[V_{j+1}([\mathbf{x} - \mathbf{D}_j]^+)]\}, \tag{3}$$

$$A_j(\mathbf{x}) = E[\Sigma_i[\Sigma_{k \geqslant j} D_{ik} - \Sigma_i x_i/M]^+], \tag{4}$$

$$\Lambda_j(\mathbf{x}) = E[\Sigma_i[D_{ij} - x_i]^+]. \tag{5}$$

Eq. (3) represents the trade-off of rebalancing immediately versus waiting to rebalance. Eq. (4) follows from Lemma 3.1. The dynamic program given in (3)–(5) can be easily expanded to cover the case when stores are unequal or when the cost of transportation is linear and the same across all locations. If we could efficiently solve the problem with unequal stores and linear but different transportation costs between retailers, the same formulation can be used to solve this more general problem. We shall analyze the simpler formulation given in (3)–(5) to understand some properties of the optimal solution.

In the following we discuss the properties of the optimal solution to the DP as a function of (a) $N$, the number of sub-periods, (b) the result in terms of fixed cost for rebalancing, and (c) the result as a function of shipment delay. Consider the $N$-stage problem with initial inventory level $\mathbf{x} = (x_1, x_2, \ldots, x_M)$. Let $V_j(\boldsymbol{x})$ be the minimum cost of the problem with $N - j + 1$ sub-periods to go and one chance of rebalancing.

**Lemma 3.4.** *The value $V_j(\boldsymbol{x})$ is a Schur-convex function of $\boldsymbol{x}$.*

**Proof.** This is proved by induction. By using (4) and the fact that symmetric convex functions are Schur-convex (Proposition C.2, p. 66, Chapter 3, Marshall and Olkin, 1979), we obtain that $A_j(\mathbf{x})$ is a Schur-convex function of $\boldsymbol{x}$. By Eq. (5) and the same reasoning, $\Lambda_j(\mathbf{x})$ is also a Schur-convex function of $\boldsymbol{x}$. By these observations as well as the preservation properties of Schur-convex function under the min operator, it follows that if $V_{j+1}$ is a Schur-convex function then so is $V_j$. □

It follows from the Schur-convexity of the cost function that among all the states in $\{\mathbf{x}; \sum_{i=1}^{M} x_i = \text{Const}\}$ the state with all components equal is the most favored state, and the state with only one component non-zero is the least favored one. Let $S_j$ be the set of states $\boldsymbol{x}$ in which it is optimal to rebalance at the beginning of the $j$-stage problem.

**Lemma 3.5.** *The objective is to minimize the expected total lost sales. If it is optimal to rebalance in sub-period $\tau$ when the inventory vector is $\mathbf{y}$ then it is optimal to rebalance in sub-period $\tau$ when the inventory vector is $\mathbf{x}$ for all $\mathbf{x} \geqslant_m \mathbf{y}$.*

**Proof.** See Appendix. □

Lemma 3.5 reveals that the set of vectors in which it is optimal to rebalance in a given sub-period is not necessarily convex. This can be pictured in two dimensions (i.e., two retailer case) as follows. Let $(a, a+2)$ be the vector with the largest value of $a$ in which it is desirable to rebalance in some sub-period. Then, from Lemma 3.5 it surely is optimal to rebalance in this sub-period when the vector is $(x, y)$, with $x + y = 2a + 2$, and $|x - y| > 2$. However, it may also be optimal to rebalance in the same sub-period when the inventory vector is either $(2a, 4)$ or $(2a + 4, 4)$ due to significant expected lost sales at the second retailer. Thus, $(2a, 4)$ and $(4, 2a + 4)$ belong to the set of vectors in which it is optimal to rebalance in the second sub-period but

$(a+2, a+4)$ does not belong to this set. However, the lemma yields a nice characterization of $S_j$ as discussed below.

**Lemma 3.6.** *Let $\boldsymbol{x}$, $\boldsymbol{y}$ be in $S_j$ and $\mathbf{x} \leqslant_m \mathbf{y}$, where $\leqslant_m$ is the majorization ordering. Let $a_{[i]}$ denote the ith largest component of a vector $\boldsymbol{a}$. Consider the vector $\boldsymbol{z}$ whose ordered components equal the convex combination of the ordered components of $\boldsymbol{x}$ and $\boldsymbol{y}$, i.e.,*

$$z_{[i]} = \alpha x_{[i]} + (1-\alpha) y_{[i]}, \quad \alpha \in (0,1).$$

*Then the vector $\boldsymbol{z}$ is in $S_j$. Similarly, if $\boldsymbol{x}$, $\boldsymbol{y}$ are not in $S_j$ then the vector $\boldsymbol{z}$ is not in $S_j$.*

**Proof.** Note that $S_j$ is defined as the set of state $\boldsymbol{x}$ for which

$$\sum_i E((D_{ij} - x_i)^+) + E(V_{j+1}(\mathbf{x} - \mathbf{D}_j)^+) \geqslant E\left(\sum_i \left(\sum_{k=j}^{N} D_{ik} - M\right)^+\right). \tag{6}$$

First note that for $\mathbf{x} \leqslant_m \mathbf{y}$ we have

$$E\left(\sum_i \left(\sum_{k=j}^{N} D_{ik} - \sum_i x_i/M\right)^+\right) = E\left(\sum_i \left(\sum_{k=j}^{N} D_{ik} - \sum_i y_i/M\right)^+\right).$$

Since, by construction $\mathbf{x} \leqslant_m \mathbf{z} \leqslant_m \mathbf{y}$ it follows from the Schur-convexity of the left hand side of (6) that (6) is satisfied when $\boldsymbol{x}$ is replaced by $\boldsymbol{z}$, proving the first part of the lemma. The second part is proved similarly. □

**Corollary 3.6.** *Consider the case of two retailers $M = 2$ and $S(\Delta) = \{\boldsymbol{x} : x_1 + x_2 = \Delta\}$. The optimal strategy takes the following form: there exists a $\delta_j$ for the j-stage problem such that given $\mathbf{x} \in S(\Delta)$, it is optimal to rebalance immediately if and only if $|x_1 - x_2| \geqslant \delta_j$. Furthermore, $\delta_j$ is an increasing function of j.*

**Proof.** When $M = 2$ all the elements in $S(\Delta)$ are ordered in majorization ordering and $\boldsymbol{x}$ is smaller than $\boldsymbol{y}$ in this ordering if and only if $|x_1 - x_2| \leqslant |y_1 - y_2|$. Let

$$a = E\left(\sum_{i=1}^{2} \left(\sum_{k=j}^{N} D_{ik} - \Delta/2\right)\right).$$

Then it is optimal to rebalance if and only if

$$\sum_i E((D_{ij} - x_i)^+) + E(V_{j+1}(\mathbf{x} - \mathbf{D}_j)^+) \geqslant a.$$

Since the left hand side of the inequality is Schur-convex, it follows that there exists $\delta_j$ such that it is optimal to rebalance immediately if and only if $|x_1 - x_2| \geqslant \delta_j$. Furthermore, $\delta_j$ is an increasing function of $j$ is a consequence of Lemma 3.8 below. □

Let us now assume that, if the system implements a rebalance it incurs a fixed cost of $c$. In this case the optimality equation becomes

$$V_j(\mathbf{x}) = \min\left\{c + E\left(\sum_{i=1}^{M} \left(\sum_{k=j}^{N} D_{ik} - \sum_i x_i/M\right)^+\right), \sum_i E((D_{ij} - x_i)^+) + E(V_{j+1}(\mathbf{x} - \mathbf{D}_j)^+)\right\}.$$

Lemma 3.5 remains satisfied, i.e., if it is optimal to rebalance in sub-period $\tau$ when the inventory level is $\boldsymbol{y}$ then it is also optimal to rebalance when the inventory level is $\mathbf{x} \leqslant_m \mathbf{y}$. The proof is identical and one only needs to show that the value function is a Schur-convex function. Let $\tau^*(\tau^{**})$ be the optimal rebalancing time for the system without (with) a fixed balance cost.

**Lemma 3.7.** *The optimal rebalancing times satisfy* $\tau^* \leqslant \tau^{**}$.

**Proof.** This result is based on the following: it is not optimal to rebalance at time $\tau$ when the inventory level is $\boldsymbol{x}$ for the system without a fixed balance cost, then it is also not optimal at time $\tau$ when the inventory level is $\boldsymbol{x}$ for the system with a fixed balance cost. The latter result is straightforward to establish. □

Consider again the special case with two retailers.

**Corollary 3.7.** *The optimal strategy for the problem with fixed rebalancing cost c takes the following form: there exists a* $\delta_j(c)$ *for the j-stage problem such that given* $\mathbf{x}$ is in $S(\Delta)$, *it is optimal to rebalance at the beginning if and only if* $|x_1 - x_2| \geqslant \delta_j(c)$. *Furthermore,* $\delta_j(c)$ *is an increasing function of c and j.*

**Proof.** The proof is similar to that of Corollary 3.6. □

Let us consider the case of positive lead-time, i.e., when a decision is made to deliver, it will not be finished until a positive lead-time $L$ later. We further assume that before the items are rebalanced they are available to their original location. The optimality condition for this case can be written as

$$V_j(\mathbf{x}) = \min\left\{ E\sum_{i=1}^{M}\left(\sum_{k=1}^{L} D_{i(j+k)} - x_i\right)^+ + \left(\sum_{k=L+1}^{N-j} D_{i(j+k)} - \sum_i \left(x_i - \sum_{k=1}^{L} D_{i(j+k)}\right)^+ \Big/ M\right)^+, \right.$$
$$\left. E\sum_i (D_{ij} - x_i)^+ . + EV_{j+1}(\mathbf{x} - \mathbf{D}_j)^+ \right\}.$$

Now we consider whether increasing the number of sub-periods decreases the likelihood of rebalancing the inventory in an earlier sub-period. This is an important result as it can be used to construct heuristics to solve the general problem.

**Lemma 3.8.** *The objective is to minimize the expected total lost sales. If it is optimal to postpone the rebalance in sub-period* $\tau$ *when the inventory vector is* $\mathbf{y}$ *and the total number of sub-periods is N then it is optimal to postpone the rebalance in sub-period* $\tau$ *when the inventory vector is* $\mathbf{y}$ *and the total number of sub-periods is* $N+1$.

**Proof.** See Appendix. □

Lemma 3.8 reveals that the set of vectors in which it is optimal to rebalance is increasing with fewer sub-periods to go, or that the frequency distribution of rebalancing over the sub-periods will tend to be skewed to the right.

### 3.3. Solution to the dynamic program

The DP shown in (2)–(4) has an extremely large state space and thus cannot be solved easily. Instead we use Monte Carlo integration to solve the problem. The algorithm takes as input the initial vector of inventories, $\boldsymbol{x}_0$ and a number of replications, denoted as $R$. Thus, "simulate to estimate" means that the relevant quantity is simulated $R$ times and averaged out over the $R$ replications. The illustrative procedure (labeled DRA) for three periods is shown below.

**Dynamic Rebalancing Algorithm (DRA)**

```
begin
For period 1
   Simulate to estimate the cost if immediately balanced, say, C(1, 1)
   Simulate and determine the first period cost if not balanced, say A(1)
   Set the accumulator of cost if not balanced in first period, C(1, 2) = 0
Generate a demand for the first period R times. Each time:
   {
      For period 2
      Determine the starting inventory vector
      Simulate to estimate the cost if immediately balanced, say, C(2, 1)
      Simulate and determine the second period cost if not balanced, say A(2)
      Set C(2, 2) = 0
      Generate a demand for the second period R times. Each time:
      {
         For period 3
         Determine the starting inventory vector
         Simulate to estimate the cost when balanced, say, A(3)
         Set C(2, 2) = C(2, 2)+A(3)
      }
   Set C(1, 2) = C(1, 2)+minimum(C(2, 1), A(2)+C(2, 2)/R)
}
Set minimum cost = minimum (C(1, 1), A(1)+C(1, 2)
end
```

The above procedure can be modified to cover the following cases/provide additional information as follows: (i) The case when there are more than three periods can be solved by adding more inner loops to DRA (though for practical implementation $M = 5$ is itself time consuming when $R = 1000$, because it takes up to 3 hours of time when $N = 6$). (ii) When there is a fixed cost of rebalancing, it might not be optimal to balance at all, thus the last period decision in that case is changed to additionally consider whether or not to rebalance. (iii) The algorithm can be used to accumulate the frequency with which the rebalancing decision is optimal in each of the periods. (iv) If an efficient transportation cost routine can be embedded within the algorithm (for example, the case discussed in Lemma 3.2) then the problem can be generalized to incorporate transportation cost. (v) If the stores face similar demand then the more general problem wherein the vendor holds back inventory and balances all inventories with an initial allocation (or at least minimizes in the majorization order all inventories if transshipment across stores is not allowed at the beginning of the period) could also be solved by searching for the optimal quantity of inventory to hold back for later rebalancing.

## 4. Numerical results

In Section 3 we established several structural properties of the optimal solution to the dynamic rebalancing problem. In this section we discuss the results of several numerical experiments that provide insights into the following issues:

1. What is the extent of benefit due to optimal dynamic rebalancing when compared to an optimal but static rebalancing policy? What are the factors that determine the magnitude of savings that can be obtained by switching to an optimal dynamic policy from a static policy?
2. Whether and under what conditions *dynamic* policies that are based on heuristic analysis perform nearly as well as the optimal policies?
3. Whether the pattern of decisions of dynamic and static policies, namely, the frequency with which rebalancing is done over the different sub-periods provide guidelines as to when to use dynamic rebalancing, for example, should the rebalancing algorithm be used for the entire horizon or can it be applied to only certain sub-periods?

### *4.1. The design of experiments*

The number of retailers, $M$, is varied from 3 to 9 in the experiments. The number of inventory reviews or sub-periods, $N$, is fixed at 5. It is a high enough number to study the impact of dynamic re-balancing in most practical cases.

Three different discrete demand distributions are used to simulate customer demand. These were chosen so that we can study the impact of the shape of the demand distribution and the variability in demand upon relative cost saving, frequency of shipments, et cetera. The mean demand at each retailer for a sub-period is equal to two in all three distributions. The demand distributions are summarized in Table 1. The retailer demand can take values 0, 1, 2, 3, or 4 (column 1) in a sub-period. The cumulative probability of observing these values in a sub-period for three different demand distributions are shown in columns 2–4. The three distributions correspond to increasing variance in demand.

We compare alternate policies under different starting inventory positions as described below.

*Alternate policies*: The DRA is cumbersome to use when either the number of sub-periods or the number of retailers is large. Therefore, we propose the use of an intuitively appealing dynamic but myopic heuristic in which (in its simple form) the re-balancing is done in sub-period $\tau$ if the expected cost of re-balancing in sub-period $\tau$ is lower than the expected cost of re-balancing in sub-period $\tau + 1$. Obviously the heuristic is not optimal unless there are exactly two sub-periods. Numerical results indicate that the main reason for the sub-optimal performance of the heuristic is its tendency to ship early. This phenomenon can be explained as follows: The expected cost of re-balancing in sub-period $\tau + 1$ is greater than the true expected cost of postponing the re-balancing decision. This makes the decision to rebalance immediately a more attractive proposition than it is in reality. However, this systematic bias can be partially corrected by adding a damping factor that inflates sub-period $\tau$'s cost compared to sub-period $(\tau + 1)$'s cost when comparing the two costs. Therefore, the heuristic is called the dampened myopic heuristic (DMDH).

In addition to the DMDH heuristic, we also compare the performance of the optimal static policy with that of DRA. The optimal static policy is determined by first computing the expected lost sales when the rebalancing is done in each of the sub-periods. The optimal static policy is defined to be the decision to rebalance in that sub-period which gives the lowest expected lost sales.

Table 1
Demand distributions used in numerical experiments

| Retailer demand in a sub-period ($D_{ij}$) | Cumulative probability | | |
|---|---|---|---|
| | Unimodal (low variance) | Uniform (medium variance) | Bimodal (high variance) |
| 0 | 0.1 | 0.2 | 0.5 |
| 1 | 0.3 | 0.4 | 0.5 |
| 2 | 0.7 | 0.6 | 0.5 |
| 3 | 0.9 | 0.8 | 0.5 |
| 4 | 1.0 | 1.0 | 1.0 |

*Starting inventory vector*: Fig. 1 and Tables 2 and 3 all depict the results of the same set of experiments in which the static, the dynamic and the DMDH performance are compared for the three demand distributions for different levels of *imbalance* in the starting inventory vector. In this set of experiments, the number of retailers is six, and the average starting inventory of the six retailers is fixed at 12 units but the imbalance of the starting vector is increased progressively. The imbalance can be seen in the starting inventory vector shown in column 1 of Tables 2 and 3. This set of experiments will be denoted the "imbalance experiments."

In contrast, in the results summarized in Table 4 the average starting inventory is changed but the starting inventory vector is kept balanced, see column 1. Thus, the experiments summarized in Tables 2 and 3 are intended to isolate the impact of the imbalance of starting inventory, and the experiments shown in Table 4 are intended to isolate the effect of the total starting inventory. This experiment will be denoted the "total inventory experiments."

### 4.2. Performance of the optimal dynamic versus the optimal static policy

The computation of the best static policy becomes self-evident upon examining the expected cost if rebalancing is carried out in the first, second, third, fourth or the fifth sub-period, see columns 3–7 in Table 3. Also, see Table 2, column 2, in which the best static rebalancing sub-period is reported. The optimal static rebalancing sub-period should be compared with the frequency with which DRA rebalances in each of the sub-periods, see columns 3–7 of Table 2. The percentage increase in cost due to adopting a static policy vis-à-vis the optimal dynamic policy, DRA, is shown in column 2 of Table 3 (for the "imbalance experiments"). From the percentage increase in cost, we see that best static policy performs rather poorly in comparison to DRA. In some cases, for example in the second and third experiments for the unimodal demand distribution shown in Table 3, the use of best static policy results in additional lost sales of 32.98% and 29.50%. The main contributor to the poor performance is the discrepancy in the rebalancing frequency when compared to DRA: In the first of the above experiments, the best static policy is to rebalance in the

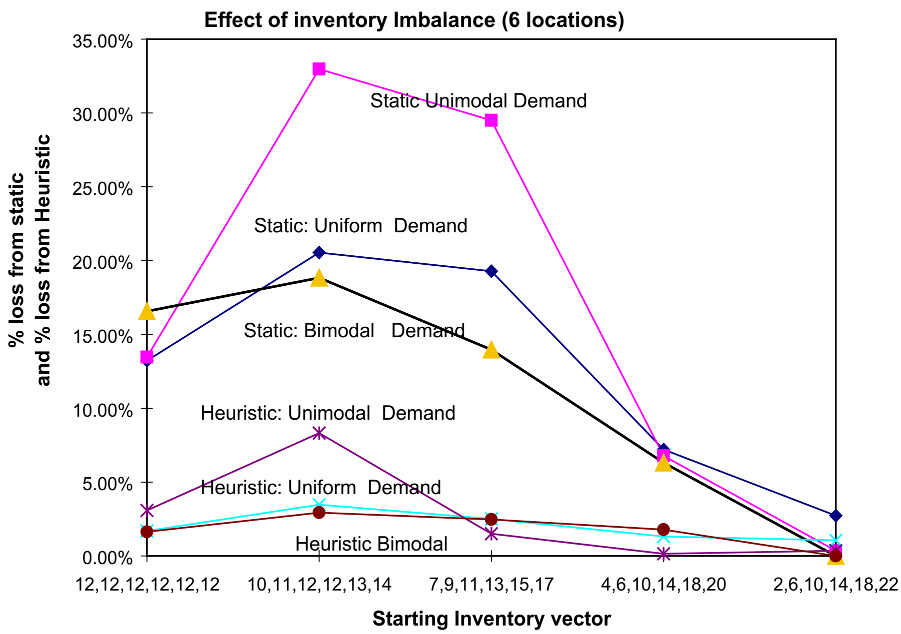


Fig. 1. Loss due to use of static policy and heuristic.

Table 2
Comparison of static, heuristic and dynamic rebalancing decisions

| Starting inventory | Optimal static period[a] | Optimal dynamic rebalancing (DRA) period frequency[b] | | | | | Heuristic (DMDH) rebalancing period frequency[c] | | | | | % loss from Heuristic[d] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Uniform demand distribution | | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 | 5 | |
| 12,12,12,12,12,12 | 5 | 0.00 | 0.00 | 0.01 | 0.25 | 0.74 | 0.00 | 0.00 | 0.01 | 0.25 | 0.735 | 1.69 |
| 10,11,12,12,13,14 | 5 | 0.00 | 0.00 | 0.03 | 0.36 | 0.61 | 0.00 | 0.03 | 0.06 | 0.35 | 0.557 | 3.47 |
| 7,9,11,13,15,17 | 4 | 0.00 | 0.03 | 0.35 | 0.49 | 0.13 | 0.00 | 0.10 | 0.35 | 0.45 | 0.102 | 2.50 |
| 4,6,10,14,18,20 | 2 | 0.00 | 0.50 | 0.47 | 0.03 | 0.00 | 0.00 | 0.60 | 0.38 | 0.02 | 2E-04 | 1.32 |
| 2,6,10,14,18,22 | 2 | 0.00 | 0.90 | 0.10 | 0.00 | 0.00 | 0.00 | 0.87 | 0.13 | 0.00 | 0.00 | 1.05 |
| *Unimodal demand distribution* | | | | | | | | | | | | |
| 12,12,12,12,12,12 | 5 | 0.00 | 0.00 | 0.01 | 0.19 | 0.79 | 0.00 | 0.00 | 0.05 | 0.18 | 0.767 | 3.09 |
| 10,11,12,12,13,14 | 5 | 0.00 | 0.00 | 0.02 | 0.30 | 0.68 | 0.00 | 0.10 | 0.04 | 0.26 | 0.6 | 8.33 |
| 7,9,11,13,15,17 | 4 | 0.00 | 0.07 | 0.29 | 0.53 | 0.12 | 0.00 | 0.07 | 0.24 | 0.58 | 0.112 | 1.50 |
| 4,6,10,14,18,20 | 2 | 0.00 | 0.70 | 0.29 | 0.01 | 0.00 | 0.00 | 0.70 | 0.29 | 0.01 | 7E-05 | 0.14 |
| 2,6,10,14,18,22 | 2 | 0.00 | 0.97 | 0.03 | 0.00 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.36 |
| *Bimodal demand distribution* | | | | | | | | | | | | |
| 12,12,12,12,12,12 | 5 | 0.00 | 0.00 | 0.04 | 0.42 | 0.53 | 0.00 | 0.00 | 0.04 | 0.43 | 0.526 | 1.64 |
| 10,11,12,12,13,14 | 4 | 0.00 | 0.00 | 0.09 | 0.44 | 0.46 | 0.00 | 0.03 | 0.13 | 0.43 | 0.411 | 2.94 |
| 7,9,11,13,15,17 | 3 | 0.00 | 0.03 | 0.42 | 0.39 | 0.15 | 0.00 | 0.03 | 0.33 | 0.48 | 0.159 | 2.47 |
| 4,6,10,14,18,20 | 1 | 0.00 | 0.47 | 0.42 | 0.10 | 0.01 | 0.00 | 0.43 | 0.47 | 0.09 | 0.012 | 1.78 |
| 2,6,10,14,18,22 | 1 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 1.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

[a] Optimal static period is the sub-period in which the static policy gives the lowest expected cost. See also Table 3.
[b] The DRA rebalances based on the inventory position. The frequency with which the DRA rebalances in each sub-period is recorded in these columns.
[c] The DMDH rebalances based on the inventory position. The frequency with which the DMDH rebalances in each sub-period is recorded in these columns.
[d] The percentage loss from use of the DMDH is computed by averaging (Cost due to DMDH/Cost of DRA-1) over the replications.

Table 3
Performance of the optimal static policy

| Starting inventory | % loss from static[a] | Static cost for given rebalancing period[b] | | | | |
|---|---|---|---|---|---|---|
| | | 1 | 2 | 3 | 4 | 5 |
| *Uniform demand distribution* | | | | | | |
| 12,12,12,12,12,12 | 13.23 | 3.78 | 2.54 | 1.95 | 1.35 | **1.11** |
| 10,11,12,12,13,14 | 20.54 | 3.78 | 2.59 | 2.03 | 1.45 | **1.40** |
| 7,9,11,13,15,17 | 19.29 | 3.78 | 2.38 | 1.79 | **1.73** | 2.87 |
| 4,6,10,14,18,20 | 7.20 | 3.78 | **2.95** | 3.11 | 4.70 | 7.76 |
| 2,6,10,14,18,22 | 2.74 | 3.78 | **2.89** | 3.73 | 5.58 | 8.90 |
| *Unimodal demand distribution* | | | | | | |
| 12,12,12,12,12,12 | 13.47 | 2.40 | 1.21 | 0.85 | 0.53 | **0.35** |
| 10,11,12,12,13,14 | **32.98** | 2.40 | 1.40 | 1.00 | 0.64 | **0.59** |
| 7,9,11,13,15,17 | **29.50** | 2.40 | 1.26 | 0.88 | **0.82** | 1.88 |
| 4,6,10,14,18,20 | 6.77 | 2.40 | **1.29** | 1.68 | 3.52 | 6.88 |
| 2,6,10,14,18,22 | 0.36 | 2.40 | **1.95** | 3.15 | 5.28 | 8.71 |
| *Bimodal demand distribution* | | | | | | |
| 12,12,12,12,12,12 | **16.57** | 5.28 | 5.83 | 4.36 | 3.52 | **3.28** |
| 10,11,12,12,13,14 | **18.84** | 5.28 | 5.49 | 4.22 | **3.57** | 3.58 |
| 7,9,11,13,15,17 | 13.97 | 5.28 | 5.31 | **4.53** | 4.54 | 5.37 |
| 4,6,10,14,18,20 | 6.31 | **5.28** | 5.79 | 5.39 | 6.71 | 8.69 |
| 2,6,10,14,18,22 | 0.00 | **5.28** | 6.70 | 6.80 | 8.40 | 10.72 |

[a] The percentage loss from use of the static policy is computed by averaging (Cost due to use of static policy/Cost of DRA-1) over the replications.

[b] The static policy rebalances based on the lowest cost of rebalancing in a fixed sub-period. The expected cost of rebalancing in each sub-period is recorded in these columns.

fifth sub-period when it is optimal to do so only 68% of the time under DRA. In the second of the examples, the best static policy is to ship 100% in the fourth sub-period compared to the optimal frequency of just 53%. Thus, factors that make DRA spread out the frequency of rebalancing in different sub-periods explain most of the poor performance of the static policy.

*(a) Imbalance of the starting vector of inventories*: From Fig. 1, the gain due to dynamic rebalancing is the largest when the starting inventory vector is (nearly) balanced. We know from the proof of Lemma 3.4 that rebalancing is more valuable when the inventory vector is more unbalanced. In contrast, the value of dynamic rebalancing vis-à-vis adopting a static policy should be intuitively greater when the initial inventories are balanced—when the inventories are highly unbalanced it is better to balance them immediately.

Even in the balanced inventory situations there are some situations (see the initial few results for the static policy shown in Fig. 1) in which the static policy does not do as poorly when compared to others. The reason is as follows: When the starting inventory is balanced and not too large, the dynamic policy waits until the last period a greater number of times before rebalancing. Therefore, if the lowest cost of rebalancing under a static policy occurs in the last period then the loss in inefficiency is not as significant, see Table 2.

*(b) Total starting inventory*: In Table 4, the results from the "total inventory experiments" are reported. In these experiments the starting inventory is balanced but the total inventory changed, see column 1. The sub-optimality of the best static policy is shown in column 2 and the expected cost for different rebalancing sub-periods in columns 4–8. We see that when the initial inventory is high and balanced, the dynamic optimal policy always does significantly better compared to the best static policy, see Table 4. The relative value of dynamic rebalancing increases with increase in the total system inventory (i.e., in systems with high

Table 4
Effect of average starting inventory (balanced case)

| Average[a] starting inventory | % loss static[b] | % loss DMDH[c] | Static cost[d] | | | | | Dynamic shipping period frequency[e] | | | | | Heuristic (DMDH) shipping period frequency[f] | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 | 5 | 1 | 2 | 3 | 4 | 5 |
| *Bimodal* | | | | | | | | | | | | | | | | | |
| 13.5 | 30.71 | 5.14 | 4.5 | 3.08 | 2.12 | 1.43 | **1.37** | 0 | 0 | 0.05 | 0.57 | 0.38 | 0 | 0.1 | 0.04 | 0.50 | 0.36 |
| 13 | 22.61 | 7.18 | 4.08 | 3.52 | 2.64 | 1.90 | **1.75** | 0 | 0 | 0.03 | 0.49 | 0.48 | 0 | 0.1 | 0.03 | 0.43 | 0.44 |
| 12.5 | 20.83 | 3.19 | 5.88 | 4.42 | 3.19 | 2.45 | **2.27** | 0 | 0 | 0.02 | 0.42 | 0.56 | 0 | 0.03 | 0.03 | 0.40 | 0.54 |
| 12 | 16.57 | 1.64 | 5.28 | 5.83 | 4.36 | 3.52 | **3.28** | 0 | 0 | 0.04 | 0.42 | 0.53 | 0 | 0.00 | 0.04 | 0.43 | 0.53 |
| 11.5 | 17.19 | 1.80 | 7.86 | 5.94 | 4.41 | 3.79 | **3.74** | 0 | 0 | 0.10 | 0.4 | 0.5 | 0 | 0.00 | 0.07 | 0.42 | 0.51 |
| 11 | 9.45 | 1.66 | 8.52 | 7.19 | 6.20 | **5.39** | 5.69 | 0 | 0 | 0.12 | 0.46 | 0.41 | 0 | 0.00 | 0.04 | 0.53 | 0.43 |
| 10.5 | 5.75 | 1.90 | 10.9 | 9.05 | 8.33 | **7.47** | 8.30 | 0 | 0.03 | 0.17 | 0.53 | 0.26 | 0 | 0.00 | 0.04 | 0.68 | 0.28 |
| 10 | 6.13 | 1.69 | 11.8 | 9.64 | 9.24 | **8.45** | 9.37 | 0 | 0.1 | 0.19 | 0.49 | 0.22 | 0 | 0.17 | 0.07 | 0.55 | 0.21 |
| *Uniform* | | | | | | | | | | | | | | | | | |
| 13 | 20.14 | **10.63** | 2.28 | 1.34 | 0.92 | 0.52 | **0.35** | 0 | 0 | 0.01 | 0.33 | 0.66 | 0 | **0.07** | **0.02** | 0.31 | 0.61 |
| 12 | 13.23 | 1.69 | 3.78 | 2.54 | 1.95 | 1.35 | **1.11** | 0 | 0 | 0.01 | 0.25 | 0.74 | 0 | 0.00 | 0.01 | 0.25 | 0.74 |
| 11 | 12.06 | 4.03 | 5.88 | 4.54 | 3.91 | **3.18** | 3.21 | 0 | 0 | 0.05 | 0.41 | 0.54 | 0 | 0.10 | 0.08 | 0.35 | 0.47 |
| 10.5 | 7.60 | 5.15 | 8.76 | 6.21 | 5.72 | **5.00** | 5.33 | 0 | 0.07 | 0.09 | 0.47 | 0.37 | 0 | 0.30 | 0.10 | 0.36 | 0.24 |
| 10 | 5.89 | 1.64 | 8.76 | 7.27 | 6.50 | **5.83** | 6.28 | 0 | 0.03 | 0.13 | 0.52 | 0.32 | 0 | 0.00 | 0.04 | 0.63 | 0.33 |
| 9 | 5.39 | 3.28 | 12.5 | 10.1 | 9.34 | **8.99** | 9.62 | 0 | 0 | 0.25 | 0.56 | 0.19 | 0 | 0.30 | 0.22 | 0.38 | 0.10 |

[a] Denotes average inventory of a balanced starting inventory vector. For e.g., the starting inventory vector for average inventory of 12 is (12,12,12,12,12,12) and the starting inventory vector for average inventory of 10.5 is (10,10,10,11,11,11).
[b] The percentage loss from use of the static policy is computed by averaging (Cost due to use of static policy/Cost of DRA-1) over the replications.
[c] The percentage loss from use of the DMDH is computed by averaging (Cost due to DMDH/Cost of DRA-1) over the replications.
[d] The expected cost of rebalancing using the static policy in each sub-period is recorded in these columns.
[e] The frequency with which the DRA rebalances in each sub-period is recorded in these columns.
[f] The frequency with which the DMDH rebalances in each sub-period is recorded in these columns.

service levels), see Table 4. This is because when there are low levels of inventory both the optimal static policy as well as the optimal dynamic policy will suggest rebalancing early.

*(c) Number of locations*: We also ran experiments in which the number of retailers ranged from 6 to 50 but the initial inventory is kept the same (12 units) at all retailers. The results from these experiments are shown in Table 5. The benefit due to rebalancing increases with the number of retailers. Without rebalancing the expected lost sales per store will remain constant. With rebalancing the expected lost sales per store drops with increase in the number of retailer inventories rebalanced, see last column of Table 5.

However, the value of dynamic rebalancing vis-a-vis static rebalancing decreases with the number of retailers. This is because the inventory vector gets distorted more quickly when there are more retailers. It is also probably quite difficult to rebalance so many retailers together. Therefore, we conjecture that sub-sets of retailer inventories should be rebalanced—not all retailers at once. Determining the optimal partitioning of retailers could be an interesting exercise because it depends on trading-off the loss from increased cost of transportation against the gain from rebalancing. In Table 5, we also report the run times for solving the DP. As can be seen, the dynamic program takes a long time to solve and this time is some what affected by the number of retailers. The main determinant of the run time are the number of replications $R$ and the number of sub-periods ($N$).

### 4.3. Performance of the DMDH heuristic

Numerical results show that when there are six retailers, the DMDH, with a damping factor equal to 1.11, achieves close to the optimal cost. Thus, in the DMDH, rebalancing is done in sub-period $\tau$ if the expected cost of re-balancing in sub-period $\tau$ is lower than 1.11 times the expected cost of re-balancing in sub-period $\tau + 1$. We note that based on Lemma 3.8 the damping factor should progressively reduce as $\tau$ increases, eventually reducing to one in the final sub-period.

The DMDH performs very well when compared to DRA and the lost sales are usually within a few percent of the optimal (see last column of Table 2 and column 4 of Table 4). To enable comparison with DRA, the shipping frequency for DMDH is reported in columns 8–12 of Table 2 and the last five columns of Table 4. The DMDH shipping frequency in different sub-periods differ very little from the optimal ones! The damping factor of 1.11 is used prevent early shipment in these experiments. Only in some situations when the heuristic suggests early shipments in sub-periods 2 or 3 there is a greater loss in efficiency (see for

Table 5
Effect of number of retailers

| Demand | Number of retailers | % loss static[a] | % loss DMDH[b] | Run time (cpu min.) | Damping factor | Optimal E (Lost sales)[c] | Optimal E (Lost sales) per store[c] |
|---|---|---|---|---|---|---|---|
| Bimodal | 6 | 16.57 | 1.64 | 248 | 1.11 | 2.81 | 0.4688 |
| | 20 | 3.83 | 1.86 | 310 | 1.22 | 6.94 | 0.3468 |
| | 35 | 0.89 | 0.68 | 367 | 1.39 | 9.85 | 0.2815 |
| | 50 | 0.40 | 0.34 | 397 | 1.54 | 14.22 | 0.2845 |
| Unimodal | 6 | 13.23 | 3.09 | 298 | 1.11 | 0.31 | 0.0515 |
| | 20 | 4.79 | 1.04 | 355 | 1.39 | 0.84 | 0.0420 |
| | 35 | 2.78 | 1.77 | 373 | 1.67 | 1.39 | 0.0397 |
| | 50 | 3.63 | 0.62 | 413 | 1.67 | 1.71 | 0.0342 |

[a] The percentage loss from use of the static policy is computed by averaging (Cost due to use of static policy/Cost of DRA-1) over the replications.
[b] The percentage loss from use of the DMDH is computed by averaging (Cost due to DMDH/Cost of DRA-1) over the replications.
[c] The expected lost sales when DRA is used.

example Table 4, the first run for uniform demand shown in boldface). Possibly a higher damping factor should be used for earlier sub-periods to counter this. The performance of the heuristic improves when there is lower starting inventory in the system (Table 4). The heuristic is computationally much more efficient when compared to DRA, especially when there are many sub-periods.

As the number of retailers increases the damping factor needs to be reduced. In Table 5, we report the optimal damping factor of the heuristic for different number of retailers. We may infer from these results that the DMDH performs quite well if the damping factor is chosen correctly. Contrariwise, when there are more retailers, myopic rebalancing, that is using the wrong choice of the damping factor, can lead to sub-optimal results and costs that exceed that of the static policy.

### 4.4. Shipping pattern of the optimal dynamic policies

It is useful to study Tables 2–4 and Fig. 2 to see if the frequency of re-balancing follows some trend. This trend if it can be predicted will allow a decision maker to plan in advance for transportation and the warehousing requirements for carrying out the rebalancing. From Table 2 (average inventory of 12 units per retailer) it is seen that if the starting vector is perfectly balanced then the majority of rebalancing shipments are made in period 5 followed by period 4. When the starting inventory vector is unbalanced the shipments are made earlier.

As seen in Table 4 and Fig. 2, at high levels of starting inventory re-balancing is done before the last sub-period because lost sales can be avoided much more easily due to the availability of ample cushion for *all* locations. When the initial inventory is low then too rebalancing is done before the last sub-period because this way left over inventory at any retailer can be avoided. Our results also suggest that a *combination* of DRA and DMDH will yield excellent results in systems with initial balanced inventories. We suggest the use the DMDH until the last 2–3 sub-periods, then to use DRA, if necessary, in the last few sub-periods.

Interestingly, in Fig. 2 the lower curve is for the unimodal demand distribution and the upper curve is for uniform demand, i.e., rebalancing is more frequently done in the last sub-period when the demand is less variable. This phenomenon can be also seen in Tables 2–4 where higher demand variance leads to rebalancing in earlier periods. This is because if the demand variance is high then significant imbalance in retailers' inventory vector develops earlier, and therefore increases the benefit due to early rebalancing. In fact, adopting the dynamic policy saves more if the demand is uniformly distributed.

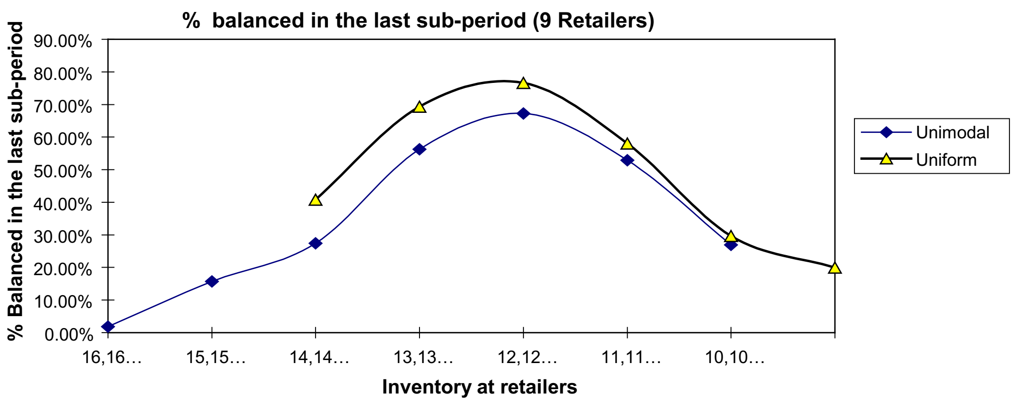


Fig. 2. Total starting inventory and frequency of rebalancing in the last sub-period.

## 5. Discussion

We have provided a thorough analysis of the dynamic rebalancing problem and established several structural properties of the optimal timing as well as rebalancing allocation. In particular, we have demonstrated that systems rebalanced using different policies can be compared analytically using suitable constructions to ascertain the impact of increasing the number of opportunities to rebalance as well as the impact of less-versus-more balanced inventory positions on the optimal time to rebalance. A key finding based on our analysis is that as the number of opportunities to rebalance increases, rebalancing tends to take place later during the period. This clearly demonstrates the value of information in supply chains.

Using numerical experiments, we have shown that dynamic rebalancing of inventories using up-to-date information on retailer inventories hold the potential to reduce lost sales quite significantly. Our results suggest that dynamic policies can lead to substantial benefits over the static policy especially in systems in which starting inventories at the retailers are balanced or when high service levels are required. The benefits are more when the initial inventories are balanced, as usually is the case when an initial allocation at the beginning of a period is followed up with a rebalancing allocation during the period. The main benefit of dynamic balancing comes from the fact that if the imbalance has not set in, then the rebalancing can be postponed. Thus, dynamic rebalancing is probably less beneficial when demand is very volatile because inventories get out-of-balance quite quickly.

We also proposed a heuristic, DMDH, that performed close to optimal in most experiments. This heuristic is myopic in nature because it compares rebalancing now versus rebalancing next period. However, we also know that rebalancing tends to occur (in systems with initially balanced inventories) in the last few sub-periods. Therefore, the myopic nature of the heuristic can be corrected by dampening the tendency to rebalance—rebalance now only if doing so yields an expected cost that is less than a factor $f$ (greater than one) times the expected cost of waiting to rebalance in the next sub-period. The factor $f$ depends not only on the number of retailers but also upon the volatility of demand. The analysis of rebalancing frequency in different sub-periods shows that the use of a *combination* of the optimal policy and DMDH will yield excellent results in systems with initial balanced inventories. We suggest that DMDH be used until the last 2–3 sub-periods, then if inventories have not yet been rebalanced, to switch to the optimal policy for the last few sub-periods.

The value of rebalancing increases with the number of retailers. However, the value of dynamic rebalancing compared to static rebalancing reduces with the number of retailers. We conjecture that sub-sets of retailer inventories should be rebalanced—not all retailers at once. The problem of determining the optimal partitioning of retailers to rebalance at a time is an open problem. It entails balancing the cost of carrying out the rebalance versus the savings from the reduction in lost sales.

## Acknowledgements

The research of Xiuli Chao is partially supported by NSF under DMI-0196084 and DMI-0200306. The research of Sridhar Seshadri is partially supported by NSF DMI-0200406. We thank the referees for their detailed comments that improved the presentation of the paper.

## Appendix

**Proof of Lemma 3.5.** Let $\tau_\alpha$ be the optimal *random* rebalance time if rebalancing is *not* done at time $\tau$ and when the current inventory vector is $\boldsymbol{y}$. The assumption states that

$$E\left(\left(\sum_{j\geqslant\tau}D_{1j}-\sum_i y_i/M\right)^+\right)$$
$$\leqslant E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\alpha-1}D_{ij}-y_i\right)^+\right)\Big/M+E\left(\left(\sum_{j\geqslant\tau_\alpha}D_{1j}-\sum_i\left(y_i-\sum_{j=\tau}^{\tau_\alpha-1}D_{ij}\right)^+\Big/M\right)^+\right). \quad (A.1)$$

Due to our assumption, that $\boldsymbol{x}$ majorizes $\boldsymbol{y}$ (see (1)),

$$E\left[\left(\sum_{j\geqslant\tau}D_{1j}-\sum_i y_i/M\right)^+\right]=E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\beta-1}D_{ij}-x_i\right)^+\right)\Big/M. \quad (A.2)$$

Let $\tau_\beta>\tau$ be the optimal time to rebalance when the inventory vector is $\boldsymbol{x}$. As $\tau_\beta$ is not optimal for $\boldsymbol{y}$,

$$E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\alpha-1}D_{ij}-y_i\right)^+\right)\Big/M+E\left(\left(\sum_{j\geqslant\tau_\alpha}D_{1j}-\sum_i\left(y_i-\sum_{j=\tau}^{\tau_\alpha-1}D_{ij}\right)^+\Big/M\right)^+\right)$$
$$\leqslant E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\beta-1}D_{ij}-y_i\right)^+\right)\Big/M+E\left(\left(\sum_{j\geqslant\tau_\beta}D_{1j}-\sum_i\left(y_i-\sum_{j=\tau}^{\tau_\beta-1}D_{ij}\right)^+\Big/M\right)^+\right). \quad (A.3)$$

Thus, all we need to show given (A.1)–(A.3) is that

$$E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\beta-1}D_{ij}-y_i\right)^+\right)\Big/M+E\left(\left(\sum_{j\geqslant\tau_\beta}D_{1j}-\sum_i\left(y_i-\sum_{j=\tau}^{\tau_\beta-1}D_{ij}\right)^+\Big/M\right)^+\right)$$
$$\leqslant E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\beta-1}D_{ij}-x_i\right)^+\right)\Big/M+E\left(\left(\sum_{j\geqslant\tau_\beta}D_{1j}-\sum_i\left(x_i-\sum_{j=\tau}^{\tau_\beta-1}D_{ij}\right)^+\Big/M\right)^+\right). \quad (A.4)$$

However, if we permute the demands over all retailers, we obtain

$$f(\mathbf{y})=E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\beta-1}D_{ij}-y_i\right)^+\right)\Big/M+E\left(\left(\sum_{j\geqslant\tau_\beta}D_{1j}-\sum_i\left(y_i-\sum_{j=\tau}^{\tau_\beta-1}D_{ij}\right)^+\Big/M\right)^+\right)$$
$$\equiv\left(\sum_\pi E\left(\sum_i\left(\sum_{j=\tau}^{\tau_\beta-1}D_{i_\pi j}-y_i\right)^+\right)\Big/M\right.$$
$$\left.+\sum_\pi E\left(\left(\sum_{j\geqslant\tau_\beta}D_{i_{i_\pi}j}-\sum_i\left(y_i-\sum_{j=\tau}^{\tau_\beta-1}D_{i_{i_\pi}j}\right)^+\Big/M\right)^+\right)\right)\Big/M!, \quad (A.5)$$

where $D_{i_\pi j}$ refers to the $j$th sub-period demand of retailer $i_\pi$ in the $\pi$th permutation. Thus, we see due to the symmetry that the function $f(\boldsymbol{y})$ defined in (A.5) is a Schur convex function of $\boldsymbol{y}$. The desired result (9) follows because $\boldsymbol{x}$ majorizes $\boldsymbol{y}$. □

**Proof of Lemma 3.8.** Let $\tau_\alpha>\tau$ be the optimal *random* rebalance time when the current inventory vector is $\boldsymbol{y}$. The assumption of the lemma is that

$$E\left(\left(\sum_{j=\tau}^{N} D_{1j} - \sum_{i} y_i/M\right)^{+}\right)$$
$$> E\left(\sum_{i}\left(\sum_{j=\tau}^{\tau_\alpha-1} D_{ij} - y_i\right)^{+}\right)\Big/M + E\left(\left(\sum_{j=\tau_\alpha}^{N} D_{1j} - \sum_{i}\left(y_i - \sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)^{+}\Big/M\right)^{+}\right). \tag{A.6}$$

It is convenient to rewrite the left hand side of (A.6) as

$$E\left(\left(\sum_{j=\tau}^{N} D_{1j} - \sum_{i} y_i/M\right)^{+}\right)$$
$$= E\left(\sum_{j=\tau}^{\tau_\alpha-1} D_{1j} - \sum_{i} y_i/M\right)^{+} + E\left(\left(\sum_{j=\tau_\alpha}^{N} D_{1j} - \left(\sum_{i} y_i/M - \sum_{j=\tau}^{\tau_\alpha-1} D_{1j}\right)^{+}\right)^{+}\right). \tag{A.7}$$

Consider the second expression on the right hand side of (A.7). By interchanging the summation and the $[.]^+$, and because $\sum_i x^+ \geqslant \left(\sum_i x\right)^+$ we obtain

$$\left(\sum_{j=\tau_\alpha}^{N} D_{1j} - \sum_{i}\left(y_i - \sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)^{+}\Big/M\right)$$
$$\leqslant \left(\sum_{j=\tau_\alpha}^{N} D_{1j} - \left(\sum_{i}\left(y_i - \sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)\Big/M\right)^{+}\right)$$
$$= \left(\sum_{j=\tau_\alpha}^{N} D_{1j} - \left(\sum_{i} y_i/M - \sum_{i}\left(\sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)\Big/M\right)^{+}\right). \tag{A.8}$$

In words, (A.8) reveals that the lost sales are smaller *after* the rebalance sub-period until the end of the period, (namely from the beginning of sub-period $\tau_\alpha$ until the end of sub-period $N$) when compared to the lost sales in the immediately rebalanced system even if inventory and demand were pooled until time $\tau_\alpha$!

The inequality in (A.8) immediately reveals that:

$$\left(D_{1(N+1)} - \left(\sum_{i}\left(y_i - \sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)^{+}\Big/M - \sum_{j=\tau_\alpha}^{N} D_{1j}\right)^{+}\right)^{+}$$
$$\leqslant \left(D_{1(N+1)} - \left(\left(\sum_{i} y_i/M - \sum_{i}\left(\sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)\Big/M\right)^{+} - \sum_{j=\tau_\alpha}^{N} D_{1j}\right)^{+}\right)^{+}. \tag{A.9}$$

We now wish to show that the right hand side of (A.9) is smaller than a similar quantity for the immediately rebalanced system.

From Theorems 2.A.12 and 2.A.6 in Shaked and Shanthikumar (1994),

$$-\sum_{i}\left(\sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)\Big/M \leqslant_{cx} -\left(\sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right). \tag{A.10}$$

Thus, from Theorem 2.A.7 in Shaked and Shanthikumar and (A.10),

$$\sum_{i} y_i/M - \sum_{i}\left(\sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)\Big/M - \sum_{j=\tau_\alpha}^{N} D_{1j} \leqslant_{cx} \sum_{i} y_i/M - \left(\sum_{j=\tau}^{\tau_\alpha-1} D_{1j}\right) - \sum_{j=\tau_\alpha}^{N} D_{1j}. \tag{A.11}$$

Let

$$X =_{\text{st}} \sum_i y_i/M - \sum_i \left(\sum_{j=\tau}^{\tau_\alpha - 1} D_{ij}\right) \Big/ M - \sum_{j=\tau_\alpha}^{N} D_{1j}, \tag{A.12}$$

$$Y =_{\text{st}} \sum_i y_i/M - \left(\sum_{j=\tau}^{\tau_\alpha - 1} D_{1j}\right) - \sum_{j=\tau_\alpha}^{N} D_{1j}, \tag{A.13}$$

where the notation $=_{\text{st}}$ stands for "stochastically equal to." From Theorem 2.A.3 in Shaked and Shanthikumar, there exist random variables defined on the same probability space such that

$$\begin{aligned} &\widehat{X} =_{\text{st}} X, \\ &\widehat{Y} =_{\text{st}} Y, \\ &E[\widehat{Y}|\widehat{X}] = \widehat{X} \text{ a.s..} \end{aligned} \tag{A.14}$$

(The qualification "a.s." stands for almost surely and is used to indicate that the said relation holds on all except possibly a set that has probability equal to zero.)

We need an important property of the constructed random variables for completing the proof. Notice that in general it is not true that the signs of $E[\widehat{Y}|\widehat{X}]$ and $\widehat{X}$ are the same. In this special construction it turns out that is true almost surely. To see this we give a proof along the lines of the positivity result for conditional expectation given in Section 9.6 (page 87) of Williams (1992). Consider the set $A_n$ on which $\left\{\widehat{Y} \leqslant -\frac{1}{n}; \widehat{X} \geqslant 0\right\}$. Let $P(A)$ denote the probabilityof the event $A$. Then

$$\int_{A_n} \widehat{Y}\,\mathrm{d}P(\omega) \leqslant -\frac{1}{n}P(A_n) \text{ and } \int_{A_n} \widehat{Y}\,\mathrm{d}P(\omega) = \int_{A_n} E[\widehat{Y}|\widehat{X}]\,\mathrm{d}P(\omega) = \int_{A_n} \widehat{X}\,\mathrm{d}P(\omega) \geqslant 0.$$

This implies that $P(A_n) = 0$. Similarly, it can be established that $\left\{\widehat{Y} \geqslant \frac{1}{n}; \widehat{X} \leqslant 0\right\}$ has probability zero. Thus, we conclude that the events $\{\widehat{Y} \geqslant 0\}$ and $\{\widehat{X} \geqslant 0\}$ differ at most by a set of measure zero. We also obtain by using Jensen's inequality (see property 1.1d, Chapter 4, Durrett (1991)), that (A.14) implies that given a constant $d$

$$E[(d - \widehat{Y})^+|\widehat{X}] \geqslant (d - E[\widehat{Y}|\widehat{X}])^+ = (d - \widehat{X})^+ \text{ a.s} \tag{A.15}$$

where we have used the fact that $(d - x)^+$ is a convex function of $x$.

Finally using this inequality and the property that the events $\{\widehat{Y} \geqslant 0\}$ and $\{\widehat{X} \geqslant 0\}$ differ at most by a set of measure zero, and noting that $D_{1(N+1)}$ is independent of both $\widehat{X}$ and $\widehat{Y}$ we obtain

$$\begin{aligned} \int_{(d,x)\in R^+\times R^+} (d-x)^+\,\mathrm{d}P(d,x) &= \int_{d\in R^+} \int_{x\in R^+} (d-x)^+\,\mathrm{d}P(x)\,\mathrm{d}P(d) \\ &\leqslant \int_{(d,x)\in R^+\times R^+} E[(d-\widehat{Y})^+|\widehat{X} = x]\,\mathrm{d}P(x)\,\mathrm{d}P(d) \\ &= \int_{(d,y)\in R^+\times R^+} (d-y)^+\,\mathrm{d}P(d,y). \end{aligned} \tag{A.16}$$

Similarly,

$$\int_{(d,x)\in R^+\times R^-} (d)\,\mathrm{d}P(d,x) = \int_{(d,y)\in R^+\times R^-} (d)\,\mathrm{d}P(d,y). \tag{A.17}$$

In other words we have produced a construction in which the system that is rebalanced at time $\tau_\alpha$ loses less expected sales *in* the $(N+1)$st sub-period compared to the system that is rebalanced immediately at time $\tau$.

This shows that the inequality in (A.6) is preserved when extended to one more sub-period, i.e., to the $(N+1)$st one. Formally, combining (A.6)–(A.17)) we obtain

$$E\left(\left(\sum_{j=\tau_\alpha}^{N+1} D_{1j} - \sum_i y_i/M\right)^+\right)$$
$$> E\left(\sum_i \left(\sum_{j=\tau}^{\tau_\alpha - 1} D_{ij} - y_i\right)^+\right)\Big/ M + E\left(\left(\sum_{j=\tau_\alpha}^{N+1} D_{1j} - \sum_i \left(y_i - \sum_{j=\tau}^{\tau_\alpha-1} D_{ij}\right)^+ \Big/ M\right)^+\right).$$

The last inequality implies that it is optimal to postpone rebalancing when there is one extra sub-period to go (because $\tau_\alpha$ need not be the optimal rebalancing time when there is one extra sub-period to go). Actually, we have proved something stronger, the immediately rebalanced system's expected lost sales in each period after the rebalance is greater than that in the system that is rebalanced later. □

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