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Queueing Systems; Oct 2001; 39, 2-3; ABI/INFORM Global
pg. 137

Queueing Systems 39, 137–155, 2001
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Optimal Configurations of General Job Shops

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Received 22 December 1999; Revised 6 April 2001

Abstract. In this paper we study the problem of minimizing the expected number of jobs in a single class general open queueing network model of a job shop. This problem was originally posed by Buzacott and Shanthikumar [2] and solved by them for a special case. We extend their work in this paper. We derive feasibility conditions that simplify the analysis of the problem. We show that the optimal configuration can be completely characterized when both the utilizations of the machine centers are high and there are a large number of servers at each machine center. We also derive conditions under which the optimization problem reduces to solving a concave or a convex program and provide conditions under which the uniform flow line and the symmetric job shop (or variants of these) are optimal configurations for the job shop.

Keywords: open queueing networks, job shop, renewal process, concave/convex optimization

1. Introduction

In this paper we study the problem of minimizing the expected number of jobs in a single class general open queueing network model of a job shop. We are given that the arrival process of jobs to the job shop is of the renewal type with mean inter-arrival time equal to $1/\lambda$ and squared coefficient of variation (scv) of the inter-arrival time equal to C_a^2. The service times at each machine center as well as the inter-arrival times to each machine center can not be approximated by exponential distributions. The jobs arriving to a machine center are processed according to the first come first served (FCFS) service protocol. There are m machine centers where machine center i has c_i identical machines or parallel servers. The service times at machine center i, S_i, are i.i.d., with mean service time equal to $1/\mu_i$ and scv equal to $C_{S_i}^2$. The utilization of each machine center is less than one. The only other parameter that is specified in this problem is that the expected number of visits that a typical job makes to machine center i is equal to v_i. The problem is to determine

(i) the fraction of jobs arriving to the network that first visit machine center i (denoted as γ_i, where $\sum_{i=1}^{m} \gamma_i = 1$), and

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(ii) the probabilities that a job upon completion of service at machine center i visits machine center $j \neq i$ (denoted as p_{ij}) such that the expected number of jobs in the job shop is minimized.

By Little's law, minimizing the expected number of jobs in the job shop is equivalent to minimizing the expected flow time of an arbitrary job in the job shop (see, e.g., [1, p. 50]). The job shop is open, and therefore every arriving job eventually leaves the shop. The p_{ij}'s are called the switching probabilities.

This problem was originally posed by Buzacott and Shanthikumar [2] and solved by them for a special case. We extend their work in this paper. As is the case for analyzing general open queueing networks, *approximations* are used to obtain tractable analytical formulae for the expected number of jobs in the job shop. For this problem, even after the approximations are made, the structure of the optimal solution as a function of the problem parameters is hard to discern.

This problem is important for several reasons. First, there is considerable flexibility in selecting the initial routing as well as the switching probabilities in service operations. Thus the techniques presented in this paper can be immediately applied in this rapidly growing sector of the economy. Second, given any job shop, a solution to this problem provides a benchmark performance measure for the expected number of jobs in the shop. Such a benchmark measure can be used, for example, to identify areas that have the greatest potential of improvement. Third, it is of interest to know the conditions under which (some variant of) the uniform flow job shop (i.e., $\gamma_1 = 1$, $p_{i,i+1} = 1$ for $i = 1, \ldots, m-1$, and all other $p_{ij} = 0$) or (some variant of) the symmetric job shop (i.e., $p_{ij} = 1/m$, $i \neq j$, $i, j = 1, \ldots, m$ and $\gamma_i = 1/m$, for $i = 1, \ldots, m$) is optimal. We derive these conditions in section 4. Fourth, as emphasized by Buzacott and Shanthikumar [2, p. 136] once structural properties of this problem are known, it becomes relatively simple to develop solution procedures for specific instances. For example, if it is known that the uniform flow shop configuration is optimal, then the search for an optimal solution can be restricted to permutations of the indices $\{1, 2, \ldots, m\}$.

In section 2, the problem is formulated as a constrained nonlinear optimization problem. In section 3, the feasibility of the problem is discussed. In section 4, we give the main results that establish the conditions under which the problem reduces to a convex or a concave optimization problem.

2. Problem formulation

We briefly describe the problem formulation in this section. We first define the traffic equations, then outline the approximation steps for determining the expected number of jobs at each machine center, and finally formulate the problem as a nonlinear program. The reader is referred to [1,2] for further details.

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2.1. *Traffic equations*

Let the arrival rate of jobs to machine center i be denoted as λ_i. We are given that the expected number of visits to machine center i is v_i. Therefore,

$$v_i = \frac{\lambda_i}{\lambda}. \tag{2.1}$$

Equating the expected number of visits into and out of machine center i, we get the traffic equations

$$v_i = \gamma_i + \sum_{j=1, j\neq i}^{m} v_j p_{ji}, \quad \text{for } i = 1, \dots, m, \tag{2.2}$$

where we assume $p_{ii} = 0$ for $i = 1, \dots, m$, i.e., self loops are not permitted. Let the utilization of machine center i be denoted as ρ_i. Then

$$\rho_i = \frac{\lambda_i}{c_i \mu_i}, \quad \text{for } i = 1, \dots, m. \tag{2.3}$$

2.2. *Approximation for the expected number of jobs*

Following Buzacott and Shanthikumar, the arrival process to machine center i is assumed to be of the renewal type. Let the interarrival times (IAT) to machine center i be denoted by τ_{ik} where τ_{ik} is the interarrival time between the $(k-1)$th and the kth job arrival times to machine center i. We denote the squared coefficient of variation of the IATs as $C^2_{\mathrm{a}_i}$, i.e., the ratio of the variance of the IAT, $\mathrm{Var}(\tau_{ik})$, to its mean square, $E(\tau_{ik})^2$. Let $C^2_{\mathrm{d}_i}$ be the scv of the departure process from machine center i. Then $C^2_{\mathrm{a}_i}$ is approximated by

$$C^2_{\mathrm{a}_i} = \frac{1}{\lambda_i} \sum_{j=1, j\neq i}^{m} \lambda_j p_{ji}\big[p_{ji} C^2_{\mathrm{d}_j} + (1 - p_{ji})\big] + \frac{\lambda \gamma_i}{\lambda_i}\big[\gamma_i C^2_{\mathrm{a}} + (1 - \gamma_i)\big]. \tag{2.4}$$

The following approximation for $C^2_{\mathrm{d}_i}$ is based on the second approximation for the mean flow time for a $GI/G/1$ model given by Buzacott and Shanthikumar (see [1, table 3.1, p. 75])

$$C^2_{\mathrm{d}_i} = 1 + \frac{\rho_i^2}{c_i}\big(C^2_{S_i} - 1\big) + \big(C^2_{\mathrm{a}_i} - 1\big)\left(\frac{(1-\rho_i^2)(2-\rho_i) + \rho_i C^2_{S_i}(1-\rho_i)^2}{2 - \rho_i + \rho_i C^2_{S_i}}\right). \tag{2.5}$$

The squared coefficients of variation of the arrival processes are computed by solving the system of linear equations (2.4) and (2.5). Let $E[W(\lambda, \mu)]_{M/M/c}$ be the expected waiting time of a job in an $M/M/c$ queue with arrival rate λ and service rate μ. Let the

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expected number of jobs at machine center i be $E[N_i]$ and the expected number of jobs in the entire shop be $E[N]$. Then we may approximate

$$E[N_i] \approx \left(\frac{E[W(\lambda_i, \mu_i)]_{M/M/c_i}}{E[W(\lambda_i, c_i\mu_i)]_{M/M/1}} \cdot \frac{\rho_i(1 + C_{S_i}^2)}{2 - \rho_i + \rho_i C_{S_i}^2} \cdot \frac{\rho_i(2 - \rho_i)C_{a_i}^2 + \rho_i^2 C_{S_i}^2}{2v_i(1 - \rho_i)} + c_i\rho_i \right), \tag{2.6}$$

where

$$E[N] = \sum_{i=1}^{m} E[N_i]. \tag{2.7}$$

As noted in the introduction, we are working only with an approximation of the system performance, viz. (2.6). To that extent our results should be interpreted to hold subject to the quality of the approximation. For a discussion on the quality of the approximation, see [1, tables 7.1–7.5, pp. 334–336]. In contrast, the feasibility condition we derive for the existence of a solution to the general job shop design problem, see section 3, does not depend upon an approximation of the system performance.

2.3. *Nonlinear optimization problem*

In this subsection we formulate the general job shop design problem as a nonlinear optimization problem. Based on (2.4), (2.6), and (2.7) the problem is

$$\mathsf{P}: \quad \min_{\mathbf{p},\gamma} E[N]$$

subject to

$$\begin{aligned} &\sum_{i=1, i\neq j}^{m} p_{ji} \leqslant 1, && j = 1, \ldots, m, \\ &\sum_{i=1}^{m} \gamma_i = 1, && \\ &\gamma_i + \sum_{j=1, j\neq i}^{m} v_j p_{ji} = v_i, && i = 1, \ldots, m, \\ &\gamma_i \geqslant 0, && i = 1, \ldots, m, \\ &p_{ij} \geqslant 0, && i, j = 1, \ldots, m, \text{ and } i \neq j. \end{aligned} \tag{2.8}$$

3. Feasibility of the general problem

In this section, we first give conditions under which problem P has a feasible solution given a set of probabilities, $\{\gamma_i, i = 1, \ldots, m\}$, that satisfy (3.1) below. We then provide conditions under which the general job shop problem is feasible. This analysis is used to establish optimality for the different cases studied in section 4. Problem P is infeasible

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if the sum of the expected number of visits to each machine center is less than one. Therefore, we always assume that $\sum_{i=1}^{m} v_i \geqslant 1$.

Theorem 1. Given γ_i such that

$$\sum_{i=1}^{m} \gamma_i = 1 \quad \text{and} \quad 0 \leqslant \gamma_i \leqslant v_i \quad \text{for } i = 1, \ldots, m, \tag{3.1}$$

then there exists a feasible solution to (2.8) if and only if for all i

$$v_i - \gamma_i \leqslant \sum_{j=1, j\neq i}^{m} v_j \tag{3.2}$$

and $\sum_{i=1}^{m} v_i \geqslant 1$.

Proof. Consider the following set of constraints which are similar to the constraint set of (2.8) except that the γ_i variables are given such that (3.1) and (3.2) hold

$$\begin{aligned} &\sum_{i=1, i\neq j}^{m} p_{ji} \leqslant 1 && \text{for } j = 1, \ldots, m, \\ &\sum_{j=1, j\neq i}^{m} v_j p_{ji} = v_i - \gamma_i && \text{for } i = 1, \ldots, m, \\ &p_{ij} \geqslant 0 && \text{for } i, j = 1, \ldots, m,\ i \neq j. \end{aligned} \tag{3.3}$$

By Farkas' lemma (see, e.g., [7, lemma 2, p. 92]) (3.3) is solvable if and only if every solution $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{R}^m$ and $\mathbf{w} = (w_1, \ldots, w_m) \in \mathbb{R}^m$ to

$$\begin{aligned} &u_j + v_j w_i \geqslant 0 && \text{for all } 1 \leqslant j \neq i \leqslant m, \\ &u_j \geqslant 0 && \text{for all } j \end{aligned} \tag{3.4}$$

satisfies

$$\sum_{j=1}^{m} u_j + \sum_{i=1}^{m} (v_i - \gamma_i) w_i \geqslant 0, \tag{3.5}$$

i.e., (3.3) is solvable if and only if $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^m$ satisfying (3.4) and

$$\sum_{j=1}^{m} u_j + \sum_{i=1}^{m} (v_i - \gamma_i) w_i < 0 \tag{3.6}$$

do not exist.

Suppose that (3.3) is solvable, but $\sum_{i=1}^{m} v_i < 1$. Then $u_j = 0$ for $j = 1, \ldots, m$ and $w_j = 1$ for $j = 1, \ldots, m$ satisfy (3.4) and (3.6), because $\sum_{i=1}^{m} \gamma_i = 1$, which is a

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contradiction. Suppose that (3.2) is violated. Then we assume without loss of generality that

$$v_1 > \gamma_1 + \sum_{j=2}^{m} v_j. \tag{3.7}$$

Then $w_1 = -1$, $w_j = 0$ for $j = 2, \ldots, m$, $u_1 = 0$ and $u_j = v_j$ for $j = 2, \ldots, m$ satisfy (3.4) and (3.6), which is again a contradiction. Therefore, the conditions of theorem 1 are necessary.

To prove their sufficiency, suppose that the conditions of the theorem are satisfied. Let $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^m$ be any solution to (3.4). Without loss of generality we can assume that

$$w_1 = \min_{1 \leqslant j \leqslant m} w_j \quad \text{and} \quad w_2 = \min_{2 \leqslant j \leqslant m} w_j.$$

Let $w_1^* = \min\{0, w_1\}$ and $w_2^* = \min\{0, w_2\}$. From (3.4) it follows that

$$u_j \geqslant -v_j w_1^* \quad \text{for } j = 2, \ldots, m \quad \text{and} \quad u_1 \geqslant -v_1 w_2^*. \tag{3.8}$$

Consequently, from $v_i - \gamma_i \geqslant 0$ for $j = 1, \ldots, m$, $w_1 \geqslant w_1^*$, $w_j \geqslant w_2 \geqslant w_2^*$ for $j = 2, \ldots, m$ and $\sum_{i=1}^{m} \gamma_i = 1$, and by using (3.8) we calculate

$$\begin{aligned}
\sum_{j=1}^{m} u_j + \sum_{i=1}^{m} (v_i - \gamma_i) w_i &\geqslant -v_1 w_2^* - \sum_{j=2}^{m} v_j w_1^* + (v_1 - \gamma_1) w_1^* + \sum_{j=2}^{m} (v_j - \gamma_j) w_2^* \\
&= \left(v_1 - \gamma_1 - \sum_{j=2}^{m} v_j \right) w_1^* - \left(v_1 - \sum_{j=2}^{m} (v_j - \gamma_j) \right) w_2^* \\
&= -w_1^* + \left(v_1 - \sum_{j=2}^{m} (v_j - \gamma_j) \right) (w_1^* - w_2^*) \\
&\geqslant -w_2^* \geqslant 0,
\end{aligned}$$

because $w_1^* \leqslant w_2^* \leqslant 0$ and by (3.2) and $\sum_{i=1}^{m} \gamma_i = 1$, $v_1 - \sum_{j=2}^{m} (v_j - \gamma_j) \leqslant 1$. Since $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{w} \in \mathbb{R}^m$ are arbitrary solutions to (3.4), by Farkas' lemma, the linear system (3.3) is solvable and theorem 1 follows. □

In section 4, we use theorem 1 to simplify the analysis of the job shop problem under some further conditions. However, in order to fully discuss the feasibility of the general job shop problem (2.8), we provide the following two lemmas.

Lemma 1. The general job shop problem (2.8) is feasible if and only if $\sum_{i=1}^{m} v_i \geqslant 1$ and there exist $\varepsilon_i \geqslant 0$ with $\varepsilon_i \leqslant v_i$ for $i = 1, \ldots, m$ and $\sum_{i=1}^{m} \varepsilon_i = 1$ such that

$$v_i - \varepsilon_i \leqslant \sum_{j=1, j \neq i}^{m} v_j \quad \text{for all } i = 1, \ldots, m. \tag{3.9}$$

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Proof. If (2.8) is feasible, let $(\boldsymbol{\gamma}, \mathbf{p})$ be any solution to (2.8), where $\boldsymbol{\gamma}$ is the m vector of the γ_i's and $\mathbf{p}$ is the $m\,(m-1)$ vector corresponding to the p_{ij}'s. Then $\mathbf{p}$ is a solution to the linear system (3.3) and thus by theorem 1 the conditions of lemma 1 are met with $\varepsilon_i = \gamma_i$ for $i = 1, \dots, m$. Suppose that the conditions of lemma 1 are satisfied. By theorem 1 the system (3.3) with $\gamma_i = \varepsilon_i$ for $i = 1, \dots, m$ is solvable and hence so is the constraint set of (2.8). □

For the next lemma, without loss of generality we assume that

$$v_1 \geqslant \cdots \geqslant v_m.$$

Lemma 2. The general job shop problem (2.8) is feasible if and only if

$$v_1 - 1 \leqslant \sum_{j=2}^{m} v_j \tag{3.10}$$

and $\sum_{i=1}^{m} v_i \geqslant 1$.

Proof. Let (2.8) be feasible. Then by lemma 1, $\sum_{i=1}^{m} v_i \geqslant 1$ and there exist $\varepsilon_i \geqslant 0$ with $\varepsilon_i \leqslant v_i$ for $i = 1, \dots, m$ and $\sum_{i=1}^{m} \varepsilon_i = 1$ such that (3.9) holds. From (3.9), $v_1 - 1 \leqslant v_1 - \varepsilon_1 \leqslant \sum_{j=2}^{m} v_j$ and hence the conditions of lemma 2 are necessary. To prove their sufficiency, we assume that they are satisfied and show that a solution to (2.8) always exists. Let

$$\varepsilon_1 = \min\{v_1, 1\} \quad \text{and} \quad \varepsilon_i = \min\left\{v_i, 1 - \sum_{j<i} \varepsilon_j\right\}, \tag{3.11}$$

where we assign values to ε_i in ascending order of their indices. Since $v_1 - 1 \leqslant \sum_{j=2}^{m} v_j$, if $v_1 \leqslant 1$, $v_1 - \varepsilon_1 = 0 \leqslant \sum_{j=2}^{m} v_j$, and if $v_1 > 1$, $v_1 - \varepsilon_1 = v_1 - 1 \leqslant \sum_{j=2}^{m} v_j$. Furthermore, for $k = 2, \dots, m$,

$$v_k - \varepsilon_k \leqslant v_k \leqslant v_1 \leqslant \sum_{j=1, j \neq k}^{m} v_j.$$

From (3.11), $\sum_{i=1}^{m} \varepsilon_i = 1$. It follows that the conditions of lemma 1 are satisfied, and hence there exists a solution to (2.8). □

Remark. Theorem 1, and lemmas 1 and 2 are of independent interest in themselves. Consider the problem, "Given a non-negative vector, can we recognize whether it is the vector of expected numbers of visits to various states by a transient Markov chain with some initial distribution." Theorem 1, and lemmas 1 and 2 provide a necessary and sufficient condition for answering this question, namely, the answer is yes only if the expected number of visits satisfy (3.10).

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4. Results

In this section, we begin with a discussion of a special case of problem P that was considered by Buzacott and Shanthikumar. We first solve this case in theorem 2 and make a correction to their original result (see [1, pp. 337–338], where the result is almost the opposite from theorem 2 below). We then extend the results to more general cases in subsections 4.2 and 4.3.

4.1. Single server, identical stations, and high utilization

Buzacott and Shanthikumar solved for the case when $\rho_i = \rho \approx 1$ for $i = 1, \ldots, m$; $v_i = 1$ for $i = 1, \ldots, m$; each machine center has a single server, and each server has the same service time distribution. Thus, denote $C^2_{S_i} = C^2_S$. Under these assumptions and using (2.6) we get

$$E[N] \approx \frac{\rho(1 + C^2_S)}{2 - \rho + \rho C^2_S} \frac{m\rho^2 C^2_S + \sum_{i=1}^m \rho(2 - \rho)C^2_{a_i}}{2(1 - \rho)} + m\rho. \tag{4.1}$$

Therefore, minimizing (4.1) is equivalent to minimizing $\sum_{i=1}^m C^2_{a_i}$. However, from (2.4) and the assumptions in this section

$$\sum_{i=1}^m C^2_{a_i} = \sum_{i=1}^m \left(\sum_{j=1, j\neq i}^m p_{ji}\left[p_{ji}C^2_{d_j} + (1 - p_{ji})\right] + \gamma_i\left[\gamma_i C^2_a + (1 - \gamma_i)\right] \right). \tag{4.2}$$

Furthermore, since $v_i = 1$, from (2.2) we get

$$\sum_{i=1}^m \left(\gamma_i + \sum_{j=1, j\neq i}^m p_{ji} \right) = m. \tag{4.3}$$

As $\rho_i \approx 1$ and $c_i = 1$, from (2.5) we approximate $C^2_{d_j} \approx C^2_S$. Thus

$$\sum_{i=1}^m C^2_{a_i} = m + \sum_{i=1}^m (C^2_S - 1) \sum_{j=1, j\neq i}^m p^2_{ji} + (C^2_a - 1)\gamma^2_i. \tag{4.4}$$

Theorem 2. In problem P, if for all i, $c_i = 1$, $C^2_{S_i} = C^2_S$, $v_i = 1$, and $\rho_i \approx 1$, then

(i) if $C^2_S \geqslant 1$ and $C^2_a \geqslant 1$ then the symmetric flow shop design (i.e., $p_{ij} = 1/m, i \neq j$, $i, j = 1, \ldots, m$ and $\gamma_i = 1/m$, for $i = 1, \ldots, m$) is an optimal solution to P.

(ii) if $C^2_S \leqslant 1$ and $C^2_a \leqslant 1$ then the uniform job shop design (i.e., $\gamma_1 = 1$, $p_{i,i+1} = 1$ for $i = 1, \ldots, m - 1$, and all other $p_{ij} = 0$) is an optimal solution to P.

Proof. (i) Since $C^2_S \geqslant 1$ and $C^2_a \geqslant 1$, the objective function (4.4) is jointly convex in the p_{ij}'s and in the γ_i's. Thus a symmetric allocation, namely, $p_{ij} = 1/m, i \neq j, i, j = 1, \ldots, m$, and $\gamma_i = 1/m$, for $i = 1, \ldots, m$, minimizes $\sum_i C^2_{a_i}$ and an extreme (uniform)

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allocation, namely, $\gamma_1 = 1$, $p_{i,i+1} = 1$ for $i = 1, \ldots, m-1$, and all other $p_{ij} = 0$ maximizes it. The objective function (4.4) evaluated at a symmetric allocation is equal to $(C_{\mathrm{a}}^2 - C_S^2)/m + (C_S^2 - 1)$, while at a uniform allocation it is equal to $m(C_S^2 - 1) + C_{\mathrm{a}}^2 - C_S^2$. Therefore,

$$\begin{aligned}\frac{C_{\mathrm{a}}^2 - C_S^2}{m} + \left(C_S^2 - 1\right) &\leqslant \sum_{i=1}^{m} \left(C_S^2 - 1\right)\left(\sum_{j=1, j\neq i}^{m} p_{ji}^2\right) + \gamma_i^2\left(C_{\mathrm{a}}^2 - 1\right) \\ &\leqslant m\left(C_S^2 - 1\right) + C_{\mathrm{a}}^2 - C_S^2. \end{aligned} \tag{4.5}$$

We thus see, in this case, the symmetric flow shop design minimizes the expected number of jobs in the shop while the uniform job shop design maximizes the expected number of jobs.

(ii) On the other hand, if $C_S^2 \leqslant 1$ and $C_{\mathrm{a}}^2 \leqslant 1$, the objective function (4.5) is jointly concave in the p_{ij}'s and in the γ_i's. Thus, the above inequalities (4.5) will change direction and the result will be exactly the opposite to the one above. □

Remark. For the cases discussed in theorem 2, it would be worthwhile to know the magnitude of error if the wrong design were chosen, i.e., if the uniform job shop design is used when the symmetric flow shop design is optimal, or vice versa. We denote $E[N_S]$ for the expected number of jobs under a symmetric job shop configuration and $E[N_{\mathrm{U}}]$ for the expected number of jobs under a uniform flow configuration. To keep the analysis tractable, let $\rho_i \approx \rho$. If the uniform flow shop were used instead of the symmetric job shop when the symmetric job shop is optimal (denoted by an asterisk), the ratio of the expected number of jobs in the system is approximately

$$\frac{E[N_{\mathrm{U}}]}{E[N_S^*]} \approx \frac{\frac{\rho(1+C_S^2)}{2-\rho+\rho C_S^2} \cdot \frac{m\rho^2 C_S^2 + \rho(2-\rho)[m + m(C_S^2-1) + C_{\mathrm{a}}^2 - C_S^2]}{2(1-\rho)} + m\rho}{\frac{\rho(1+C_S^2)}{2-\rho+\rho C_S^2} \cdot \frac{m\rho^2 C_S^2 + \rho(2-\rho)[m + ((C_{\mathrm{a}}^2 - C_S^2)/m) + (C_S^2 - 1)]}{2(1-\rho)} + m\rho}. \tag{4.6}$$

Set $m = \lceil 1/(1-\rho) \rceil$. By evaluating the limit of (4.6) as ρ approaches 1 from below, we find that this ratio approaches $2C_S^2/(C_S^2+1)$ which is greater than or equal to 1 since $C_S^2 \geqslant 1$ (see theorem 2(i)). In the worst case (as $C_S^2 \to \infty$) this ratio is two. Also note that $m\rho$ is uncontrollable. Thus, we may simplify our discussion of the ratio in (4.6) by eliminating the second term from both the numerator as well as from the denominator, i.e., we consider the ratio of the number of jobs in the queue

$$\frac{C_S^2 + \frac{(2-\rho)}{\rho}\left(C_S^2 + \frac{C_{\mathrm{a}}^2 - C_S^2}{m}\right)}{C_S^2 + \frac{(2-\rho)}{\rho}\left(1 + \frac{C_{\mathrm{a}}^2 - C_S^2}{m} + \frac{(C_S^2 - 1)}{m}\right)}. \tag{4.7}$$

As before, when $\rho \approx 1$ and m is large the ratio is $2C_S^2/(C_S^2+1)$. In addition, from (4.7), as ρ becomes smaller, the ratio and thus the relative magnitude of error increases. In very light traffic and when there are many machine centers, the ratio can become as large as C_S^2.

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In the case where a symmetric allocation is used when a uniform flow shop configuration is optimal the ratio in question is just the reciprocal of (4.6) or (4.7). Here the ratio approaches $(C_S^2 + 1)/2C_S^2$ which is greater than or equal to 1 since $C_S^2 \leqslant 1$ (see theorem 2(ii)). This ratio can be made as large as desired by reducing C_S^2. Thus, in the worst case, the ratio is unbounded. Therefore, it is relatively more important to use a proper configuration when there are many machine centers and either when the variability of the service times is low or when the utilization is low.

4.2. *High machine utilization*

In this subsection, we generalize the problem solved in section 4.1. More precisely, we assume that each machine center has utilization $\rho_i \approx 1$, but permit machine centers to have different expected number of visits, and also different service time distributions. However, we assume that each machine center has a large number of machines. From (2.6) we see that the only variables that directly influence the objective function are the $C_{\mathrm{a}_i}^2$'s. Under the assumptions that $\rho_i \approx 1$ and $c_i \gg 1$, from (2.5) we approximate

$$C_{\mathrm{d}_i}^2 \approx 1 + \frac{1}{c_i}\left(C_{S_i}^2 - 1\right) \approx 1. \tag{4.8}$$

The above approximation can be interpreted as "the superposition of many independent streams is approximately Poisson" (see [3, section 9.2; 4, section 2.8]). Therefore from (2.4) and (2.2) we approximate

$$C_{\mathrm{a}_i}^2 \approx 1 + \left(C_{\mathrm{a}}^2 - 1\right)\frac{\lambda \gamma_i^2}{\lambda_i}. \tag{4.9}$$

Define

$$K_i = \frac{\rho_i(2-\rho_i)}{2(1-\rho_i)} \cdot \frac{\rho_i(1+C_{S_i}^2)}{2-\rho_i+\rho_i C_{S_i}^2} \cdot \frac{E[W(\lambda_i, \mu_i)]_{M/M/c_i}}{E[W(\lambda_i, c_i\mu_i)]_{M/M/1}}. \tag{4.10}$$

Define

$$L_i = \max\left\{0,\, v_i - \sum_{j=1, j\neq i}^{m} v_j\right\}, \qquad U_i = \min\{v_i, 1\}. \tag{4.11}$$

Then by (2.6), (4.9) and theorem 1, the decision problem can be reformulated as

$$\min \sum_{i=1}^{m} \frac{K_i}{\lambda_i v_i}\left(C_{\mathrm{a}}^2 - 1\right)\gamma_i^2 \tag{4.12}$$

such that

$$\sum_{i=1}^{m} \gamma_i = 1, \tag{4.13}$$

$$L_i \leqslant \gamma_i \leqslant U_i \quad \text{for } i = 1, \ldots, m, \tag{4.14}$$

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where the new constraint set arises from theorem 1 and the fact that in this case the p_{ij}'s are not decision variables. The objective function in (4.12) is convex in the γ_i's if $C_a^2 \geqslant 1$ and concave otherwise. Consider the following Lagrangian function

$$\mathcal{L} = \sum_{i=1}^{m} \left(\frac{K_i}{\lambda_i v_i} (C_a^2 - 1)\gamma_i^2 - s_i(\gamma_i - L_i) - t_i(U_i - \gamma_i) \right) - \mu \left(\sum_{i=1}^{m} \gamma_i - 1 \right), \quad (4.15)$$

where μ is the Lagrangian multiplier for constraint (4.13), s_i and t_i are the Lagrangian multipliers for constraints (4.14). In the next theorem we characterize the optimal solution to the decision problem (4.12)–(4.14).

Theorem 3. According to the assumptions in this section, if $C_a^2 \geqslant 1$ then the unique optimal solution to (4.12), γ^*, can be written as a function of μ^*, as shown below

$$\gamma_i^*(\mu^*) = L_i \qquad \text{if } \frac{\mu^* \lambda_i v_i}{K_i} \leqslant L_i, \quad (4.16)$$

$$\gamma_i^*(\mu^*) = \frac{\mu^* \lambda_i v_i}{K_i} \qquad \text{if } L_i \leqslant \frac{\mu^* \lambda_i v_i}{K_i} \leqslant U_i, \quad (4.17)$$

$$\gamma_i^*(\mu^*) = U_i \qquad \text{if } \frac{\mu^* \lambda_i v_i}{K_i} \geqslant U_i, \quad (4.18)$$

where

$$\mu^* \in \left[0, \max_i \left[U_i \frac{K_i}{\lambda_i v_i} \right] \right] \quad \text{and} \quad \sum_{i=1}^{m} \gamma_i(\mu^*) = 1.$$

The optimal solution can be determined by performing a bisection search for μ^* in $[0, \max_i [U_i K_i/(\lambda_i v_i)]]$.

Proof. Since the minimization problem (4.12) has a strictly convex objective function and linear constraints, the optimal solution can be obtained by solving for the first order condition of the Lagrangian function (4.15). Differentiating the Lagrangian function (4.15) with respect to γ_i gives the first order conditions

$$\frac{K_i}{\lambda_i v_i} \gamma_i - \mu - s_i + t_i = 0 \quad \text{for } i = 1, \ldots, m. \quad (4.19)$$

If for i neither the upper nor the lower bounds (4.14) are binding, then $\gamma_i = \mu \lambda_i v_i / K_i$. On the other hand, if the lower bound is binding then $\gamma_i = L_i$ and if the upper bound is binding then $\gamma_i = U_i$. Since $0 \leqslant \gamma_i \leqslant 1$, the feasible choices for μ are restricted to

$$\mu \in \left[0, \max_i \left[U_i \frac{K_i}{\lambda_i v_i} \right] \right]. \quad (4.20)$$

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However,

$$F(\mu) = \sum_{i=1}^{m} \gamma_i(\mu) \tag{4.21}$$

is a nondecreasing continuous function in μ. Therefore, the optimal solution is found by using a bisection search for the unique μ such that $F(\mu) = 1$. □

In order to discuss the case when $C_{\mathrm{a}}^2 < 1$, we first define the sets

$$U(\gamma) = \{i \mid \gamma_i = U_i\}, \qquad L(\gamma) = \{i \mid \gamma_i = L_i\}. \tag{4.22}$$

Denote the cardinality of the set A as $|A|$. Thus, according to the assumptions of this section, if $C_{\mathrm{a}}^2 < 1$, then the objective function (4.12) is concave in the γ_i's. Thus, an optimal solution must be an extreme point solution of the feasible set, i.e., it has the property that either

$$\text{(i)} \quad |U(\gamma)| + |L(\gamma)| = m \quad \text{and} \quad \sum_{i \in U(\gamma)} U_i + \sum_{i \in L(\gamma)} L_i = 1 \quad \text{or} \tag{4.23}$$

$$\text{(ii)} \quad |U(\gamma)| + |L(\gamma)| = m - 1 \quad \text{and} \quad \sum_{i \in U(\gamma)} U_i + \sum_{i \in L(\gamma)} L_i + \gamma_k = 1, \tag{4.24}$$

where $k \notin U(\gamma)$, $k \notin L(\gamma)$ and $0 < \gamma_k < 1$. Therefore, it suffices to compare all such solutions and choose the one that has the smallest objective function value. In particular, if $v_i \geqslant 1$ for all i, then as long as $v_i \leqslant \sum_{j=1, j \neq i}^{m} v_j$ for $i = 1, \ldots, m$ (see theorem 1), the optimal solution is

$$\gamma_k = 1, \qquad \gamma_i = 0 \quad \text{for } i \neq k, \tag{4.25}$$

where $k = \operatorname{argmax}_i \{K_i/(\lambda_i v_i)\}$.

4.3. *Lower machine utilization*

In this subsection, we further generalize the problem solved in section 4.1 by not restricting the ρ_i's to be close to 1, and also by not requiring the c_i's to be large. We define the following quantities

$$r_i = \frac{(1 - \rho_i^2)(2 - \rho_i) + \rho_i C_{s_i}^2 (1 - \rho_i)^2}{2 - \rho_i + \rho_i C_{s_i}^2}, \tag{4.26}$$

$$\ell_i = \frac{\rho_i^2}{c_i}\left(C_{s_i}^2 - 1\right). \tag{4.27}$$

We then note that from (2.5),

$$C_{\mathrm{d}_i}^2 = 1 + \ell_i + \left(C_{\mathrm{a}_i}^2 - 1\right) r_i. \tag{4.28}$$

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Then from (2.4), we get a system of m equations defining $C^2_{\mathrm{a}_i}$, namely,

$$C^2_{\mathrm{a}_i} - 1 = \sum_{j=1}^{m} \frac{v_j}{v_i} p^2_{ji}\big(\ell_j + r_j(C^2_{\mathrm{a}_j} - 1)\big) + \frac{\gamma_i^2}{v_i}\big(C^2_{\mathrm{a}} - 1\big). \tag{4.29}$$

We define the following matrices:

$$\mathbf{A} = \begin{pmatrix} 0 & p^2_{21}\frac{v_2}{v_1} r_2 & \cdots & p^2_{m1}\frac{v_m}{v_1} r_m \\ p^2_{12}\frac{v_1}{v_2} r_1 & 0 & \cdots & p^2_{m2}\frac{v_m}{v_2} r_m \\ \vdots & & \ddots & \\ p^2_{1m}\frac{v_1}{v_m} r_1 & p^2_{2m}\frac{v_2}{v_m} r_2 & \cdots & 0 \end{pmatrix}, \tag{4.30}$$

$$\mathbf{B} = \begin{pmatrix} 0 & p^2_{21}\frac{v_2}{v_1} \ell_2 & \cdots & p^2_{m1}\frac{v_m}{v_1} \ell_m \\ p^2_{12}\frac{v_1}{v_2} \ell_1 & 0 & \cdots & p^2_{m2}\frac{v_m}{v_2} \ell_m \\ \vdots & & \ddots & \\ p^2_{1m}\frac{v_1}{v_m} \ell_1 & p^2_{2m}\frac{v_2}{v_m} \ell_2 & \cdots & 0 \end{pmatrix}, \tag{4.31}$$

$$\mathbf{g} = \begin{pmatrix} \frac{\gamma_1^2}{v_1} \\ \vdots \\ \frac{\gamma_m^2}{v_m} \end{pmatrix}, \tag{4.32}$$

$$\mathbf{c}^2_{\mathrm{a}} = \begin{pmatrix} C^2_{\mathrm{a}_1} \\ \vdots \\ C^2_{\mathrm{a}_m} \end{pmatrix}. \tag{4.33}$$

Then (4.29) can be written in matrix form as

$$\mathbf{c}^2_{\mathrm{a}} - \mathbf{e} = \mathbf{A}\big(\mathbf{c}^2_{\mathrm{a}} - \mathbf{e}\big) + \mathbf{B}\mathbf{e} + \big(C^2_{\mathrm{a}} - 1\big)\mathbf{g}, \tag{4.34}$$

where $\mathbf{e}$ is a column vector of m ones.

Note that the matrix $\mathbf{A}$ is non-negative. Therefore, by the Perron–Frobenius Theorem (e.g., [6, chapter 6]), there exists an eigenvalue $\lambda_0 \geqslant 0$ such that any other eigenvalue of this matrix, λ, has the property that $|\lambda| \leqslant \lambda_0$. Furthermore, the eigenvalue λ_0 is bounded by the minimum and maximum column sum and/or row sum of $\mathbf{A}$. However, by analyzing r_j as a function of $C^2_{S_j}$, we find that $(1-\rho_j)^2 \leqslant r_j \leqslant (1-\rho_j^2)$. Furthermore,

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from (2.2), $\sum_{j=1, j\neq i}^{m} p_{ji}^2 v_j / v_i \leqslant 1$. Thus, as long as $\rho_j > 0$ for all j, every column sum of **A** is strictly less than 1. Therefore,

$$\lambda_0 < 1. \tag{4.35}$$

Thus, all of the eigenvalues of **A** are strictly inside the unit circle. Then, $(\mathbf{I} - \mathbf{A})$ is invertible and we may express $(\mathbf{I} - \mathbf{A})^{-1}$ as (c.f. [6, chapter 6])

$$(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots. \tag{4.36}$$

Then, we reformulate (4.34) in order to isolate $\mathbf{c}_{\mathrm{a}}^2 - \mathbf{e}$ as

$$\mathbf{c}_{\mathrm{a}}^2 - \mathbf{e} = (\mathbf{I} - \mathbf{A})^{-1}\big((C_{\mathrm{a}}^2 - 1)\mathbf{g} + \mathbf{B}\mathbf{e}\big). \tag{4.37}$$

Then from (4.10) and (2.6), the objective function is

$$\min \sum_{i=1}^{m} \frac{K_i}{v_i} C_{\mathrm{a}_i}^2. \tag{4.38}$$

However, from (4.37), it is easier to solve the minimization problem with the objective function reformulated as

$$\min \sum_{i=1}^{m} \frac{K_i}{v_i} (C_{\mathrm{a}_i}^2 - 1) \tag{4.39}$$

instead. This leads us to the following theorem.

Theorem 4. If $C_{\mathrm{a}}^2 \geqslant 1$ and $C_{S_j}^2 \geqslant 1$ for all j, then the objective function (4.39) is convex in $(\mathbf{p}, \boldsymbol{\gamma})$. If $C_{\mathrm{a}}^2 \leqslant 1$ and $C_{S_j}^2 \leqslant 1$ for all j, then the objective function (4.39) is concave in $(\mathbf{p}, \boldsymbol{\gamma})$.

Proof. If $C_{S_j}^2 \geqslant 1$, then $\ell_j = (\rho_j^2/c_j)(C_{S_j}^2 - 1) \geqslant 0$. Thus all entries of the matrix **B** are non-negative. When $C_{\mathrm{a}}^2 \geqslant 1$ then for all j, γ_j^2 has a non-negative coefficient in (4.37), (also see (4.29)). Let $(\mathbf{p}^{(1)}, \boldsymbol{\gamma}^{(1)})$ and $(\mathbf{p}^{(2)}, \boldsymbol{\gamma}^{(2)})$ be two feasible solutions. For any i, j and $0 \leqslant \lambda \leqslant 1$,

$$\big(\lambda p_{ji}^{(1)} + (1-\lambda) p_{ji}^{(2)}\big)^2 \leqslant \lambda p_{ji}^{(1)2} + (1-\lambda) p_{ji}^{(2)2}, \tag{4.40}$$

$$\big(\lambda \gamma_i^{(1)} + (1-\lambda) \gamma_i^{(2)}\big)^2 \leqslant \lambda \gamma_i^{(1)2} + (1-\lambda) \gamma_i^{(2)2}. \tag{4.41}$$

Let $\mathbf{A}^{(1)}$ and $\mathbf{A}^{(2)}$ be the matrices defined in (4.30) for the solutions $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$, respectively and let $\mathbf{g}^{(1)}$ and $\mathbf{g}^{(2)}$ be the vectors defined in (4.32) for the solutions $\boldsymbol{\gamma}^{(1)}$ and $\boldsymbol{\gamma}^{(2)}$, respectively. Let $\mathbf{A}^{(3)}$ and $\mathbf{g}^{(3)}$ be the matrices defined in (4.30) and (4.32) for the solutions $\lambda\mathbf{p}^{(1)} + (1-\lambda)\mathbf{p}^{(2)}$ and $\lambda\boldsymbol{\gamma}^{(1)} + (1-\lambda)\boldsymbol{\gamma}^{(2)}$, respectively. It follows from (4.40) and (4.41) that for all positive integers n

$$\mathbf{A}^{(3)n} \leqslant \lambda \mathbf{A}^{(1)n} + (1-\lambda)\mathbf{A}^{(2)n}, \tag{4.42}$$

$$\mathbf{A}^{(3)n}\mathbf{g}^{(3)} \leqslant \lambda \mathbf{A}^{(1)n}\mathbf{g}^{(1)} + (1-\lambda)\mathbf{A}^{(2)n}\mathbf{g}^{(2)}. \tag{4.43}$$

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If we define $\mathbf{B}^{(1)}$, $\mathbf{B}^{(2)}$ and $\mathbf{B}^{(3)}$ as we have done for $\mathbf{A}$, then by (4.40), (4.41) and the fact that all elements of $\mathbf{B}$ are non-negative

$$\mathbf{B}^{(3)} \leqslant \lambda \mathbf{B}^{(1)} + (1-\lambda)\mathbf{B}^{(2)}. \tag{4.44}$$

Then it follows from (4.42) and (4.44) that

$$\mathbf{A}^{(3)n}\mathbf{B}^{(3)} \leqslant \lambda \mathbf{A}^{(1)n}\mathbf{B}^{(1)} + (1-\lambda)\mathbf{A}^{(2)n}\mathbf{B}^{(2)}. \tag{4.45}$$

Let

$$\mathbf{c}_{\mathrm{a}}^{2(i)} - \mathbf{e} = \left(\mathbf{I} - \mathbf{A}^{(i)}\right)^{-1}\left(\left(C_{\mathrm{a}}^2 - 1\right)\mathbf{g}^{(i)} + \mathbf{B}^{(i)}\mathbf{e}\right). \tag{4.46}$$

Thus, by (4.43), (4.45) and (4.46)

$$\mathbf{c}_{\mathrm{a}}^{2(3)} - \mathbf{e} \leqslant \lambda\left(\mathbf{c}_{\mathrm{a}}^{2(1)} - \mathbf{e}\right) + (1-\lambda)\left(\mathbf{c}_{\mathrm{a}}^{2(2)} - \mathbf{e}\right). \tag{4.47}$$

Since the objective function (4.39) has positive coefficients, it follows that the objective function (4.39) is convex.

If on the other hand, $C_{\mathrm{a}}^2 \leqslant 1$ and $C_{S_j}^2 \leqslant 1$ for all j, then $\mathbf{g}$ has a nonpositive coefficient in (4.37) and (4.46). Furthermore, all entries of the matrix $\mathbf{B}$ are nonpositive (because $\ell_j \leqslant 0$), while the matrix $\mathbf{A}$ remains non-negative. Then, (4.42) and (4.43) still hold but the direction of the inequality changes in (4.44) and (4.45). Thus, it follows that the objective function is concave. □

The above theorem when combined with theorem 1 provides us with the ability to solve the problem at hand using convex or concave optimization algorithms. Furthermore, it also provides us with the machinery to characterize solutions of several specific cases. These cases are discussed in the following lemmas.

Lemma 3. If $v_i = v \geqslant 1/m$, $\rho_i = \rho < 1$, $c_i = c$, $C_{\mathrm{a}}^2 \geqslant 1$, and $C_{S_j}^2 = C_S^2 \geqslant 1$ then a symmetric allocation is an optimal solution.

Proof. If $v_i = v$, $\rho_i = \rho < 1$, $c_i = c$, and $C_{S_j}^2 = C_S^2$ then $K_i = K$. Thus, from Theorem 4 the decision problem will be a convex minimization problem with equal weights to each $C_{\mathrm{a}_i}^2$ in the objective function. Since $v \geqslant 1/m$, by lemma 2 the problem is feasible. Consider any feasible solution $(\mathbf{p}^1, \boldsymbol{\gamma}^1)$. Since there are m stations, there exist $m!$ permutations of the indices to the stations. If we were to consider all $m!$ permutations of the given feasible solution, the objective function value in each case will be the same since the weights in the objective function are equal. By convexity though, the average of all of these permuted solutions will provide a feasible solution that has an objective function value less than or equal to every one of the permuted solutions. This average solution is

$$\gamma_i = \frac{1}{m}, \quad \text{for } i = 1, \ldots, m, \quad p_{ij} = \frac{mv - 1}{(m-1)mv}, \quad \text{for } i \neq j,$$

and $p_{ii} = 0$. Now, if we follow the above procedure for all feasible solutions, we can conclude that the symmetric allocation solution is optimal. □

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In order to discuss the next lemma, we modify the constraint set of problem P (2.8) by including variables $L_1, \ldots, L_m$ for the "leakage" flow from each of the m machine centers. Also, we assume here that $v_i = v$. The constraint set is then formulated as

$$
\begin{aligned}
& v \sum_{i=1, i\neq j}^{m} p_{ji} + L_j = v, && j = 1, \ldots, m, \\
& \sum_{i=1}^{m} \gamma_i = 1, && \\
& \gamma_i + v \sum_{j=1, j\neq i}^{m} p_{ji} = v, && i = 1, \ldots, m, \\
& \gamma_i, L_i, \geqslant 0, && i = 1, \ldots, m, \\
& p_{ij} \geqslant 0, && i, j = 1, \ldots, m \text{ and } i \neq j.
\end{aligned}
$$

We define

$$
\gamma_i^0 = \frac{\gamma_i}{v}, \qquad L_i^0 = \frac{L_i}{v}. \tag{4.48}
$$

Then the constraint set becomes

$$
\begin{aligned}
& \sum_{i=1, i\neq j}^{m} p_{ji} + L_j^0 = 1, && j = 1, \ldots, m, \\
& \sum_{i=1}^{m} \gamma_i^0 = \frac{1}{v}, && \\
& \gamma_i^0 + \sum_{j=1, j\neq i}^{m} p_{ji} = 1, && i = 1, \ldots, m, \\
& \gamma_i^0, L_i^0 \geqslant 0, && i = 1, \ldots, m, \\
& p_{ij} \geqslant 0, && i, j = 1, \ldots, m \text{ and } i \neq j.
\end{aligned}
$$

These are $2m+1$ equality constraints with coefficients equal to only zeros and ones. The right-hand side coefficients include $2m$ elements equal to ones and one element equal to $1/v$. Thus, the constraint set has the form of a coefficient matrix of a transportation problem.

Lemma 4. If $v_i = v \geqslant 1$, where $v \neq (k+1)/k$ for any positive integer k, $\rho_i = \rho < 1$, $C_{\mathrm{a}}^2 \leqslant 1$, and $C_{S_j}^2 \leqslant 1$ then an optimal solution has the property that all variables p_{ij} take

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the values 0, $1/v, 1 - 1/v$, 1 with at most $2m - 1$ of them positive, exactly one $\gamma_i = 1$, and exactly one $L_j = 1$. In other words, if we denote

$$\mathbf{P} = \begin{pmatrix} 0 & p_{21} & p_{31} & \cdots & p_{m1} \\ p_{12} & 0 & p_{32} & \cdots & p_{m2} \\ \vdots & & \ddots & & \vdots \\ p_{1m} & p_{2m} & p_{3m} & \cdots & 0 \end{pmatrix} \tag{4.49}$$

then an optimal solution has the property that

(i) $m - 1$ of the row and column sums of $\mathbf{P}$ are equal to 1 and

(ii) exactly one row and exactly one column sum are equal to $1 - 1/v$.

Therefore, we can find an optimal solution by enumerating all feasible solutions that have the above form.

Proof. Since $v \geqslant 1$, by theorem 1 this problem is feasible. Consider the following subset of constraints of the original constraint set (2.8)

$$\sum_{i=1}^{m} p_{ji} \leqslant 1 \quad \text{for } j = 1, \ldots, m, \tag{4.50}$$

$$\sum_{i=1}^{m} \gamma_i = 1. \tag{4.51}$$

From (4.48), we reformulate (4.50) and (4.51) as

$$\sum_{i=1}^{m} p_{ji} + L_j^0 = 1 \quad \text{for } j = 1, \ldots, m, \tag{4.52}$$

$$\sum_{i=1}^{m} \gamma_i^0 = \frac{1}{v}. \tag{4.53}$$

As observed earlier, this new system is in the form of a constraint set to a transportation problem. The right hand side vector $\mathbf{b}$ of this system contains $2m$ ones and the last element equals $1/v$, i.e.,

$$\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \frac{1}{v} \end{pmatrix}. \tag{4.54}$$

The matrix correpsonding to the linear constraints consists of only zeros and ones. Thus, let $\mathbf{B}$ be a nonsingular submatrix of $2m + 1$ independent columns, i.e., a basis. Since the constraint set is totally unimodular, the elements of $\mathbf{B}^{-1}$ can only take the values 0, ± 1

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(cf. [7, p. 204]). Therefore, the variables p_{ij}, γ_i^0, and L_i^0 corresponding to the positive part of a basic feasible solution $\mathbf{B}^{-1}\mathbf{b}$ can only take the values $1/v$, $1 - 1/v$, 1.

Since $v \neq (k+1)/k$, the constraint $\sum_{i=1}^{m} \gamma_i^0 = 1/v$ implies that exactly one $\gamma_i^0 = 1/v$. Note, the same will hold true for L_i^0 since $\sum_{i=1}^{m} L_i^0 = 1/v$ must hold as well. Therefore, there exist at most $2m - 1$ remaining positive elements in a basic feasible solution. From the analysis of $\mathbf{B}^{-1}\mathbf{b}$ these positive elements can only consist of the values $1/v$, $1 - 1/v$, 1 as mentioned above.

Since $C_{\mathrm{a}}^2 \leqslant 1$ and $C_{S_j}^2 \leqslant 1$, the objective function is concave (see theorem 4). Thus, an optimal solution occurs at an extreme point of the feasible set, i.e., at one of the basic feasible solutions discussed above. □

Remarks (When $C_{\mathrm{a}}^2 \leqslant 1$ and $C_{S_j}^2 \leqslant 1$). (i) In the special case where $v = 1$, the optimal configuration is a uniform flow job shop.

(ii) When $v > 1$, in all of our numerical results, the optimal configuration is a reentrant flow line [5], i.e., the optimal solution is one in which the machine centers are arranged in a flow line, jobs go through the flow line and either exit the shop at the last station with probability $1/v$ or reenter the first station with probability $1 - 1/v$.

5. Conclusions

We have discussed the problem of minimizing the *approximate* expected number of jobs in a single class general open queueing network model of a job shop. We have provided conditions under which the problem is feasible. We have re-solved the special case originally posed and solved by Buzacott and Shanthikumar. Furthermore, we have solved and provided insight into more general cases.

We have completely determined the optimal configurations when $\rho_i \approx 1$ and the c_i's are large without imposing any other restrictions. In the more difficult case when $\rho_i < 1$, (i) we have given conditions under which the problem is convex and stated when the problem is concave, and (ii) we have given complete solutions for special cases. Based on our analysis, we may also conclude that when the approximate expected number of visits to the machine centers are equal, in the majority of cases, (i) when the service and arrival processes are less variable ($C_{\mathrm{a}}^2 \leqslant 1$ and $C_{S_i}^2 \leqslant 1$) a uniform flow line or a variant of this configuration is optimal, and (ii) when the service and arrival processes are more variable ($C_{\mathrm{a}}^2 \geqslant 1$ and $C_{S_i}^2 \geqslant 1$) a symmetric job shop or a variant of this configuration is optimal.

In future work, we plan to extend this analysis to multi-class open networks of queues.

Acknowledgements

We thank Professor Manfred Padberg for several helpful suggestions and in particular for a compact proof of theorem 1. We are grateful to the Associate Editor and the two referees for many useful suggestions. One of their suggestions resulted in the discussion of

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the magnitude of performance difference between the uniform flow shop and symmetric job shop configurations while another suggestion led to the remark on the applicability of our results to transient Markov chains.

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