

Hedging Inventory Risk Through Market Instruments

Vishal Gaur, Sridhar Seshadri

Department of Information, Operations and Management Science, Leonard N. Stern School of Business,
New York University, Suite 8-160, 44 West 4th Street, New York, New York 10012
{vgaur@stern.nyu.edu, sseshadr@stern.nyu.edu}

We address the problem of hedging inventory risk for a short life cycle or seasonal item when its demand is correlated with the price of a financial asset. We show how to construct optimal hedging transactions that minimize the variance of profit and increase the expected utility for a risk-averse decision maker. We show that for a wide range of hedging strategies and utility functions, a risk-averse decision maker orders more inventory when he or she hedges the inventory risk. Our results are useful to both risk-neutral and risk-averse decision makers because (1) the price information of the financial asset is used to determine both the optimal inventory level and the hedge, (2) this enables the decision maker to update the demand forecast and the financial hedge as more information becomes available, and (3) hedging leads to lower risk and higher return on inventory investment. We illustrate these benefits using data from a retailing firm.

Key words: demand forecasting; financial hedging; newsboy model; real options; risk aversion

History: Received: April 2, 2003; accepted: October 29, 2004. This paper was with the authors 8 months for 3 revisions.

1. Introduction

The demand for discretionary purchase items, such as apparel, consumer electronics, and home furnishings, is widely believed to be correlated with economic indicators. Our analysis not only supports this belief but also shows that the correlation can be quite significant. For example, *The Redbook Average* monthly time-series data¹ for the period November 1999 to November 2001 have a correlation coefficient of 0.90 with the same-period returns on the S&P 500 index ($R^2 = 81\%$, see Figure 1). Furthermore, using sectorwise data, we find that the value of R^2 is correlated with the fraction of discretionary items sold as a percentage of total sales. For example, discretionary items constitute a larger fraction of total sales for apparel stores and department stores than discount stores. Correspondingly, apparel stores and department stores have a higher correlation of demand with the S&P 500 index than discount stores (see Table 1). Our results are also supported by firm-level analysis.

¹ *The Redbook Average* is a seasonally adjusted sales-weighted average of year-to-year same-store sales growth in a sample of 60 large U.S. general merchandise retailers representing about 9,000 stores (Instinet Research 2001a, b). It is released by Instinet Research on the first Thursday of every month.

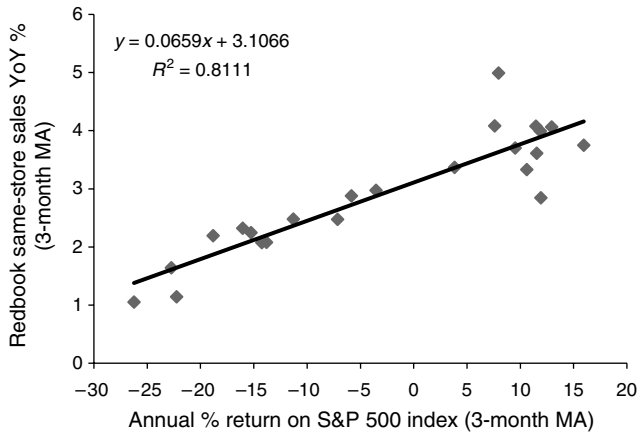
Figure 2 shows that for The Home Depot Inc.,² sales per customer transaction and sales per square foot both have statistically significant correlation with the value of the S&P 500 index. Their R^2 values are equal to 79.11% and 39.92%, respectively.

These findings present an opportunity to use financial market information to improve demand forecasting and inventory planning, and use financial contracts to mitigate (hedge) the risk in carrying inventory. This paper addresses these problems for discretionary purchase items based on a forecasting model that incorporates the subjective assessment of the retailer and the price information of a financial asset.

We show how to construct static hedging strategies in both the mean-variance framework and the more general utility-maximization framework. In the mean-variance framework, we determine the optimal portfolio that minimizes the variance of profit for a

² Home Depot is a retail chain selling home construction and home furnishing products. We use public data from the first quarter of fiscal 1997 to the second quarter of fiscal 2001, a total of 22 quarterly observations. The data are obtained from the 10-K and 10-Q reports that Home Depot files with the Securities and Exchange Commission.

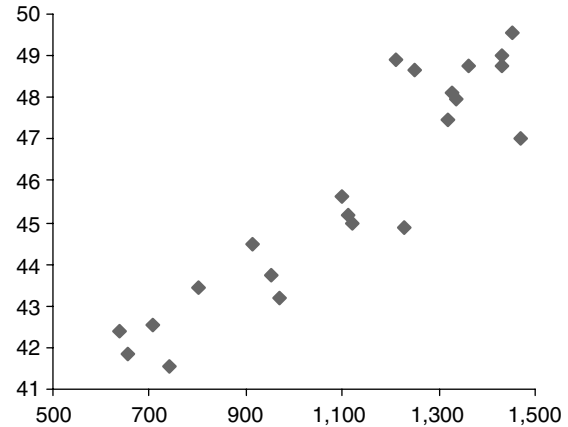
Figure 1 *Redbook Same-Store Sales Growth Rate vs. Annual Return on the S&P 500 Index*



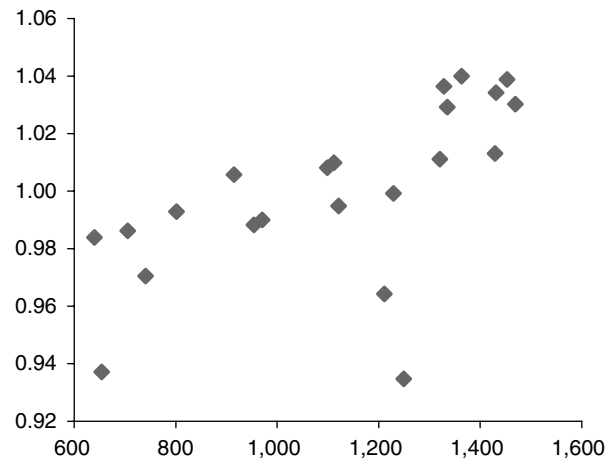
Notes. The *Redbook* year-on-year same-store sales growth rate is computed as the growth rate of same-store sales during a month with respect to the same month in the previous year. We use time-series data for 25 months from November 1999 to November 2001. We compute three-month moving averages to reduce autocorrelation.

given inventory level. This hedge could be complex to create in practice. Therefore, we motivate a heuristic hedging strategy and evaluate its performance with respect to the optimal hedge. We similarly derive the structure of the optimal hedging strategy in the utility-maximization framework. Finally, we analyze the impact of hedging on the expected utility of the decision maker and on the optimal inventory decision. The overall contribution of this paper is to show how to generate a solution to the hedging problem and analyze how the hedging solution affects inventory levels. We note, however, that most of the results

Figure 2 *Quarterly Sales of Home Depot vs. Corresponding Values of the S&P 500 Index*



(a) Sales per customer transaction



(b) Sales per square foot

Table 1 *Results for the Regression of Sectorwise Redbook Same Store Sales Growth Rate on the Annual Return on the S&P 500 Index*

	R^2	F-statistic	p-value	Intercept	Slope
Apparel	0.411	7.6618	0.0183	-0.583 2.168	0.351 0.127
Department stores	0.468	9.6673	0.0099	2.088 1.182	0.215 0.069
Discount stores	0.020	0.2298	0.6410	3.596 0.890	-0.025 0.052

Notes. (1) The analysis uses monthly data for 13 months from November 2000 to November 2001 for each sector. The dependent variable is the growth rate of same-store sales during the month with respect to the same month in the previous year. The independent variable is the annual return on the S&P 500 index for the same period. (2) The standard errors of the parameters are reported below the corresponding parameter estimates.

derived in this paper are either standard or obtained by a combination of standard results.

Researchers in inventory theory have considered both risk-neutral and risk-averse decision makers, but none of these researchers has studied the impact of hedging on decision making. According to the received theory, a risk-neutral decision maker is unaffected by the variance of profit, thus is indifferent toward hedging inventory risk (for example, see Hadley and Whitin 1963, Lee and Nahmias 1993, Nahmias 1993, Porteus 2002, Zipkin 2000). For a risk-averse decision maker, it is well known that the expected utility-maximizing inventory level is less

than the expected value-maximizing inventory level (for example, see Agrawal and Seshadri 2000a, b; Chen and Federgruen 2000; Eeckhoudt et al. 1995; and the papers cited therein). Although it seems reasonable to conjecture that risk-averse decision makers will prefer to hedge inventory risk, it is less obvious that the hedge will also lead to an increase in the quantity ordered. We show that hedging impacts both types of decision makers:

(1) Hedging reduces the variance of profit and increases expected utility. The reduction in the variance of profit is directly proportional to the correlation of demand with the price of the asset.

(2) It provides an incentive to a risk-averse decision maker to order a quantity that is closer to the expected value-maximizing quantity. This result holds for a wide range of hedging strategies and for all increasing concave utility functions with constant or decreasing absolute risk aversion.

(3) The hedging transactions do not require additional investment. On the contrary, the funds required to finance inventory at the beginning of the planning period are offset by the cash flows from the hedging transactions, so that the net inventory investment of the firm is reduced.

The last result shows that hedging is useful even to a risk-neutral decision maker although he or she may not be interested in reducing the variance of profit in a perfect market.³ Hedging is especially useful to small privately owned firms, e.g., the so-called Mom and Pop retail stores, because risk reduction provides them access to capital, reduces the cost of financial distress, and enables the owners to diversify their risk and increase their return on investment.

We present a numerical study using data from a retailing firm to quantify the impact of our results on forecasting demand, optimal inventory planning, risk reduction, and return on investment. Because the forecast is a function of the asset price, the retailer can dynamically update it with changes in asset price or the volatility of the asset. Therefore, it improves the optimal inventory decision and impacts the expected profit of the retailer. The increase in the expected

profit in our numerical study is as high as 7%. The numerical study also shows that hedging reduces the variance of profit by 12.5% to 56.5%. The amount of reduction is a function of the correlation of demand with the asset price, the volatility of the asset, and the lead time. Compared with the optimal hedge, the heuristic hedge proposed by us achieves 6% less reduction in the variance of profit. Dynamic hedging using the heuristic strategy reduces the variance of profit to within 0.5% of the optimal hedge.

This paper is also related to the real-options literature. For example, Dasu and Li (1997), Huchzermeier and Cohen (1996), Kogut and Kulatilaka (1994), and Kouvelis (1999) consider the valuation of real options wherein the cash flows from real assets depend on the price of a traded security, such as the exchange rate. Other authors, including Birge (2000), Brennan and Schwartz (1985), McDonald and Siegel (1986), Triantis and Hodder (1990), and Trigeorgis (1996) consider the valuation of real options using the assumption that the cash flows from a basecase scenario or a portfolio of marketed securities, or both, can be used to replicate the cash flows from the real option. When this assumption holds, the value of the option can be set equal to the value of the replicating portfolio. (This assumption is called the Marketed Asset Disclaimer. See Copeland and Antikarov 2001, p. 94.) Our paper differs from this literature in two aspects. First, we do not focus on valuation. Instead, we focus on the interaction between real options and financial hedging by analyzing how the optimal inventory decision changes with hedging and with the degree of correlation of demand with the underlying asset. Second, we use neither the marketed asset disclaimer nor the assumption that the cash flows corresponding to each inventory level are traded in a perfect market to construct the hedge. Instead, as set out in the first paragraph of the paper, we justify the application of risk-neutral valuation by demonstrating the correlation of demand with financial assets, and show in §4 how to incorporate this information in a forecasting model to plan inventory.

The paper is organized as follows. We set up the framework of our analysis in §2 by using a model in which demand is perfectly correlated with the price of an underlying asset. In §3, we analyze the model with partial demand correlation and establish

³ When there are market imperfections, for example, if bankruptcy is costly, then even a risk-neutral decision maker may prefer to purchase insurance.

the properties of the hedged payoff function resulting from the newsvendor model. Section 4 presents a numerical example to illustrate the results of our model. Section 5 concludes the paper with directions for future research.

2. Demand Perfectly Correlated with the Price of a Marketable Security

We consider a single-period, single-item inventory model with stochastic demand, i.e., the newsvendor model. To establish the basic ideas, we first consider the case when the demand forecast for the item is perfectly correlated with the price at time T of an underlying asset that is actively traded in the financial markets. The analysis in this section is based on the theory of valuing real options.

Let p denote the selling price of the item, c the unit cost, s the salvage value, I the stocking quantity, and D the demand. The firm purchases quantity I at time 0 and demand occurs at a future time T . Demand in excess of I is lost, and any excess inventory is liquidated at the salvage price of s . The firm's cash flows at times 0 and T , respectively, are

$$\begin{aligned}\Pi_U^0(I) &= -cI, \quad \text{and} \\ \Pi_U^T(I) &= p \min\{D, I\} + s(I - D)^+.\end{aligned}\quad (1)$$

We use the subscript U to denote *unhedged cash flows*, i.e., cash flows before any financial transactions, and the subscript H to denote *hedged cash flows*, i.e., cash flows including financial transactions.

Let S_0 be the current price of the financial asset, S_T be its price at time T , and r be the risk-free rate of return per annum. We assume that the financial market is complete and has a unique risk-neutral pricing measure (RNPM).⁴ Let E_N denote the expectation under the risk-neutral probability measure. Thus, we have $S_0 = e^{-rT} E_N S_T$. To distinguish expectation under the RNPM from expectation under the decision maker's subjective probability measure, we shall denote expectation under the subjective measure as $E[\cdot]$, and conditional expectation under the subjective measure over a random variable ζ as $E_\zeta[\cdot]$.

⁴ This assumption can be relaxed further because market completion is not a necessary condition. All we need is that the no-arbitrage principle should hold in the market and that the claim S_T should have a unique price at time 0 (see Pliska 1999, chap. 1).

We use the correlation of demand with S_T in three ways: to value the newsvendor profit function, to construct transactions to hedge inventory risk, and to exploit the benefits of hedging. Because the demand is perfectly correlated with S_T , we specify it as $D = a + bS_T$, where a and b are constants. We assume that $b > 0$ to ensure that the demand is nonnegative, and that $I > \max\{a, 0\}$, otherwise $\Pi_U^T(I)$ will be a deterministic quantity equal to $p \max\{a, 0\}$ that requires no risk analysis. We also assume that $p > ce^{rT} > s$, otherwise the newsvendor problem has trivial solutions at either $I = 0$ or $I = \infty$.

Substituting the demand forecast in (1) and simplifying the expression for Π_U^T , we get

$$\begin{aligned}\Pi_U^T(I) &= (p - s) \min\{a + bS_T, I\} + sI \\ &= (p - s)bS_T + [(p - s)a + sI] \\ &\quad - (p - s)b \max\{S_T - (I - a)/b, 0\}.\end{aligned}\quad (2)$$

Equation (2) reveals that the random payoff from the newsvendor model can be represented as a portfolio comprising only financial assets. This portfolio is said to replicate the newsvendor payoff function. It follows from standard valuation theory that this portfolio can be used not only to value $\Pi_U^T(I)$, but also to hedge the inventory risk. Because demand is perfectly correlated with S_T , the hedge is perfect and completely eliminates the uncertainty in newsvendor profits. The hedging transactions at time 0 are

(1) Borrow and sell $(p - s)b$ units of the underlying asset at the current price S_0 . The borrowed asset is to be replaced at time T by purchasing $(p - s)b$ units of the asset from the market at price S_T .

(2) Buy $(p - s)b$ call options on this asset with exercise price $(I - a)/b$ and settlement date T .

(3) Borrow a sum of money equal to $[(p - s)a + sI]e^{-rT}$ at the risk-free rate to be repaid at time T .

These hedging transactions have several benefits. Through hedging, the net payoff to the retailer in all states of nature at time T is zero, and the hedged profit is realized at time 0 itself. This profit is given by

$$\begin{aligned}\Pi_H(I) &= (p - s)bS_0 + e^{-rT}[(p - s)a + sI] \\ &\quad - (p - s)be^{-rT}E_N[\max\{S_T - (I - a)/b, 0\}] - cI.\end{aligned}$$

The hedged profit is identical to the expected newsvendor profit for any inventory I , i.e.,

$$E_N[\Pi_U(I)] = \Pi_H(I) \quad \text{and} \quad \text{Var}[\Pi_H(I)] = 0,$$

where $\Pi_U(\cdot)$ denotes the total unhedged profit discounted to time 0. Therefore, the expected value-maximizing inventory decision remains the same regardless of the decision to hedge the market exposure. Notably, because the hedged profit is realized at time 0, we find that hedging reduces the retailer's investment to zero in all cases when there is perfect correlation. Moreover, there is no need to revise the hedge over time.

3. Demand Partially Correlated with the Price of a Marketable Security

Let the demand forecast for T periods hence be given by

$$D = a + bS_T + \epsilon',$$

where ϵ' is an error term independent of S_T such that $E[\epsilon'] = 0$ and $E[\epsilon'^2] < \infty$. In this model, a is a function of the firm's subjective forecast, b gives the slope of demand with respect to S_T , and ϵ is the firm's subjective forecast error. Section 4, Equation (20), shows how to incorporate both the subjective forecast and the market information in the same forecasting model.

Define $\epsilon = \epsilon'/b$ and $\tilde{S}_T = S_T + \epsilon'/b = S_T + \epsilon$, so that demand can equivalently be written as $D = a + b\tilde{S}_T$. As in §2, we assume that $b > 0$ and $I > \max\{a, 0\}$. Additionally, we assume that a is sufficiently large that the probability of demand being negative is negligible. Thus, analogous to the case of perfectly correlated demand, the unhedged newsvendor cash flow can be written in terms of \tilde{S}_T as

$$\begin{aligned} \Pi_U^0(I) &= -cI, \quad \text{and} \\ \Pi_U^T(I) &= (p-s)b\tilde{S}_T - (p-s)b \max\{\tilde{S}_T - (I-a)/b, 0\} \\ &\quad + (p-s)a + sI. \end{aligned} \quad (3)$$

Suppose that the cash flow at time T is hedged by short selling $(p-s)b$ units of a portfolio derived from the underlying asset. Let X_T denote the cash flows of the hedging portfolio at time T , and X_0 denote its price at time 0. Because the financial market is arbitrage free and frictionless, we have $X_0 = e^{-rT}E_N[X_T]$. The hedged cash flow is no longer deterministic, because ϵ cannot be replicated in the financial market. It is therefore natural to split the hedged

cash flow into components at time 0 and at time T as shown below:

$$\begin{aligned} \Pi_H^0(I) &= -cI \quad \text{purchase inventory} \\ &\quad + (p-s)bX_0 \quad \text{short sell } X_T, \end{aligned} \quad (4)$$

$$\begin{aligned} \Pi_H^T(I) &= (p-s)b\tilde{S}_T - (p-s)b \max\{\tilde{S}_T - (I-a)/b, 0\} \\ &\quad + [(p-s)a + sI] \quad \text{realize sales} \\ &\quad - (p-s)bX_T \quad \text{cover short sale of } X_T. \end{aligned} \quad (5)$$

Let the present value of the total hedged profit discounted at the risk-free rate be $\Pi_H(I)$. From (4) and (5),

$$\Pi_H(I) = \Pi_H^T(I)e^{-rT} + \Pi_H^0(I).$$

Consider the expectation under the RNPM of the hedged profit similar to the case of perfectly correlated demand. Using the fact $E_N[X_T] = e^{rT}X_0$, we find that

$$E_N[\Pi_U(I)] = E_N[\Pi_H(I)].$$

Therefore, it follows that if I^* maximizes the expected value under the RNPM of the newsvendor profit function $E_N[\Pi(I)]$, then I^* also maximizes the expected value under the same probability measure of the hedged newsvendor profit function $E_N[\Pi_H(I)]$, regardless of the hedging transactions used.

However, it is no longer possible to hedge the inventory risk perfectly. As a consequence, unlike the case in §2, we cannot provide a no-arbitrage argument for the use of RNPM to compute the optimal inventory level. Moreover, different decision makers could prefer different hedging transactions depending on their utility functions. Therefore, from here forward, we assume that the decision maker is risk averse. We use the subjective probability measure in our analysis. In §3.1, we analyze the hedged payoff (5) in the mean-variance framework. In §3.2, we provide results for more general utility functions.

3.1. Minimum Variance Hedge

Minimum variance hedging is a common solution concept used for a risk-averse decision maker (see, for example, Hull 2002, chap. 2). Therefore, in this section, we first determine the optimal hedging portfolio X_T that minimizes the variance of the payoff at time T for a given inventory level I , and then consider a heuristic hedge comprising fewer financial transactions. Finally, in §3.1.1, we examine dynamic

hedging, i.e., the rebalancing of the hedging portfolio when new information regarding the asset price and the demand forecast becomes available over time. The optimal solution to the problem of minimizing the variance of $\Pi_H^T(I)$ with respect to X_T for given I can be obtained from a standard result in probability theory; see, for example, §9.4 in Williams (1991). Specifically,

LEMMA 1. *The variance of $\Pi_H^T(I)$ is minimized by setting $X_T^* = E_\epsilon[\Pi_U^T(I) | S_T]$.*

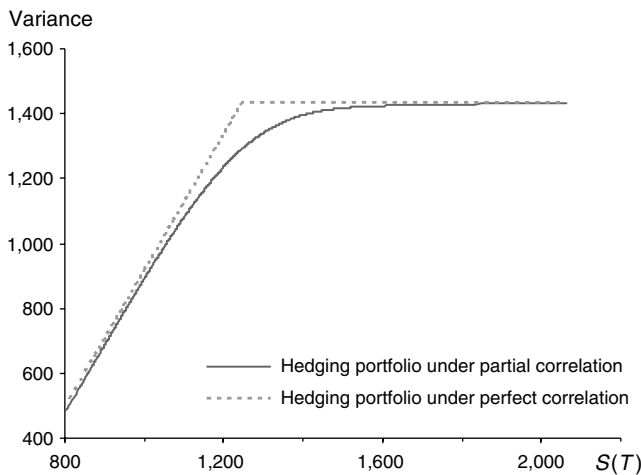
PROOF. Omitted. \square

Figure 3 depicts the cash flows of the minimum variance hedge as a function of S_T . For comparison, it also shows the hedging portfolio when $\epsilon = 0$. It can easily be shown that the hedge is a concave increasing function of S_T . Thus, the hedge can be approximated by short selling the underlying asset and purchasing a series of call options with settlement date T and different exercise prices. As a first-order approximation, consider a hedging portfolio consisting of a short sale of the asset and a purchase of call options with a single exercise price, s_p . Let

$$X_T = (p-s)b\alpha S_T - (p-s)b\beta \max\{S_T - s_p, 0\}.$$

We refer to this portfolio as the *heuristic hedge* with parameters α , β , and s_p , where α and β are the hedge ratios. Let $C(s_p)$ denote the cost at time 0 of purchasing a European call option on the underlying financial asset with an exercise price of s_p and

Figure 3 Cash Flows of the Minimum Variance Hedging Portfolio X_T^* as a Function of S_T



Notes. This example uses dataset B described in §4 with $I = 6$.

settlement date T . From Hull (2002, chap. 11), $C(s_p) = e^{-rT} E_N[\max\{S_T - s_p, 0\}]$.

The decision maker seeks to determine α , β , and s_p , such that the variance of $\Pi_H^T(I)$ is minimized. For simplicity, we shall ignore the constant scale factor $(p-s)b$ in (5). We first determine α and β for given s_p .

$$\min_{\alpha, \beta} \text{Var}[\tilde{S}_T - \max\{\tilde{S}_T - (I-a)/b, 0\} - \alpha S_T + \beta \max\{S_T - s_p, 0\}]. \quad (6)$$

Because ϵ is independent of S_T , it follows that

$$\text{Cov}(\tilde{S}_T, S_T) = \text{Var}(S_T) \quad \text{and}$$

$$\text{Cov}(\tilde{S}_T, \max\{S_T - s_p, 0\}) = \text{Cov}(S_T, \max\{S_T - s_p, 0\}).$$

Expanding (6) and using this simplification, we obtain

$$\begin{aligned} \min_{\alpha, \beta} & [(1-\alpha)^2 \text{Var}(S_T) + \beta^2 \text{Var}[\max\{S_T - s_p, 0\}] \\ & + 2\alpha \text{Cov}(S_T, \max\{\tilde{S}_T - (I-a)/b, 0\}) \\ & - 2\alpha\beta \text{Cov}(S_T, \max\{S_T - s_p, 0\}) \\ & + 2\beta \text{Cov}(S_T, \max\{S_T - s_p, 0\}) \\ & - 2\beta \text{Cov}(\max\{S_T - s_p, 0\}, \max\{\tilde{S}_T - (I-a)/b, 0\}) \\ & + \text{terms independent of } \alpha, \beta \text{ and } s_p]. \quad (7) \end{aligned}$$

Let A, B, C, D , and E denote $\text{Var}(S_T)$, $\text{Cov}(S_T, \max\{S_T - s_p, 0\})$, $\text{Cov}(\max\{S_T - s_p, 0\}, \max\{\tilde{S}_T - (I-a)/b, 0\})$, $\text{Cov}(S_T, \max\{\tilde{S}_T - (I-a)/b, 0\})$, and $\text{Var}[\max\{S_T - s_p, 0\}]$, respectively. Ignoring the terms independent of α , β , and s_p , we rewrite (7) as

$$\min_{\alpha, \beta} [(1-\alpha)^2 A + 2\beta(1-\alpha)B - 2\beta C + 2\alpha D + \beta^2 E]. \quad (8)$$

This problem is similar to the minimization of the squared error in regression. Therefore, by applying standard procedures, we obtain the following proposition:

PROPOSITION 1. *If $s_p > 0$, then the function in (8) is strictly convex in α and β , and the minimum variance hedge is obtained by setting*

$$\alpha = 1 - \frac{DE - BC}{AE - B^2} \quad (9)$$

$$\beta = \frac{AC - BD}{AE - B^2}. \quad (10)$$

PROOF: Omitted. \square

Now consider the choice of s_p . As pointed out by a referee, it is no longer optimal in general to set $s_p = (I - a)/b$ as it was in the case of perfectly correlated demand in §2. Furthermore, we found examples in which the variance function is not jointly quasi-convex in α , β , and s_p . Therefore, the optimal value of s_p has to be determined numerically by doing a line search on s_p with the values of α and β , as given by Proposition 1. This method gives the correct answer, because the variance is strictly convex in α and β for a given s_p .

The following example illustrates how the variance of the hedged profit behaves as a function of s_p .

EXAMPLE. Let $S_0 = \$660$, $r = 10\%$ per annum, and S_T have a log-normal distribution with $\mu = 10\%$ per annum and $\sigma = 20\%$ per annum. Let $T = 6$ months. Let the demand for the item be $10S_T + \epsilon$, where ϵ is normally distributed with mean 0 and standard deviation 600. Let $p = \$1$, $c = \$0.60$, and $s = \$0.10$.

Let $I = 7,000$. The variance of the unhedged profit function is equal to 371,280. Figure 4 shows the variance of the hedged profit as a function of s_p for different values of α and β . Whereas $(I - a)/b$ is equal to 700, the optimal value of s_p differs from 700 for each set of α and β values. For example, when $\alpha = 1.1$ and

$\beta = 0.8$, then the minimum variance is 161,301 and is realized at $s_p = 630$; when $\alpha = 0.65$ and $\beta = 0.9$, then the minimum variance is 155,145 and is realized at $s_p = 770$. The global optimal solution is $\alpha = \beta = 0.75$ and $s_p = 722$, and gives a variance of 146,400.

Now let $I = 8,950$. It can be shown that for $\alpha = 0.851$ and $\beta = 0.02$, the variance function has a local maximum at $s_p = 472$ and a local minimum at $s_p = 762$. Thus, the variance function is not quasiconvex in s_p .

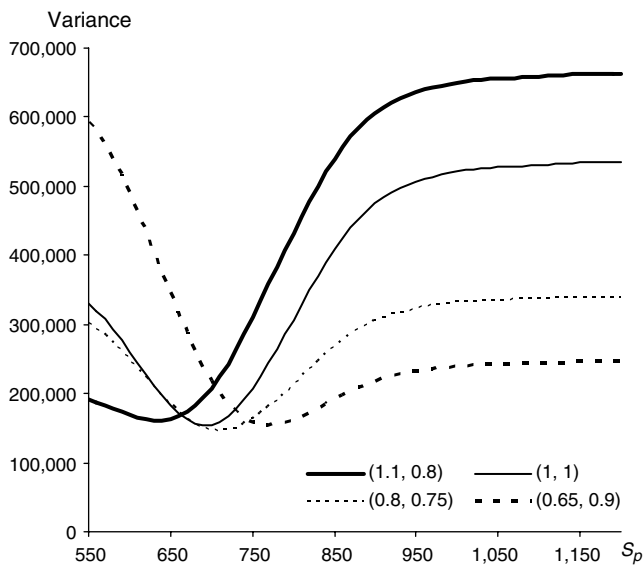
The minimum variance hedge, as given by Lemma 1, gives a lower bound for the variance of the hedged profit. Thus, the effectiveness of the heuristic hedge can be ascertained by benchmarking it against this lower bound. We provide such comparisons in §4.

We note that the optimal values of α and β in the heuristic hedge are always nonnegative. Thus, the hedging transactions always consist of a short sale of the asset and a purchase of call options (the interested reader is referred to Appendix B in Gaur and Seshadri 2004). This implies that the payoffs from these two transactions offset each other, so that the market exposure of the firm is reduced. Moreover, the sale of the asset at time 0 provides cash to finance the investment in inventory. Thus, the net investment required by the firm is reduced and its return on investment is increased. The numerical study in §4 shows the impact of minimum variance hedging on risk, inventory levels, and return on investment.

Our method for computing the hedge parameters can easily be generalized to the case when several call options with different exercise prices are considered in the hedging portfolio. Even for this case, the variance function remains convex in the hedge ratios and the formulas for the optimal hedge ratios can be computed similarly.

3.1.1. Dynamic Hedging. When demand is partially correlated with the price of the underlying financial asset, the demand forecast may change with time as new information is revealed. Suppose that there are $T + 1$ trading time instants, $t = 0, \dots, T$. At each time t , the price, S_t , of the underlying asset is observed. Also suppose that the forecast error at time t is given by ϵ_t , where $E[\epsilon_t] = 0$. In this manner, the demand forecast is updated with time. Thus, the decision maker can use dynamic hedging, i.e., he or she can trade between times $t = 0$ and $t = T$ to rebalance the hedging portfolio.

Figure 4 Variance of Hedged Profit as a Function of s_p for Different Values of Hedge Ratios α and β



Notes. The legend gives the values of α and β , respectively.

Such information revelation has implications on whether the hedging strategy is self-financing. A trading strategy is said to be *self-financing* if the time t values of the portfolio just before and just after any time t transactions are equal. In a self-financing trading strategy, no money is added to or withdrawn from the portfolio at times $t = 1$ to $T - 1$. Such a trading strategy has a unique value at time $t = 0$ (see Pliska 1999, chap. 3). In our case, the following proposition gives the implications of information revelation on the hedging strategy:

PROPOSITION 2. *If new information is revealed at times $t = 0, \dots, T - 1$ about S_T but not about ϵ , then the minimum variance trading strategy, X_T^* , defined in Lemma 1 is self-financing. If, instead, new information is revealed at times $t = 0, \dots, T - 1$ about both S_T and ϵ , then the minimum variance trading strategy, X_T^* , defined in Lemma 1 is not in general self-financing.*

Thus, if both the asset price and the subjective forecast error are updated with time, then dynamic hedging can result in net cash flows different from zero at intermediate time instants. The hedging strategy can no longer be uniquely valued at time 0 because it depends on the decision maker's utility function. Nevertheless, dynamic hedging can give further reduction in the variance of the profit. Section 4 evaluates the benefits of such dynamic hedging.

The benefits of revising the hedge possibly increase when inventory commitments can be changed or can be made at more than one time epoch. Analysis of these issues is deferred to future work.

3.2. Risk Aversion

This section analyzes whether a risk-averse decision maker will choose to hedge, the form of the hedging portfolio for such a decision maker, and the impact of hedging on operational decisions.

Consider a risk-averse decision maker with a concave utility function, $u: \mathfrak{R} \rightarrow \mathfrak{R}$. Let W_0 denote the initial wealth of the decision maker before investing in inventory or undertaking any financial transactions. To facilitate comparison between the hedged and unhedged payoffs, we transfer all payoffs to time T by investing the certain payoffs at time 0 at the risk-free rate, denoted r . Thus, the expected utility of the

decision maker after purchasing inventory I takes the form

$$E[u(\Pi_U(I))] = E[u(W_0 e^{rT} + (p-s)b \min\{S_T + \epsilon, (I-a)/b\} + (p-s)a - (ce^{rT} - s)I)].$$

Without loss of generality, we scale all cash flows by $1/(p-s)b$. Let c_1 denote $(ce^{rT} - s)/\{(p-s)b\}$, and W denote $\{W_0 e^{rT} + (p-s)a\}/\{(p-s)b\}$. Thus, we write $\Pi_U(I) = \min\{S_T + \epsilon, (I-a)/b\} - c_1 I$ and

$$E[u(\Pi_U(I))] = E[u(W + \min\{S_T + \epsilon, (I-a)/b\} - c_1 I)]. \quad (11)$$

The decision maker can access the financial market and construct a portfolio derived from the underlying asset S_T at zero transaction cost. Given this alternative, the decision maker may or may not prefer to invest in inventory depending on the parameters of the newsvendor model. The following proposition specifies the range of parameter values under which the decision maker prefers to invest in inventory.

PROPOSITION 3. *Any risk-averse decision maker with utility function, $u(\cdot)$, prefers to invest in inventory I than to invest solely in the financial market if $c_1 I < E_N[E_\epsilon[\min\{S_T + \epsilon, (I-a)/b\} | S_T]]$. In particular, the decision maker prefers to invest in inventory I if $c_1 I < E_N[\min\{S_T, (I-a)/b\}] - E[|\epsilon|]$ and does not prefer to invest in inventory I if $c_1 I > E_N[\min\{S_T, (I-a)/b\}]$.*

This result can be explained as follows: The payoff $E_\epsilon[\min\{S_T + \epsilon, (I-a)/b\} | S_T]$ can be constructed from derivative instruments with S_T as the underlying asset. This payoff is larger than $\Pi_U + c_1 I$ in the second-order stochastic dominance sense and costs $E_N[E_\epsilon[\min\{S_T + \epsilon, (I-a)/b\} | S_T]]$. Thus, if $c_1 I > E_N[E_\epsilon[\min\{S_T + \epsilon, (I-a)/b\} | S_T]]$, then we obtain a financial asset that is preferred to the profits from inventory I and costs less than the investment of $c_1 I$ in inventory. Thus, any risk-averse decision maker chooses to invest in this asset rather than in inventory level I .

In the rest of the analysis, we assume that $c_1 I$ is such that the decision maker prefers to invest in inventory I . Let there be a hedging portfolio with the random payoff at time T denoted as X_T , and the price at time 0 denoted as X_0 . We assume that X_T is a

fair gamble, i.e., $e^{-rT}X_T$ is a martingale and the decision maker must construct the hedge subject to this assumption. Thus, we require that

$$E[X_T - X_0 e^{rT}] = 0. \quad (12)$$

If, instead, X_T had a positive risk premium, then there will be two effects of investing in X_T on the decision maker's expected utility: a wealth effect and a risk-reduction effect. By assuming that X_T is a fair gamble, we examine the risk-reduction effect while controlling for the wealth effect.

First, consider the problem of determining X_T such that the expected utility of the decision maker is maximized for a given inventory level I , i.e.,

$$\begin{aligned} \max \quad & E[u(W + \Pi_U(I) - X_T + X_0 e^{rT})] \\ \text{such that} \quad & E[X_T - X_0 e^{rT}] = 0. \end{aligned} \quad (13)$$

The following proposition specifies the form of the hedge using the Karush-Kuhn-Tucker conditions (KKT). It provides broad insights but is primarily useful for the purposes of constructing an optimal hedge:

PROPOSITION 4. *For a strictly concave utility function, $u(\cdot)$, the optimal solution of Problem (13) is given by X_T^{**} that satisfies the following equations for some $\lambda \in \Re$:*

$$\begin{aligned} E_\epsilon[u'(W + \Pi_U(I) - X_T^{**} + X_0^{**} e^{rT})] &= \lambda, \\ E[X_T^{**} - X_0^{**} e^{rT}] &= 0. \end{aligned}$$

For example, note that for a quadratic utility function, Proposition 4 yields the same solution as given in Lemma 1. To see this, let $u(x) = ax - bx^2/2$ where $a, b > 0$. Then, Proposition 4 gives

$$\begin{aligned} E_\epsilon[a - b(W + \Pi_U(I) - X_T^{**} + X_0^{**} e^{rT})] \\ = a - b(W - X_T^{**} + X_0^{**} e^{rT}) - bE_\epsilon[\Pi_U(I)] = \lambda. \end{aligned}$$

Because a, b, W are constants, $X_T^{**} = E_\epsilon[\Pi_U(I)] = X_T^*$ is an optimal solution to the above problem. As stated in §3.1, Figure 3 depicts X_T^* as a function of S_T . Optimal hedges for other utility functions could be similarly derived.

We now consider the questions whether the expected utility of the decision maker increases with hedging, and whether the optimal inventory level increases in the degree of hedging. In practice, simpler hedges than the hedge in Proposition 4 may be

used. Therefore, we consider a fairly general class of hedges in the remaining analysis. We assume that X_T is an increasing function of S_T because it should offset the subject cash flows as closely as possible. The minimum variance hedge considered in §3.1 satisfies this assumption. The heuristic hedge in §3.1 also satisfies this assumption when $\beta \leq \alpha$ regardless of the value of s_p . Furthermore, the hedging strategy permits the use of several call options with different exercise prices in order to match Π_U more closely. To facilitate the analysis, we also assume that X_T is a piecewise continuous function of S_T , and is differentiable with respect to S_T almost everywhere.

The following properties of utility functions are useful in the remaining analysis. The Arrow and Pratt measure of absolute risk aversion of a utility function $u(\cdot)$ of wealth w is defined as the ratio $R_A(w) = -u''(w)/u'(w)$ (Arrow 1971). Note that $R_A(w)$ is always nonnegative for an increasing concave-utility function. The utility function is said to display decreasing absolute risk aversion (DARA) when $R_A(w)$ is decreasing in w , and constant absolute risk aversion (CARA) when $R_A(w)$ is a constant. Both DARA and CARA further imply that $u''(w)$ is decreasing in absolute value (i.e., $u'''(w) \geq 0$). Absolute prudence is defined as the ratio $-u'''(w)/u''(w)$ (Kimball 1990). Decreasing or constant absolute prudence implies that $u'''(w)$ is decreasing in w . Many commonly used classes of utility functions, such as the power utility function, the negative exponential utility function, and the logarithmic utility function, satisfy the properties of constant or decreasing absolute prudence (DAP).

Suppose that the risk-averse firm shorts an amount α of the portfolio X_T . Thus, the expected utility of the decision maker from the investment in inventory and from the hedging transactions is given by

$$\begin{aligned} E[u(\Pi_H(I, \alpha))] &= E[u(W + \min\{S_T + \epsilon, (I - a)/b\} \\ &\quad - c_1 I - \alpha X_T + \alpha X_0 e^{rT})]. \end{aligned} \quad (14)$$

Proposition 5 shows that a risk-averse decision maker prefers the hedged newsvendor payoff to the unhedged newsvendor payoff for any given I .

PROPOSITION 5. *For any concave and differentiable utility function, $u(\cdot)$,*

$$\frac{d}{d\alpha} E[u(\Pi_H(I, \alpha))] |_{\alpha=0} \geq 0. \quad (15)$$

Although this result by itself is not surprising, it should be considered as a counterpart to Proposition 3. Together they show that, under appropriate conditions, a risk-averse decision maker will both invest in inventory and hedge the risk using financial instruments. We now examine the implications of hedging on the optimal inventory level. Propositions 6 and 7 give two sufficient conditions under which the optimal stocking quantity chosen by a risk-averse decision maker increases when he or she decides to hedge the newsvendor risk. Given the utility function, Proposition 6 gives a sufficient condition on the structure of the hedging portfolio, derived using the form of the newsvendor payoff and Jensen's inequality. Given the hedging portfolio, Proposition 7 gives a sufficient condition on the utility function.

PROPOSITION 6. *For any increasing, concave, and differentiable utility function, $u(\cdot)$, the value of inventory that maximizes $E[u(\Pi_H)]$ is greater than the value of inventory that maximizes $E[u(\Pi_U)]$ if X_T is such that*

$$E[\{u'(\Pi_H(I)) - u'(\Pi_U(I))\} \cdot 1\{\epsilon \leq (I - a)/b - S_T\}] \leq 0, \quad (16)$$

and

$$E[u'((I - a)/b - c_1 I - \alpha X_T + \alpha X_0 e^{rT}) - u'((I - a)/b - c_1 I)] \geq 0. \quad (17)$$

If $u(\cdot)$ is additionally CARA or DARA, then (17) is automatically satisfied.

Proposition 6 is primarily useful for constructing an optimal hedge. The intuitive content of the proposition is that when S_T , and thus, Π_U is small (i.e., when (16) applies), then the hedge increases the utility of the decision maker. When S_T is large (i.e., when (17) applies), however, then the hedge decreases the utility of the decision maker.

Proposition 7 considers hedging portfolios that are increasing in S_T , and gives a sufficient condition on the decision maker's utility function. Thus, Proposition 7 supplements the result in Proposition 6. Let $I^*(\alpha) \equiv \arg \max_I \{E[u(\Pi_H(I, \alpha))]\}$ denote the stocking quantity that maximizes expected utility for a given value of α . Also, let $\bar{\alpha}$ be the largest value of α such that $E_\epsilon[\Pi_H(I, \alpha) | S_T]$ is nondecreasing in S_T . We focus attention on hedging ratios in the range

$0 \leq \alpha \leq \bar{\alpha}$ because higher values of α correspond to overhedging.

PROPOSITION 7. *For any increasing, concave, and differentiable utility function, $u(\cdot)$, with constant or decreasing absolute risk aversion and constant or decreasing absolute prudence, $dI^*/d\alpha \geq 0$ for $0 \leq \alpha \leq \bar{\alpha}$.*

We relate these results to the research on the impact of risk aversion on operational decisions, specifically on the newsvendor model. Eeckhoudt et al. (1995) show that the optimal inventory level for a risk-averse newsvendor is lower than that for a risk-neutral newsvendor under both CARA and DARA preferences. Similar conclusions are to be found in Agrawal and Seshadri (2000a, b) and Chen and Federgruen (2000). Propositions 6 and 7 add to the above research by showing that financial hedging changes the optimal inventory decision for a risk-averse newsvendor under various conditions. In particular, financial hedging increases the optimal inventory level for the risk-averse newsvendor, and brings it closer to the risk-neutral profit-maximizing quantity. Thus, it increases expected profit, decreases the effect of risk aversion, and brings the market closer to efficiency.

We also note that Eeckhoudt et al. (1995) make similar assumptions on the utility function as in Proposition 7, and show that the optimal inventory level decreases when an uncorrelated background risk is added. Our result differs from their results because the new risk X_T is correlated with the investment in inventory. Therefore, although Eeckhoudt et al. (1995) find that optimal inventory decreases with the addition of the background risk, we find that the optimal inventory increases with hedging.

4. Numerical Example

In this section, we quantify the impact of our method on expected profit, risk reduction, and return on investment using a numerical example. We show how the benefits of hedging change with the degree of correlation of demand with the price of the underlying financial asset, with the volatility of the asset price, and with dynamic hedging. The example is based on sales data for computer game CDs sold at a consumer electronics retailing chain.

We are given the following datasets.

(1) A fit sample consisting of historical monthly forecasts and sales data for 42 items for one year

Table 2 Results of the Estimation of Forecasting Models (18) and (19) for Item 1

Forecasting model	Dataset	m_1	m_2	b	R^2	F -statistic	Standard error
Equation (18)	A	-117.128	1.049	0.162	0.736	296.3	19.63
		17.724	0.046	0.086			
	B	-112.273	1.059	0.157	0.628	179.8	25.302
Equation (19)	A	22.844	0.059	0.111	0.626	358.3	23.291
		47.51	1.029				
		3.136	0.054				
	B	47.257	1.039		0.541	252.2	28.043
		3.776	0.065				

Notes. (1) The F -test for each regression model is statistically significant at $p < 0.001$. (2) The standard errors of the parameters are reported below the corresponding parameter estimates.

aggregated across all stores in the chain. The total number of observations is 216 because the items have short life cycles and not all items are sold in each month.

(2) A test sample consisting of demand forecasts for 10 items for one month in the subsequent year.

Let $t = 1, \dots, t_0$ denote time indices in the fit sample, and T denote the time index in the test sample, $T > t_0$. Let x_{it} and y_{it} , respectively, denote the forecast and the unit sales for item i in month t aggregated across all stores, and S_t denote the value of the S&P 500 index at the end of month t . We first fit a forecasting equation to the fit sample. Then, using the estimated parameters, we compute the optimal inventory level, the hedging parameters, the expected profit, and the standard deviation of profit for the test sample.

To estimate the effect of correlation of demand with S_t on the performance variables, we perturb the fit sample by adding independent and identically distributed (i.i.d.) errors ξ_{it} to y_{it} . Let $\hat{y}_{it} = y_{it} + \xi_{it}$. Here, ξ_{it} has normal distribution with mean 0. Results are computed for various values of the standard deviation of ξ_{it} to ascertain the effect of decreasing correlation of demand with S_t . In this paper, we report results for two datasets, the original dataset and the perturbed dataset, with the standard deviation of ξ_{it} equal to 15. These datasets are labeled A and B, respectively.

To estimate the effect of the volatility of S_T on the performance variables, we consider six scenarios for each dataset assuming that the inventory decision is taken 1, 2, ..., 6 months in advance of time T . Let l denote the lead time and $T - l$ denote the time when

the order is placed. The longer the lead time, the greater is the volatility of S_T .

For both datasets in the fit sample, we estimate the following forecasting equations using linear regression.

$$\hat{y}_{it} = m_1 + m_2 x_{it} + b S_t + \epsilon_{it} \quad (18)$$

$$\hat{y}_{it} = m'_1 + m'_2 x_{it} + \epsilon'_{it} \quad (19)$$

Here, the error terms, ϵ_{it} and ϵ'_{it} , are assumed to be i.i.d. normally distributed.⁵ The coefficients m_1, m_2, m'_1, m'_2 , and b are assumed to be identical across items and over time.⁶ Note that the comparison between the forecasting Equations (18) and (19) remains fair when ξ_{it} are added to y_{it} . Table 2 shows the estimation results for (18) and (19). The coefficient of S_t is statistically significant ($p < 0.001$ in each case), showing that (18) is more appropriate than (19) for modeling demand. However, note that although (18) has a higher R^2 , it does not imply a lower forecast error because S_t is a random variable. Thus, the benefit of using (18) is that it provides a method to incorporate market information in forecasting demand, not that it reduces forecast error.

We compute the volatility of the S&P 500 index using 90 days of historical data prior to the time

⁵ The sales of a given item at a given store may not be normally distributed, because they are truncated by the inventory level. However, normal distribution is a fair approximation for the sum of sales across a large number of stores.

⁶ The coefficient b is identical across items, because we do not expect items in the product category to have different degrees of correlation with economic factors. Likewise, the coefficients m_1, m_2, m'_1, m'_2 are identical across items, because we do not expect dissimilar biases in the forecasts of different items.

Table 3 Optimal Inventory Level and Expected Profit for Item 1 Obtained Using Forecasting Models (18) and (19) for Each Dataset for Different Degrees of Volatility of S_t

Dataset	Lead time l (months)	Optimal inventory		Expected profit		Increase in expected profit		
		Model (18)	Model (19)	Model (18)	Model (19)	Mean	Standard error	Percent increase
A	1	179	153	1,299.53	1,221.66	77.87	20.41	6.37
A	2	181	153	1,274.60	1,198.19	76.41	22.62	6.38
A	3	183	153	1,250.20	1,175.55	74.65	24.11	6.35
A	4	183	153	1,226.23	1,153.56	72.66	24.34	6.30
A	5	183	153	1,202.73	1,132.09	70.63	24.98	6.24
A	6	184	153	1,179.56	1,111.04	68.52	25.55	6.17
B	1	181	155	1,270.68	1,208.27	62.41	19.88	5.17
B	2	182	155	1,248.83	1,186.23	62.61	21.64	5.28
B	3	183	155	1,226.83	1,164.70	62.13	23.00	5.33
B	4	184	155	1,204.84	1,143.63	61.22	24.05	5.35
B	5	184	155	1,182.90	1,122.93	59.97	24.49	5.34
B	6	184	155	1,161.00	1,102.54	58.46	24.51	5.30

of inventory decision for month T by the method given in Hull (2002, §11.3). The volatility is given by the standard deviation of $\log(S_d/S_{d-1})$, where S_d is the closing value of the index for day d . The value of the daily standard deviation is obtained as 1.3984%, and the annual standard deviation as 22.1982%, assuming 252 trading days in the year. The risk-free rate of return is assumed to be 5% per year.

Optimal Inventory Level and Expected Profit.

Using the estimates of Model (18), the demand forecast for item i in the test sample can be written as

$$D_{iT} = a + bS_T + \epsilon_{iT}, \tag{20}$$

where $a = m_1 + m_2x_{iT}$. We compute the optimal inventory level and the profit with and without hedging in this model. As a benchmark, we compute the inventory level and profit for Model (19). Note that in (19) the demand forecast for item i in the test sample is given by $m'_1 + m'_2x_{iT} + \epsilon'_{iT}$.

Table 3 compares the inventory levels and the expected profits obtained using demand distributions estimated from the two forecasting models. Results are reported for one item in the test sample. Other items give similar insights. We find that using (18) instead of (19) changes the inventory decision significantly and increases the expected profit by 5.1% to 6.6% for different datasets. The values of the standard deviation show that the increase in expected profit is statistically significant. The reasons for the increase are as follows.

(1) The two models use different probability distributions for the forecast error. In (19), the forecast error ϵ'_{it} is normally distributed, whereas in (18) the distribution of the forecast error is a convolution of the lognormal distribution of S_t and the normal distribution of ϵ_{it} . Because the lognormal distribution is skewed to the right, the convolution results in a higher inventory level. See Figure 5 for a Q-Q plot of the demand distribution.

(2) In Model (18), up-to-date information from the financial markets has been used to augment the firm’s historical data. Thus, forecasts based on (18) adjust to changes in S_t . When the market moves up (down), the forecasts are revised upward (downward).

Figure 5 Q-Q Plot of the Demand for Item 1 Shows that the Demand Distribution Is Skewed to the Right

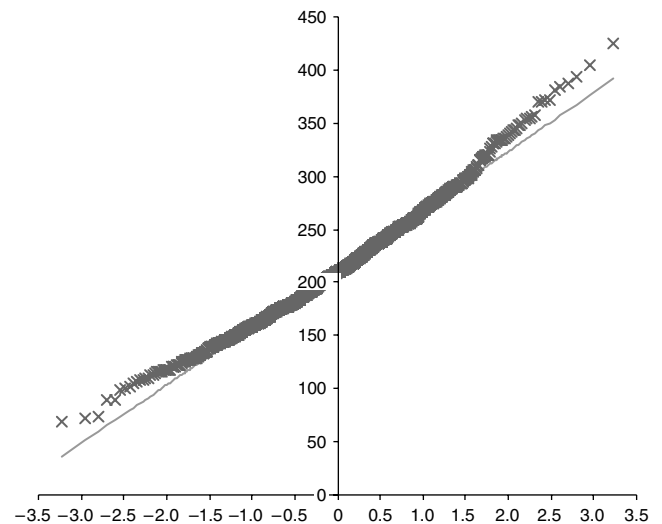


Table 4 Variance of Profit with Different Hedging Strategies

Dataset	Lead time l (months)	Lower bound (LB)		Static hedge – LB		Dynamic hedge – LB	
		Variance	Standard error (σ)	Variance	σ (Static hedge – LB)	Variance	σ (Dynamic hedge – LB)
A	1	80.16	2.87	85.71	0.29	80.40	0.23
A	2	67.5	3.81	73.73	0.51	67.88	0.36
A	3	58.61	4.10	65.02	0.65	59.07	0.42
A	4	52.47	4.20	59.15	0.97	52.98	0.46
A	5	47.47	4.14	54.28	1.02	47.99	0.46
A	6	43.44	4.04	50.72	1.05	43.97	0.46
B	1	87.65	1.98	92.66	0.39	87.79	0.15
B	2	78.56	3.00	84.13	0.57	78.83	0.27
B	3	71.43	3.56	77.22	0.64	71.79	0.35
B	4	65.67	3.87	71.55	0.83	66.10	0.41
B	5	61.09	4.05	67.39	0.94	61.58	0.45
B	6	57.39	4.15	63.96	1.20	57.92	0.48

Notes. All variances are expressed as percentages of the variance of unhedged profit. The lower bound is obtained from the minimum variance hedge in §3.1. The static hedge is identical to the heuristic hedge in §3.1. The dynamic hedge is constructed by dynamically rebalancing the heuristic hedge once at time $(T - l)/2$ as new information is revealed.

By comparing the results for different values of l , we find that Model (18) enables the decision maker to respond to the increase in volatility of S_t by increasing the inventory level whereas Model (19) does not. Furthermore, hedging gives a greater reduction of risk as the volatility of S_t increases, as shown below.

Risk and Investment. Table 4 compares the variance of unhedged profit at the optimal inventory level with the variance of hedged profit. It compares results from the minimum variance hedge of §3.1 and from the heuristic hedge of §3.1 with no rebalancing of the hedging portfolio (static hedge), and with a rebalancing of the hedging portfolio once at time $T - l/2$ (dynamic hedge). Because the minimum variance hedge gives a lower bound on the variance of hedged profit, it provides a benchmark for the static and dynamic hedges.

The static hedge parameters, α , β , and s_p , are computed as described in §3.1. The dynamic hedge is computed by dynamic programming. Thus, at time $T - l/2$, the following actions take place: (i) the asset price $S_{T-l/2}$ and the preliminary forecast error, $\epsilon_{T-l/2}$ are observed⁷; (ii) the hedge parameters, $(\alpha_1, \beta_1, \text{ and } s_{p1})$, are computed. These hedge parameters yield

⁷We assume that the subjective forecast is reevaluated at time $T - l/2$, and that the forecast error is a sum of two components, $\epsilon_{T-l/2}$ observed at time $T - l/2$, and ϵ_T observed at time T . Here, we let $\text{Var}[\epsilon_{T-l/2}] = \text{Var}[\epsilon_T] = \text{Var}[\epsilon]/2$.

Table 5 Initial Investments Required Without Hedging and With Static Hedging for the Inventory Levels Corresponding to Tables 3 and 4

Dataset	Lead time l (months)	Initial investment		Percent difference
		Without hedging	With static hedging	
A	1	3,502.58	1,573.79	55.07
A	2	3,535.56	1,484.95	58.00
A	3	3,562.47	1,449.96	59.30
A	4	3,563.47	1,424.20	60.03
A	5	3,576.19	1,417.16	60.37
A	6	3,588.22	1,417.31	60.50
B	1	3,524.59	1,761.46	50.02
B	2	3,547.35	1,675.63	52.76
B	3	3,570.36	1,634.52	54.22
B	4	3,590.07	1,613.15	55.07
B	5	3,597.64	1,596.92	55.61
B	6	3,596.34	1,582.51	56.00

the cash flows at time $T - l/2$. Thus, the hedge parameters at time 0, α_0, β_0 , and s_{p0} , are then computed in order to hedge the cash flows at time $T - l/2$.

From Proposition 2, rebalancing the hedge is not a self-financing activity, because it uses information about $\epsilon_{T-l/2}$. Thus, we reinvest the cash flow at time $T - l/2$ at the risk-free rate to evaluate the variance of hedged profit at time T . Identical series of sample paths are used to evaluate all hedging strategies. Many simulation runs are conducted to compute average performance statistics and estimate the statistical significance of the results.

All figures in Table 4 are expressed as percentages of the variance of unhedged profit. We find that the lower bound on the variance of hedged profit varies between 87.7% and 43.4%. Thus, the potential reduction in variance that can be obtained by hedging varies between 12.4% and 56.6%. Static hedging has a gap of about 6% with respect to the lower bound. This gap is statistically significant at $p = 0.01$. Dynamic hedging realizes almost the full potential for variance reduction. Its gap with respect to the lower bound is about 0.4%, and is not statistically significant. This performance is notable, because both dynamic and static hedging use only two financial instruments. In particular, the results on dynamic hedging show that even though the decision maker is unable to modify the inventory level after time t , he or she can still use new information to manage the exposure to risk.

We further find that the percent reduction in variance increases significantly with the volatility of S_t . For example, for dataset A the percent reduction in variance under the minimum variance hedge is 20% when $l = 1$ and 56.6% when $l = 6$. Thus, hedging is more beneficial when the market volatility is higher, or, equivalently, when the lead time is longer. As expected, we also find that the percent reduction in variance decreases when demand is less correlated with S_t .

Table 5 shows the initial investment in inventory with and without hedging for each scenario corresponding to Table 5. Note that hedging reduces the initial investment by about 60% because the inflow from the short sale of the stock offsets the cash required for buying inventory and call options.

Furthermore, the investment decreases as the volatility of S_t increases. This is surprising: We would expect both the amount of inventory and the price of the call option to increase with volatility, resulting in larger investment. We find that α increases with volatility, however. Thus, a larger quantity of the underlying asset is sold short, offsetting the additional investment required in inventory and call options.

Therefore, from Tables 4 and 5 we conclude that the benefits of hedging increase with the volatility of S_t . Interestingly, this implies that items with longer lead times will benefit more from hedging than items with shorter lead times.

Risk-Averse Decision Maker. To evaluate the effect of hedging on the optimal inventory decision of the risk-averse decision maker, we assume the expected utility representation $E[u(w)] = E[w] - \rho \text{Var}[w]$, where w denotes wealth. The value of ρ is taken as 0.01. Table 6 presents the inventory levels that maximize the expected utility for each of the 10 items in the test sample with and without hedging. Observe that hedging increases the optimal inventory level. It brings the inventory level closer to the expected value maximizing quantity, restoring efficiency in the market.

5. Conclusions

We have shown how to generate a solution to the hedging of inventory risk using the newsvendor model when demand is correlated with the price of

Table 6 Comparison of Optimal Inventory Level, Expected Profit, Standard Deviation of Profit, and Expected Utility for Each Item for a Risk-Averse Decision Maker Without and With Hedging

Item	Optimal inventory level		Expected profit		Standard deviation of profit		Expected utility	
	Without hedging	With hedging	Without hedging	With hedging	Without hedging	With hedging	Without hedging	With hedging
1	159	165	1,241.92	1,288.15	82.50	89.94	1,199.37	1,235.72
2	159	165	1,286.76	1,314.75	61.97	68.18	1,248.36	1,268.27
3	159	165	1,282.37	1,311.24	61.04	68.44	1,245.11	1,264.39
4	160	164	1,285.8	1,305.33	67.40	68.67	1,240.38	1,258.17
5	158	164	1,276.8	1,306.02	64.91	74.79	1,234.67	1,250.09
6	158	162	1,276.61	1,296.39	70.19	74.87	1,227.34	1,240.33
7	163	171	1,315.53	1,340.02	60.47	53.98	1,278.97	1,310.88
8	163	175	1,322.04	1,349.92	52.85	33.29	1,294.11	1,338.84
9	165	175	1,328.63	1,350.78	55.74	19.85	1,297.56	1,346.84
10	164	176	1,326.99	1,351.23	53.48	15.85	1,298.4	1,348.72

Note. These results are obtained using the original dataset, i.e., $\xi_{it} = 0$.

a financial asset. Hedging reduces the variance of profit and the investment in inventory, increases the expected utility of a risk-averse decision maker, and increases the optimal inventory level for a broad class of utility functions. Our numerical analysis shows that hedging is more beneficial when the price of the underlying asset is more volatile or the product has a longer order lead time. Dynamic hedging provides additional risk reduction even when the retailer cannot change her initial inventory commitment.

Our forecasting model could be extended to incorporate macroeconomic variables such as interest rates and foreign exchange rates that provide demand signals. It might also be customized for specific businesses by using more securities from the equities market, such as sector-specific indices or portfolios of firms in similar businesses. Furthermore, the evolution of the price of the underlying asset may be used to update the demand forecast and modify order quantities even in the absence of early demand data.

An important aspect of our analysis is that the demand risk is not fully spanned by the financial market. Our analysis of the effects of financial hedging on operational decisions under such a scenario may be extended to other problems that have been considered in the real-options literature, such as production switching (Dasu and Li 1997, Huchzermeier and Cohen 1996, Kouvelis 1999), capacity planning (Birge 2000), and global contracting (Scheller-Wolfe and Tayur 1999).

Acknowledgments

The authors thank Marti Subrahmanyam and seminar participants at Case Western Reserve University, Northwestern University, Rutgers University, the University of Michigan, and the University of Texas Dallas for helpful comments. They also thank Garrett J. van Ryzin, the senior editor, and two anonymous referees for many helpful suggestions. The work of the second author is partially supported by NSF Grant DMI-0200406.

Appendix

PROOF OF PROPOSITION 2. According to Lemma 1, the optimal hedge at time t is given by $E_{\epsilon_t}[\Pi_U^T(I) | S_t]$. If no additional information about ϵ is available at time t compared with time 0, then the optimal hedge at time t equals $E_{\epsilon}[\Pi_U^T(I) | S_t]$. However, $E_{\epsilon}[\Pi_U^T(I) | S_t]$ is a martingale with respect to the filtration generated by $\{S_t\}$. Thus, the dynamic hedging strategy, $E_{\epsilon}[\Pi_U^T(I) | S_t]$, is self-financing.

If, however, additional information about ϵ is available at time t , then $E_{\epsilon_t}[\Pi_U^T(I) | S_t]$ is not a martingale with respect to the filtration generated by $\{S_t\}$. Thus, the dynamic hedging strategy is not, in general, self-financing. \square

PROOF OF PROPOSITION 3. We show that if

$$c_1 I > E_N[E[\min\{S_T + \epsilon, (I - a)/b | S_T\}]],$$

then there exists a portfolio X_T that is preferred to Π_U by all risk-averse decision makers. Let

$$X_T = E[\min\{S_T + \epsilon, (I - a)/b | S_T\}].$$

Then X_T is a strictly increasing function of S_T . The new vendor profit function can be written in terms of X_T , as

$$\Pi_U = X_T - c_1 I + \delta(S_T),$$

where $E[\delta(S_T) | X_T] = E[\delta(S_T) | S_T] = 0$. From the conditional Jensen's inequality,

$$E[u(\Pi_U)] = E[E[u(X_T - c_1 I + \delta) | S_T]] \leq E[u(X_T - c_1 I)].$$

Now, note that the time-zero cost of portfolio X_T is $X_0 = e^{-rT} E_N[X_T]$. Thus, if

$$c_1 I > E_N[E[\min\{S_T + \epsilon, (I - a)/b | S_T\}]],$$

then investment of an amount less than $e^{-rT} c_1 I$ in X_T yields a higher expected utility than the inventory investment.

The second part of the proposition follows from the fact that $E_N[\min\{S_T, (I - a)/b\}]$ and $E_N[\min\{S_T, (I - a)/b\}] - E[|\epsilon|]$, respectively, are upper and lower bounds on

$$E_N[E[\min\{S_T + \epsilon, (I - a)/b | S_T\}]]. \quad \square$$

PROOF OF PROPOSITION 4. Let $\lambda \in \Re$ be the Lagrangian multiplier for the constraint $E[X_T - X_0 e^{rT}] = 0$. The decision maker solves the problem

$$\max E[u(W + \Pi_U(I) - X_T + X_0 e^{rT}) + \lambda X_T - \lambda X_0 e^{rT}].$$

Because u is concave, the first-order conditions of optimality are sufficient. The optimal solution is obtained by maximizing the utility function pointwise at each value of S_T . Thus, the first-order conditions are

$$E_{\epsilon}[u'(W + \Pi_U(I) - X_T + X_0 e^{rT})] = \lambda, \quad (21)$$

$$E[X_T - X_0 e^{rT}] = 0. \quad (22)$$

A feasible solution to this system of equations can be found as follows: Fix λ . For each value of S_T , find X_T such that (21) is satisfied. This value of X_T exists and is unique because u' is strictly decreasing. Substitute X_T into (22). If $E[X_T - X_0 e^{rT}] > 0$, then reduce λ (and correspondingly reduce all X_T) until a solution is obtained. Otherwise, increase λ and correspondingly increase all X_T .

Suppose that there exist two distinct solutions to (21)–(22), denoted $(\lambda, X_T^{**}(\lambda))$ and $(\lambda', X_T^{**}(\lambda'))$. Clearly, $\lambda = \lambda'$.

This further shows that $X_T^{**}(\lambda)$ is equal to $X_T^{**}(\lambda')$ almost everywhere. Thus, the two solutions are equal except possibly on a set of measure zero. \square

PROOF OF PROPOSITION 5. The first derivative of the expected utility function evaluated at $\alpha = 0$ gives

$$E[u'(W + \min\{S_T + \epsilon, (I - a)/b\} - c_1I)\{-X_T + X_0e^{rT}\}].$$

Here, $u'(\cdot)$ is a decreasing function of S_T at $\alpha = 0$, and $-X_T + X_0e^{rT}$ is also decreasing in S_T . Thus, they have a positive covariance, which gives

$$\begin{aligned} & E[u'(W + \min\{S_T + \epsilon, (I - a)/b\} - c_1I)\{-X_T + X_0e^{rT}\}] \\ & \geq E[u'(W + \min\{S_T + \epsilon, (I - a)/b\} - c_1I)] \\ & \quad \cdot E[-X_T + X_0e^{rT}] \\ & = 0, \end{aligned}$$

where the last equality follows from (12). \square

PROOF OF PROPOSITION 6. The value of inventory that maximizes $E[u(\Pi_H)]$ is greater than the value of inventory that maximizes $E[u(\Pi_U)]$ if

$$E\left[\{u'(\Pi_H(I)) - u'(\Pi_U(I))\} \frac{\partial \Pi_U}{\partial I}\right] \geq 0.$$

Here, we have used the fact that $\partial \Pi_H / \partial I = \partial \Pi_U / \partial I$. Let $G(\epsilon)$ denote the distribution function of ϵ . Conditioning on S_T and taking expectation with respect to ϵ , we get

$$\begin{aligned} & E_\epsilon \left[\left\{ u'(\Pi_H(I)) - u'(\Pi_U(I)) \right\} \frac{\partial \Pi_U}{\partial I} \middle| S_T \right] \\ & = -c_1 \int_{-\infty}^{(I-a)/b} \{u'(\Pi_H(I)) - u'(\Pi_U(I))\} dG(\epsilon) \\ & \quad + \{u'((I-a)/b - c_1I - \alpha X_T + \alpha X_0e^{rT}) \\ & \quad - u'((I-a)/b - c_1I)\} \Pr\{\epsilon \geq (I-a)/b - S_T\}. \end{aligned}$$

Consider the expectation over S_T of the second term in the above equation. We use the facts that

$$\Pr\{\epsilon \geq (I-a)/b - S_T\} \geq \Pr\{\epsilon \geq (I-a)/b\},$$

and that S_T is independent of ϵ to write

$$\begin{aligned} & E\left[\{u'((I-a)/b - c_1I - \alpha X_T + \alpha X_0e^{rT}) - u'((I-a)/b - c_1I)\} \right. \\ & \quad \cdot \Pr\{\epsilon \geq (I-a)/b - S_T\}] \\ & \geq E\left[\{u'((I-a)/b - c_1I - \alpha X_T + \alpha X_0e^{rT}) \right. \\ & \quad \left. - u'((I-a)/b - c_1I)\} \Pr\{\epsilon \geq (I-a)/b\}\right] \\ & \geq [E\{u'((I-a)/b - c_1I - \alpha X_T + \alpha X_0e^{rT}) \\ & \quad - u'((I-a)/b - c_1I)\} \Pr\{\epsilon \geq (I-a)/b\}]. \end{aligned} \quad (23)$$

Thus, if

$$E[u'((I-a)/b - c_1I - \alpha X_T + \alpha X_0e^{rT})] \geq u'((I-a)/b - c_1I),$$

then the second term is nonnegative. This inequality combined with the first term gives sufficient conditions on the hedging portfolio under which the hedged optimal inventory level is larger than the unhedged optimal inventory level.

When $u(\cdot)$ is CARA or DARA, then $u'(\cdot)$ is convex in wealth. Thus, applying Jensen's inequality to (23), we get

$$\begin{aligned} & E[u'((I-a)/b - c_1I - \alpha X_T + \alpha X_0e^{rT})] - u'((I-a)/b - c_1I) \\ & \geq u'((I-a)/b - c_1I - \alpha E[X_T - X_0e^{rT}]) - u'((I-a)/b - c_1I) \\ & = 0. \quad \square \end{aligned}$$

The following lemma is a standard result. It is useful for proving Proposition 7.

LEMMA 2. Let X be any random variable, and $f: \mathfrak{R} \rightarrow \mathfrak{R}$ be a decreasing function such that $E[f(X)] = 0$. Then, (i) for any decreasing nonnegative function $w(x)$, $E[w(X)f(X)] > 0$; (ii) for any increasing nonnegative function $w(x)$, $E[w(X)f(X)] < 0$.

PROOF. Consider (i). Let $G(X)$ denote the cumulative distribution function of X . Because $f(x)$ is decreasing in x , there exists x_0 such that $f(x) > 0$ for all $x < x_0$ and $f(x) < 0$ for all $x > x_0$. Then,

$$\begin{aligned} E[w(x)f(X)] & = \int_0^{x_0} w(x)f(x) dG(x) + \int_{x_0}^{\infty} w(x)f(x) dG(x) \\ & > \int_0^{x_0} w(x_0)f(x) dG(x) + \int_{x_0}^{\infty} w(x_0)f(x) dG(x) \\ & \geq w(x_0)E[f(X)] \\ & \geq 0. \end{aligned}$$

Now consider (ii). We have

$$\begin{aligned} E[w(x)f(X)] & = \int_0^{x_0} w(x)f(x) dG(x) + \int_{x_0}^{\infty} w(x)f(x) dG(x) \\ & < \int_0^{x_0} w(x_0)f(x) dG(x) + \int_{x_0}^{\infty} w(x_0)f(x) dG(x) \\ & \leq w(x_0)E[f(X)] \\ & \leq 0. \quad \square \end{aligned}$$

PROOF OF PROPOSITION 7. Let the hedged profit be denoted Π (the subscript H is ignored for simplicity). We need to show that $\partial^2 E[u(\Pi)] / \partial I \partial \alpha$ is greater than or equal to zero. We have

$$\begin{aligned} \frac{\partial^2}{\partial I \partial \alpha} E[u(\Pi)] & = E\left[u''(\Pi) \frac{\partial \Pi}{\partial I} \frac{\partial \Pi}{\partial \alpha}\right] \\ & = E\left[E_\epsilon \left[u''(\Pi) \frac{\partial \Pi}{\partial I} \middle| S_T \right] \frac{\partial \Pi}{\partial \alpha}\right] \quad (24) \\ & = E\left[-c_1 E_\epsilon [u''(\Pi) | S_T] \frac{\partial \Pi}{\partial \alpha} - \frac{1}{b} R_A(\Pi_0) u'(\Pi_0) \right. \\ & \quad \left. \cdot \Pr\{\epsilon > (I-a)/b - S_T\} \frac{\partial \Pi}{\partial \alpha}\right], \quad (25) \end{aligned}$$

where (24) follows because $\partial\Pi/\partial\alpha$ is independent of ϵ . For (25), note that

$$\frac{\partial\Pi}{\partial I} = -c_1 + \frac{1}{b}1\{\epsilon > (I - a)/b - S_T\}. \quad (26)$$

Also note that Π is independent of ϵ for $\epsilon > (I - a)/b - S_T$, and we write

$$\Pi_0 = (I - a)/b - c_1I - \alpha X_T + \alpha X_0.$$

Finally, $u''(\Pi) = -R_A(\Pi)u'(\Pi)$.

Consider $E_\epsilon[u''(\Pi) | S_T]$. Let $G(\epsilon)$ denote the distribution function of ϵ . We have

$$\begin{aligned} E_\epsilon[u''(\Pi) | S_T] &= \int_{-\infty}^{(I-a)/b-S_T} u''(S_T + \epsilon - c_1I - \alpha X_T + \alpha X_0) dG(\epsilon) \\ &\quad + \int_{(I-a)/b-S_T}^{\infty} u''((I-a)/b - c_1I - \alpha X_T + \alpha X_0) dG(\epsilon). \end{aligned}$$

Because $u''(\cdot) < 0$, the above expression shows that $E_\epsilon[u''(\Pi) | S_T]$ is negative for all S_T . Furthermore, differentiating $E_\epsilon[u''(\Pi) | S_T]$ with respect to S_T , we get

$$\begin{aligned} \frac{d}{dS_T} E_\epsilon[u''(\Pi) | S_T] &= \int_{-\infty}^{(I-a)/b-S_T} u'''(S_T + \epsilon - c_1I - \alpha X_T + \alpha X_0) \left(1 - \alpha \frac{dX_T}{dS_T}\right) dG(\epsilon) \\ &\quad - \int_{(I-a)/b-S_T}^{\infty} u'''((I-a)/b - c_1I - \alpha X_T + \alpha X_0) \alpha \frac{dX_T}{dS_T} dG(\epsilon). \end{aligned}$$

Let

$$f(\epsilon, S_T) = 1\{\epsilon < (I - a)/b - S_T\} - \alpha dX_T/dS_T.$$

$f(\cdot)$ is decreasing in ϵ . Furthermore, $E_\epsilon[f(\cdot) | S_T]$, which is equal to the slope of $E_\epsilon[\Pi | S_T]$ with respect to S_T , is positive for all $\alpha \in [0, \bar{\alpha}]$. Thus, $f(\cdot)$ is a decreasing function with a nonnegative conditional expectation with respect to ϵ .

In addition, $u'''(\cdot) > 0$ because $u(\cdot)$ is CARA or DARA. Furthermore, from the assumption of constant or decreasing absolute prudence, we have that $u'''(\cdot)$ is decreasing in ϵ . Combining these observations and applying Lemma 2(i), we find that $E_\epsilon[u''(\Pi) | S_T]$ is increasing in S_T .

Consider the first term in (25): $\partial\Pi/\partial\alpha$ is decreasing in S_T and has zero expectation (due to the first condition for an optimum with respect to α); $-c_1E_\epsilon[u''(\Pi) | S_T]$ is decreasing in S_T and is nonnegative for all S_T . Thus, all conditions of Lemma 2(i) are satisfied. Therefore, applying Lemma 2(i), we find that the first term in (25) is nonnegative.

Consider the second term in (25). Here, Π_0 is decreasing in S_T . Thus, $R_A(\Pi_0)$, $u'(\Pi_0)$, and $\Pr\{\epsilon > (I - a)/b - S_T\}$ are increasing in S_T ; $\partial\Pi/\partial\alpha$ is decreasing in S_T and has zero expectation. Therefore, applying Lemma 2(ii), we find that the second term in (25) (i.e., $(1/b)R_A(\Pi_0)u'(\Pi_0)\Pr\{\epsilon > (I - a)/b - S_T\}\partial\Pi/\partial\alpha$) is negative.

Thus, $dl^*/d\alpha \geq 0$ for $0 \leq \alpha \leq \bar{\alpha}$. \square

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