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Information Sharing in a Supply Chain Under ARMA Demand

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In this paper we study how the time-series structure of the demand process affects the value of information sharing in a supply chain. We consider a two-stage supply chain model in which a retailer serves autoregressive moving-average (ARMA) demand and a manufacturer fills the retailer's orders. We characterize three types of situations based on the parameters of the demand process: (i) the manufacturer benefits from inferring demand information from the retailer's orders; (ii) the manufacturer cannot infer demand, but benefits from sharing demand information; and (iii) the manufacturer is better off neither inferring nor sharing, but instead uses only the most recent orders in its production planning. Using the example of ARMA(1,1) demand, we find that sharing or inferring retail demand leads to a 16.0% average reduction in the manufacturer's safety-stock requirement in cases (i) and (ii), but leads to an increase in the manufacturer's safety-stock requirement in (iii). Our results apply not only to two-stage but also to multistage supply chains.

Key words: single-item inventory model; nonstationary demand; information sharing; supply chain management; electronic data interchange
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1. Introduction

In this paper, we examine how the value of sharing demand information in a supply chain depends on the time-series structure of the demand process. The question is motivated by the work of Lee et al. (2000) and Raghunathan (2001), who study the value of information sharing in a two-stage supply chain model consisting of a single retailer and a manufacturer. In their model, the retailer serves an AR(1) demand with a nonnegative autocorrelation coefficient, and places orders with the manufacturer using a periodic-review order-up-to policy. Both the manufacturer and the retailer know the parameters of the demand process; however, the retailer might choose not to share information about the actual realizations of demand with the manufacturer.

Lee, So, and Tang (hereafter referred to as LST) analyze this model and conclude that information sharing results in significant inventory reduction and cost savings to the manufacturer. However, they qualify their results by noting that information sharing could be less valuable if the manufacturer uses the historical stream of orders from the retailer to forecast demand. Raghunathan (2001) further develops this idea and shows that the value of information sharing indeed decreases monotonically with each time period, converging to zero in the limit, under AR(1) demand with a nonnegative autocorrelation coefficient.

In this paper, we extend the results of Raghunathan to cases in which demand is generated by a higher-order autoregressive process, viz., a finite-order autoregressive process (AR(p), $p \geq 1$) or an autoregressive moving-average process (ARMA(p, q), $p \geq 1$, $q \geq 1$). This extension is valuable to managers for two reasons: First, real-life demand patterns often follow higher-order autoregressive processes due to the presence of seasonality and business cycles. For example, the monthly demand for a seasonal item can be an AR(12) process. More general ARMA processes are found to fit demand for long lifecycle goods such as fuel, food products, machine tools, etc. as observed in Chopra and Meindl (2001) and Nahmias (1993). Second, recent research has shown that ARMA demand processes occur naturally in multistage supply chains.

For example, Zhang (2004) studies the time-series characterization of the order process for a decision maker serving invertible ARMA demand and using a periodic-review order-up-to policy. He proposes the "ARMA-in-ARMA-out" principle, i.e., if demand follows an ARMA(p, q) process, then the order process is asymptotically an ARMA$(p, \max\{p, q-l\})$ process, where l is the replenishment lead time. Thus, it is of interest to know whether the results on the value of information sharing under AR(1) demand extend to general autoregressive moving-average processes.

We make the following contributions in this paper: First, we give a time-series characterization of the retailer's order process when demand follows a *noninvertible* ARMA process and an order-up-to policy is used.[1] We find that a noninvertible ARMA(p, q) demand process leads to a noninvertible ARMA$(p, p+q)$ order process. This characterization is exact for both finite and infinite time horizons. It is also necessary for analyzing the value of information sharing in a supply chain because of our finding that even when demand is an invertible ARMA process and Zhang's results are used, the orders can be a noninvertible ARMA process. In such cases, results for noninvertible demand will be required to characterize the order process in a multistage supply chain.

Second, we determine conditions on the parameters of the demand process under which the retailer's demand history can be inferred from the retailer's orders. In such situations, we say that the demand is *inferable*. Our definition of inferability differs from that of invertibility of demand: Invertibility implies forecasting demand from historical values of demand, while inferability implies forecasting demand from historical orders. Using this characterization, we obtain a rule by which a decision maker can determine when information sharing is useful in the supply chain and when using historical order information suffices. Consider the application of this rule to AR(1) demand. Raghunathan (2001) proves that the retailer's demand can be inferred from the retailer's order process for AR(1) demand with autocorrelation coefficient, ρ, greater than or equal to zero. We show that Raghunathan's insight that the value of information sharing tends to 0 as time t goes to infinity extends to the case when $\rho \in (-0.5, 0)$, but does not carry over when $\rho \leq -0.5$. Thus, even for AR(1) demand, information sharing can be valuable if $\rho \in (-1, -0.5]$.

Third, we numerically quantify the benefit to the manufacturer from the retailer's demand information using the example of ARMA$(1, 1)$ demand. In this part of the paper, we only assume that the manufacturer has access to demand information, but we do not distinguish whether there is sharing of demand information between the retailer and the manufacturer, or whether the manufacturer infers demand information from the retailer's historical orders. We find that when the autoregressive and moving-average coefficients have the same sign, the use of downstream demand information leads to a 16.0% average reduction in the standard deviation of the manufacturer's lead-time demand, and hence in the manufacturer's safety-stock requirement.[2] In these cases, the benefit of demand information is often much higher under ARMA$(1, 1)$ demand than under AR(1) demand, and is increasing in the magnitudes of the autoregressive and moving-average coefficients. On the other hand, when the autoregressive and moving-average coefficients have opposite signs and demand is not invertible, downstream demand information can lead to an increase in the standard deviation of the manufacturer's lead-time demand. In such cases, we find that the manufacturer is better off treating the retailer's orders as an independent noninvertible ARMA process, and forecasting lead-time demand using only the most recent p orders. Our results in this part of the paper, together with our results on the inferability of demand, help classify all pertinent cases: those in which there is value to sharing demand information, cases where inferring demand information is beneficial, and those in which the manufacturer is best served by treating the order process as an independent noninvertible time-series.

Information sharing can be valuable in a supply chain for various reasons. Chen (2003) reviews the research on this topic for different supply chain models. For example, Cachon and Fisher (2000) analyze the ability of a warehouse to make better ordering and allocation decisions using downstream inventory information in a one-warehouse multiretailer system. Chen (1998) analyzes the cost difference between echelon-stock and installation-stock ordering policies in a serial supply chain, where the echelon-stock policy is made possible by availability of downstream information. Gavirneni et al. (1999) study the effect of demand information in the context of a retailer using an (s, S) policy in a two-stage serial supply chain. Aviv and Federgruen (1998) quantify the value of information sharing and vendor-managed inventory in a decentralized one-supplier multiretailer system. While the above papers consider stationary demand, Graves (1999) considers a serial supply chain

[1] Informally, invertibility is the property by which we can determine the underlying driving process (i.e., the white noise in the demand process) by observing the output process generated by it (in this case, the historical demand process).

[2] This average is taken over the following set of values of ρ and λ: $|\rho| < 1$, $|\lambda| \leq 2$. Here, ρ and λ denote the autocorrelation and moving-average coefficients, respectively, as defined in §2. The lead times are assumed to be two time periods.

with ARIMA(0, 1, 1) demand, and shows that the demand process for the upstream stage is also of the ARIMA(0, 1, 1) form, but is more variable than for the downstream stage. Similar to Raghunathan's conclusion, Graves finds that there is no benefit from letting the upstream stages see exogenous demand in this model. Our paper adds to this literature by addressing the question of the value of information sharing under ARMA demand, and identifying cases in which there is value from information sharing or inferring demand.

The rest of this paper is organized as follows. Section 2 introduces the model setup; §3 presents the time-series characterization of the retailer's order process given the demand process; §4 provides conditions on the parameters of the demand process under which demand can be inferred from the order process; §5 investigates the value of sharing demand information in the supply chain; and §6 summarizes the implications of our results.

2. Model Setup

The model setup and assumptions are the same as those of LST (2000) and Raghunathan (2001) except for the assumptions about demand. We consider a supply chain with one retailer and one manufacturer. At discrete time periods, $t = 0, 1, 2, \ldots,$ the retailer faces external demand, D_t, for a single item. Let D_t follow an ARMA(p, q) process,

$$D_t = d + \rho_1 D_{t-1} + \rho_2 D_{t-2} + \cdots + \rho_p D_{t-p} + \epsilon_t - \lambda_1 \epsilon_{t-1} - \lambda_2 \epsilon_{t-2} - \cdots - \lambda_q \epsilon_{t-q}, \quad (1)$$

where ϵ_t is a sequence of uncorrelated random variables with mean zero and variance σ^2, and $\rho_1, \ldots, \rho_p$ and $\lambda_1, \ldots, \lambda_q$ are known constants. If $p = 1$ or $q = 1$, we drop the respective subscript and denote the parameter simply as ρ or λ.

We assume that D_t is covariance stationary, that is, $E[D_t]$ exists and is constant for all t, Var$[D_t]$ is finite, and the covariance of D_t and D_{t+h} depends on h but not on t. Let the cumulative error at time t be defined as $\xi_t = \epsilon_t - \lambda_1 \epsilon_{t-1} - \lambda_2 \epsilon_{t-2} - \cdots - \lambda_q \epsilon_{t-q}$.

Let the replenishment lead times from the external supplier to the manufacturer, and from the manufacturer to the retailer, be L and l periods, respectively. In each period t, the retailer satisfies demand D_t from its on-hand inventory with complete backlogging of excess demand. Then, the retailer places order Y_t with the manufacturer. The manufacturer satisfies this order from its own on-hand inventory, also with complete backlogging (see Chen 2003). The shipment of this order is received at the retailer at the beginning of period $t + l + 1$. The sequence of events at the manufacturer is similar. The manufacturer places its orders with an external supplier with ample stock to replenish its own inventory. We assume that both the retailer and the manufacturer use myopic order-up-to inventory policies where negative order quantities are allowed, and d is sufficiently large so that the probability of negative demand or negative orders is negligible.

With respect to the information structure, we assume that the parameters of the demand process are common knowledge to the retailer and the manufacturer, but the demand realizations are private knowledge to the retailer. When there is no information sharing, the manufacturer receives an order of Y_t at the end of time period t from the retailer. On the other hand, when there is information sharing, the manufacturer receives the order Y_t as well as information about D_t at the end of time period t from the retailer.

Because the choice of time 0 is arbitrary, we shall define the demand process for $t = 0, \pm 1, \pm 2, \ldots.$ Also, we denote an infinite sequence of real numbers, such as $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2, \ldots,$ by $\{a_j\}$. An infinite sequence is said to be *absolutely summable* if the limit $\lim_{n\to\infty} \sum_{j=-n}^{n} |a_j|$ is finite. The following properties of a time-series process are useful; see Fuller (1996) or Box et al. (1994).

Property 1. D_t is covariance stationary if and only if it can be expressed as an infinite moving average of $\{\epsilon_t\}$, $D_t = \sum_{j=0}^{\infty} a_j \epsilon_{t-j}$, where $\{a_j\}$ is an absolutely summable sequence of real numbers.

Property 2. D_t is invertible if and only if all the roots of the equation formed by the coefficients of lagged ϵ_t terms,

$$m^q - \lambda_1 m^{q-1} - \lambda_2 m^{q-2} - \cdots - \lambda_q = 0,$$

are less than one in absolute value.

3. Computation of the Order Process

To derive expressions for the order process, Y_t, we consider the following alternative representation of D_t. Let $\mathbf{D}(t)$ denote the p-vector $(D_t, D_{t-1}, \ldots, D_{t-p+1})'$, $\boldsymbol{\xi}_t$ denote the p-vector $(\xi_t, 0, \ldots, 0)'$, and $\mathbf{P}$ be the $p \times p$ matrix

$$\begin{pmatrix} \rho_1 & \cdots & \rho_p \\ \mathbf{I}_{p-1} & & \mathbf{0}_{p-1} \end{pmatrix},$$

where $\mathbf{I}_{p-1}$ is a $p-1 \times p-1$ identity matrix and $\mathbf{0}_{p-1}$ is a column of $p-1$ zeros. Also let $\mathbf{e} = (1, 0, \ldots, 0)'$ be a p-vector with 1 in the first row and 0's in the remaining $p-1$ rows. Then,

$$\mathbf{D}(t) = d\mathbf{e} + \mathbf{P}\mathbf{D}(t-1) + \boldsymbol{\xi}_t = d\sum_{i=0}^{k-1} \mathbf{P}^i \mathbf{e} + \mathbf{P}^k \mathbf{D}(t-k) + \sum_{i=0}^{k-1} \mathbf{P}^i \boldsymbol{\xi}_{t-i}$$

for $k \geq 1$. Particularly, D_t can be expressed as a function of $\mathbf{D}(t-k)$ for $k \geq 1$ and arbitrary t as follows:

$$D_t = \mathbf{P}^k(1{:})\mathbf{D}(t-k) + d\sum_{i=0}^{k-1} P^i_{(11)} + \sum_{i=0}^{k-1} P^i_{(11)}\xi_{t-i}, \quad (2)$$

where $\mathbf{P}^k(1{:})$ is the first row and $P^k_{(ij)}$ is the ijth element of the matrix $\mathbf{P}^k$.

Because we are interested in the retailer's order quantity to the manufacturer when lead time equals l, consider the total demand over the lead time,

$$\sum_{k=1}^{l+1} D_{t+k} = \sum_{k=1}^{l+1}\left(\mathbf{P}^k(1{:})\mathbf{D}(t) + d\sum_{i=0}^{k-1} P^i_{(11)} + \sum_{i=0}^{k-1} P^i_{(11)}\xi_{t+k-i}\right).$$

The conditional expectation of $\sum_{k=1}^{l+1} D_{t+k}$ given $D_t, D_{t-1}, \ldots$ is

$$\begin{aligned} m_t &= E\left[\sum_{k=1}^{l+1} D_{t+k} \,\Big|\, D_t, D_{t-1}, \ldots\right] \\ &= \sum_{k=1}^{l+1}\left(\mathbf{P}^k(1{:})\mathbf{D}(t) + d\sum_{i=0}^{k-1} P^i_{(11)}\right) \\ &\quad + E\left[\sum_{k=1}^{l+1}\sum_{i=0}^{k-1} P^i_{(11)}\xi_{t+k-i} \,\Big|\, D_t, D_{t-1}, \ldots\right]. \end{aligned}$$

Here, $\sum_{k=1}^{l+1}\sum_{i=0}^{k-1} P^i_{(11)}\xi_{t+k-i}$ can be expressed as a linear function of $\epsilon_{t-q+1}, \ldots, \epsilon_t, \epsilon_{t+1}, \ldots, \epsilon_{t+l+1}$. It is clear that $E[\epsilon_s \mid D_t, D_{t-1}, \ldots] = 0$ for $t+1 \leq s \leq t+l+1$ because demand in periods $t+1$ through $t+\ell+1$ has not been observed. However, the problem of estimating $E[\epsilon_s \mid D_t, D_{t-1}, \ldots]$ for $t-q+1 \leq s \leq t$ remains because demand in those periods has been observed and therefore gives some information about these error terms. In this estimation problem, given data from time 1 until time t, there are t unknowns, namely, $\epsilon_1, \ldots, \epsilon_t$, whereas only $t-q$ equations, namely, the ARMA equations for $D_{q+1}, \ldots, D_t$.[3]

Because we consider noninvertible demand processes, we assume that the conditional expectations of the most recent q error terms, $\epsilon_{t-q+1}, \ldots, \epsilon_t$, are set to zero. When demand is invertible, an alternative assumption used in the time-series literature is to set the conditional expectations of $\epsilon_1, \ldots, \epsilon_q$ to zero, and estimate the remaining error terms recursively by using the observed demand data. Under this alternative assumption, the estimate of ϵ_t converges almost surely to the true value as t tends to infinity for invertible demand, but is not a convergent series for noninvertible demand (see Fuller 1996, pp. 79–93).[4] Zhang (2004) provides a time-series characterization of Y_t under this alternative assumption. We refer to the computation of the order process by Zhang for invertible demand as "Zhang's approach" (set the first q error terms equal to zero), and the method proposed by us for noninvertible demand as "our approach" (set the last q error terms equal to zero). We discuss and compare the results from the two approaches in §5.

We note that our approach gives the same results as Zhang when demand is AR(p). Additionally, our approach can also be used as an alternative to Zhang's approach when D_t is invertible. Thus, in this section and §4, we do not explicitly require the demand process to be noninvertible.

Therefore, we have

$$\begin{aligned} m_t &= E\left[\sum_{k=1}^{l+1} D_{t+k} \,\Big|\, D_t, \ldots, D_{t-p+1}\right] \\ &= \sum_{k=1}^{l+1}\left(\mathbf{P}^k(1{:})\mathbf{D}(t) + d\sum_{i=0}^{k-1} P^i_{(11)}\right). \end{aligned}$$

Likewise, the conditional variance of $\sum_{k=1}^{l+1} D_{t+k}$ given $D_t, \ldots, D_{t-p+1}$ is

$$\begin{aligned} v_t &= \mathrm{Var}\left[\sum_{k=1}^{l+1} D_{t+k} \,\Big|\, D_t, \ldots, D_{t-p+1}\right] \\ &= \mathrm{Var}\left[\sum_{k=1}^{l+1}\sum_{i=0}^{k-1} P^i_{(11)}\xi_{t+k-i} \,\Big|\, D_t, \ldots, D_{t-p+1}\right]. \end{aligned}$$

The covariance stationarity of ξ_t implies that v_t can be written as $v_t = C\sigma^2$, where C is a positive constant independent of t. In the case of AR(p) demand, we obtain the following closed-form expression for v_t:

$$v_t = \sigma^2 \sum_{k=1}^{l+1}\left(\sum_{i=0}^{k-1} P^i_{(11)}\right)^2.$$

Now consider the retailer's ordering decision. Let S_t denote the retailer's order-up-to level in period t. Then,

$$Y_t = D_t + S_t - S_{t-1},$$

where

$$S_t = m_t + z\sigma\sqrt{C}$$

for some critical fractile z. The value of z depends on the desired service level and the distribution of ϵ_t. Because ξ_t is assumed to be independent of m_t, the value of z is independent of m_t. Thus, the retailer's order quantity at time t can be written as

$$Y_t = D_t + \sum_{k=1}^{l+1}(\mathbf{P}^k(1{:})[\mathbf{D}(t) - \mathbf{D}(t-1)]). \quad (3)$$

[3] The ARMA equations for $D_1, \ldots, D_q$ cannot be used because they contain the terms $D_{-p+1}, \ldots, D_0$, which are also unknown.

[4] Fuller (1996) also describes other forecasts for covariance-stationary invertible time-series that differ in their finite sample properties, but lead to the same asymptotic convergence.

Let $\sum_{k=1}^{l+1} \mathbf{P}^k(1:)$ be denoted by the vector $(\alpha_1, \ldots, \alpha_p)$. Using this, (3) can be expressed as

$$Y_t = (1+\alpha_1)D_t - (\alpha_1 - \alpha_2)D_{t-1} - (\alpha_2 - \alpha_3)D_{t-2} - \cdots - (\alpha_{p-1} - \alpha_p)D_{t-p+1} - \alpha_p D_{t-p}. \quad (4)$$

The following properties of Y_t follow from this equation.

Theorem 1. (i) *If D_t is covariance stationary, then Y_t is covariance stationary.* (ii) *If D_t is ARMA(p, q), then Y_t is ARMA$(p, p+q)$. In particular, if D_t is AR(p), then Y_t is ARMA(p, p).*

Proof. From (4), Y_t can be written as a moving average of $\{D_t\}$,

$$Y_t = \sum_{j=0}^{p} b_j D_{t-j},$$

where $b_0 = 1 + \alpha_1$, $b_i = \alpha_i - \alpha_{i+1}$ for $1 \le i \le p-1$, and $b_p = \alpha_p$. Because $\{b_j\}$ is a finite sequence, it is absolutely summable. Further, from Property 1, D_t can be written as a moving average of $\{\epsilon_t\}$ with absolutely summable coefficients. Thus, (i) follows by Fuller (1996, Corollary 2.2.2.2, p. 34).

To prove (ii), let $\mathscr{B}$ denote the backward shift operator, i.e., $\mathscr{B}D_t = D_{t-1}$. Also, let $\phi(\mathscr{B}) = 1 - \rho_1\mathscr{B} - \rho_2\mathscr{B}^2 - \cdots - \rho_p\mathscr{B}^p$, $\psi(\mathscr{B}) = 1 - \lambda_1\mathscr{B} - \lambda_2\mathscr{B}^2 - \cdots - \lambda_q\mathscr{B}^q$, and $\varphi(\mathscr{B}) = (1+\alpha_1) - (\alpha_1 - \alpha_2)\mathscr{B} - (\alpha_2 - \alpha_3)\mathscr{B}^2 - \cdots - \alpha_p\mathscr{B}^p$. Using $\phi(\mathscr{B})$ and $\psi(\mathscr{B})$, we rewrite the demand process (1) as

$$D_t = \phi^{-1}(\mathscr{B}) \cdot [d + \psi(\mathscr{B}) \cdot \epsilon_t]. \quad (5)$$

Similarly, using $\varphi(\mathscr{B})$, we rewrite the order process (4) as

$$Y_t = \varphi(\mathscr{B}) \cdot D_t. \quad (6)$$

Substituting from (5) into (6), we get $Y_t = \varphi(\mathscr{B}) \cdot \phi^{-1}(\mathscr{B}) \cdot [d + \psi(\mathscr{B}) \cdot \epsilon_t]$, and thus,

$$\phi(\mathscr{B}) \cdot Y_t = d + \varphi(\mathscr{B}) \cdot \psi(\mathscr{B}) \cdot \epsilon_t. \quad (7)$$

Because $\varphi(\mathscr{B}) \cdot \psi(\mathscr{B})$ is a polynomial of degree $p+q$, we find that Y_t is an ARMA$(p, p+q)$ process. The result for AR(p) demand follows by setting $\psi(\mathscr{B}) = 1$. □

3.1. Demand Propagation in the Supply Chain

Theorem 2 applied iteratively to a multistage supply chain shows that, if the demand at the lowest stage (numbered as the first stage) is ARMA(p, q), and all players in the supply chain use an order-up-to policy, then the demand at the kth stage is ARMA$(p, (k-1)p + q)$. For example, even when the retailer's demand is AR(1), the dependent demand for components and subassemblies at the kth upstream level in the supply chain is an ARMA$(1, k)$ process. In addition, as demand progresses upstream in a supply chain, the coefficients of its autoregressive component are preserved.

To see this, note that if D_t is AR(p), then Y_t has the representation

$$Y_t = d + \sum_{i=1}^{p} \rho_i Y_{t-i} + (1+\alpha_1)\epsilon_t - \sum_{i=1}^{p-1} (\alpha_i - \alpha_{i+1})\epsilon_{t-i} - \alpha_p \epsilon_{t-p}. \quad (8)$$

Alternatively, if D_t is ARMA(p, q), then Y_t has the representation

$$Y_t = d + \sum_{i=1}^{p} \rho_i Y_{t-i} + \varphi(\mathscr{B}) \cdot \psi(\mathscr{B}) \cdot \epsilon_t. \quad (9)$$

In either case, the coefficients of lagged D_t terms in (1), ρ_i, appear again in (8) and (9) as the coefficients of lagged Y_t terms. Further, the coefficients of ϵ_t terms in (8) and (9) are obtained by multiplying the original coefficients of ϵ_t in D_t represented by the polynomial $\psi(\mathscr{B})$ with the coefficients of D_t terms in (4) represented by $\varphi(\mathscr{B})$. Because $\psi(\mathscr{B})$ is a polynomial of degree q and $\varphi(\mathscr{B})$ is a polynomial of degree p, Y_t contains $p+q$ lagged ϵ_t terms. Therefore, Y_t has the same autoregressive component as D_t and a higher-order moving-average component.

4. Inferring D_t from Y_t

In this section, we use the characterization of the order process in §3 to show the conditions under which it is possible for the manufacturer to infer D_t accurately simply by observing Y_t. In brief, (4) gives a rule by which the retailer can determine its order quantity after observing the demand realization in each period. However, this equation may also be used to express the demand process, D_t, in terms of historical values of the order process, Y_t. We state this as a definition.

Definition 1. Demand process $\{D_t\}$ is inferable from $\{Y_t\}$, or simply that D_t is inferable, if the manufacturer's forecast of D_t obtained by observing Y_t converges almost surely (i.e., with probability 1) to the actual realization of D_t as t tends to infinity.

When D_t is inferable, there is no need to share demand information between the retailer and the manufacturer; when D_t is not inferable, the manufacturer may benefit from sharing of the retailer's demand information. Let $\mathbf{Y}(t)$ denote the p-vector $(Y_t, 0, \ldots, 0)'$ and $\mathbf{A}$ be the $p \times p$ matrix

$$\begin{pmatrix} \dfrac{\alpha_1 - \alpha_2}{1+\alpha_1} & \cdots & \dfrac{\alpha_p}{1+\alpha_1} \\ \mathbf{I}_{p-1} & & \mathbf{0}_{p-1} \end{pmatrix}.$$

Then, from (4), $\mathbf{D}(t)$ can be written as

$$\mathbf{D}(t) = \mathbf{A}\mathbf{D}(t-1) + \frac{1}{1+\alpha_1}\mathbf{Y}(t) = \mathbf{A}^k\mathbf{D}(t-k) + \frac{1}{1+\alpha_1}\sum_{i=0}^{k-1}\mathbf{A}^i\mathbf{Y}(t-i)$$

for $k \geq 1$ and $t = 0, \pm 1, \pm 2, \ldots$. Therefore, for $t \geq p$, D_t can be expressed in terms of the historical values of $\{Y_t\}$ and initial vector $\mathbf{D}(p-1)$ as

$$D_t = \sum_{j=1}^{p} A_{(1j)}^{t-p+1} D_{p-j} + \frac{1}{1+\alpha_1}\sum_{k=0}^{t-p} A_{(11)}^{k} Y_{t-k}, \tag{10}$$

where $A_{(ij)}^k$ is the ijth element of the matrix $\mathbf{A}^k$. Suppose that the manufacturer has a possibly noisy estimate, $\widetilde{\mathbf{D}}(p-1)$, of the initial vector. Using this estimate, the manufacturer can compute an estimate, $\widetilde{D}_t$, of the demand at time t using (10). The manufacturer's estimate of the retailer's demand converges to the true value over time if the contribution of the initial vector, $\widetilde{\mathbf{D}}(p-1)$, to $\widetilde{D}_t$ tends to zero as t goes to infinity. The following theorem provides the conditions under which this happens.

Theorem 2. *D_t is inferable from Y_t, that is, independent of the choice of $\widetilde{\mathbf{D}}(p-1)$, $\widetilde{D}_t$ converges to D_t almost surely as t goes to infinity, if the roots of the equation*

$$(1+\alpha_1)m^p - (\alpha_1 - \alpha_2)m^{p-1} - (\alpha_2 - \alpha_3)m^{p-2} - \cdots - \alpha_p = 0 \tag{11}$$

are less than one in absolute value.

Proof. Please see the appendix. □

Thus, Theorem 2 gives a rule by which a manufacturer can determine whether it can accurately estimate the retailer's demand from historical orders. If D_t is inferable, then the manufacturer can forecast its lead-time demand, $\sum_{k=1}^{L+1} Y_{t+k}$, by applying (4) and (10) recursively. The result in Theorem 2 implies that, even without explicit demand information from the retailer, the manufacturer's forecast of its lead-time demand converges to the same value as if there were complete information sharing between the retailer and the manufacturer. *Therefore, the need for information sharing diminishes as t increases if the roots of* (11) *are all less than* 1 *in absolute value.*

In the case of AR$(1,q)$ demand, Theorem 2 reduces to the following result.

Corollary 1. *Let D_t be $AR(1,q)$ with the representation $D_t = d + \rho D_{t-1} + \epsilon_t - \lambda_1\epsilon_{t-1} - \cdots - \lambda_q\epsilon_{t-q}$, where $|\rho| < 1$ and ϵ_t are uncorrelated $(0, \sigma^2)$ random variables. Then, D_t is inferable for all l if and only if $\rho \in (-0.5, 1)$.*

Proof. When D_t is AR$(1,q)$, (4) reduces to $Y_t = (1+\alpha_1)D_t - \alpha_1 D_{t-1}$, where $\alpha_1 = \rho + \rho^2 + \cdots + \rho^{l+1}$, and has the characteristic equation $(1+\alpha_1)\lambda - \alpha_1 = 0$ with the unique root $\lambda = \alpha_1/(1+\alpha_1)$. Thus, from Theorem 2, D_t is inferable if and only if

$$\left|\frac{\alpha_1}{1+\alpha_1}\right| = \left|\frac{\rho(1-\rho^{l+1})}{1-\rho^{l+2}}\right| < 1.$$

This gives us two inequalities:

$$-1 < \frac{\rho(1-\rho^{l+1})}{1-\rho^{l+2}} < 1.$$

The second inequality gives $\rho - \rho^{l+2} < 1 - \rho^{l+2}$, which is true for all $\rho < 1$. The first inequality gives $-1 + \rho^{l+2} < \rho - \rho^{l+2}$, which is rewritten as

$$2\rho^{l+2} - \rho < 1. \tag{12}$$

The left-hand side achieves its largest value when $l = 0$. Thus, D_t is inferable for all l if and only if $2\rho^2 - \rho < 1$, which implies that $\rho \in (-0.5, 1)$. □

Remark 4.1. In Corollary 1, consider the cases of even and odd values of l separately. For l odd, inequality (12) holds for all $\rho \in (-1, 1)$, so that D_t is inferable for all $\rho \in (-1, 1)$. For l even, the left-hand side of (12) is decreasing in l. Thus, for l even, the range of values of ρ for which D_t is inferable increases as l increases. For example, when $l = 2$, D_t is inferable for all $\rho \in (-0.64, 1)$, when $l = 4$, D_t is inferable for all $\rho \in (-0.72, 1)$, and so on.

Remark 4.2. Raghunathan (2001) showed that for AR(1) demand with $\rho \geq 0$, the value of information sharing declines monotonically as t increases and tends to 0 as t tends to infinity. Corollary 1 extends this result to the case when $\rho < 0$. It shows that his result extends to the case when $\rho \in (-0.5, 0)$, but not to the case when $\rho \leq -0.5$. Negative autocorrelation of demand can arise when the demand is generated by a process with negative feedback, e.g., due to high-low pricing in supermarkets, or due to technological evolution leading to business cycles in the semiconductor industry. Thus, even for AR(1) demand, information sharing can be valuable in such cases.

Remark 4.3. For general ARMA(p,q) demand processes, the property of invertibility gives a simple method for verifying the condition in Theorem 2. Recall the definition of invertibility in Property 2. For Y_t, the coefficients of lagged ϵ_t terms are obtained by applying the backshift operator polynomials, $\varphi(\mathscr{B})$ and $\psi(\mathscr{B})$, as in (7). Define $\tilde{\varphi}(m) = (1+\alpha_1)m^p - (\alpha_1 - \alpha_2)m^{p-1} - (\alpha_2 - \alpha_3)m^{p-2} - \cdots - \alpha_p$ and $\tilde{\psi}(m) = m^q - \lambda_1 m^{q-1} - \lambda_2 m^{q-2} - \cdots - \lambda_q$. Then, Y_t is invertible if and only if the roots of the equation

$$\tilde{\varphi}(m)\cdot\tilde{\psi}(m) = \left[(1+\alpha_1)m^p - (\alpha_1-\alpha_2)m^{p-1} - (\alpha_2-\alpha_3)m^{p-2} - \cdots - \alpha_p\right] \times \left[m^q - \lambda_1 m^{q-1} - \lambda_2 m^{q-2} - \cdots - \lambda_q\right] = 0 \tag{13}$$

are less than one in absolute value. Here, $\tilde{\varphi}(m)$ is generated by the transformation of D_t into Y_t, while $\tilde{\psi}(m)$ is generated by the moving-average terms of D_t, as defined in (1). It follows that *if the order process is invertible, then the manufacturer can infer D_t from Y_t without explicitly solving for the condition in Theorem* 2. Thus, the manufacturer can utilize standard time-series analysis software to estimate the parameters of the order process and verify if it is invertible. We find that invertibility gives a stronger result when Zhang's (2004) approach is used, that is, invertibility of Y_t is both necessary and sufficient for the inferability of demand. These results are useful in interpreting the numerical examples in the next section.

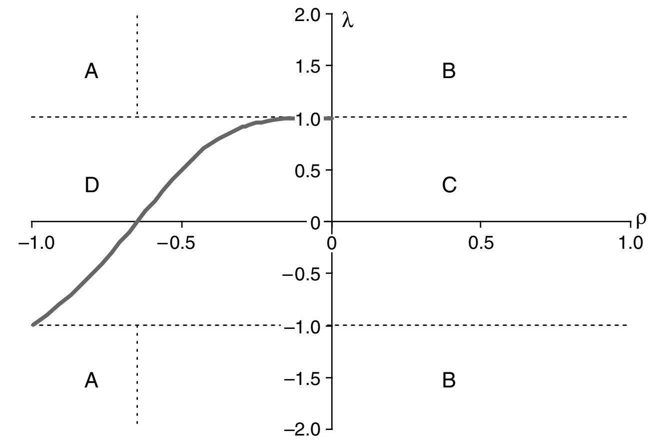
5. Value of Information Sharing

In this section, we provide decision rules for sharing of demand information and computing order quantities for t sufficiently large by integrating our results for noninvertible ARMA demand processes with Zhang's (2004) results for invertible ARMA demand processes. We then present a numerical study quantifying the benefits to the manufacturer from sharing of demand information between the retailer and the manufacturer, or from inferring demand using historical orders.

Table 1 summarizes how the time-series structure of the demand process impacts information sharing and ordering decisions. As an illustration, Figure 1 gives the range of parameter values for ARMA(1, 1) demand in which each of the implications in Table 1 hold.

When D_t is invertible, the retailer computes Y_t using the approach in Zhang (2004). In this case, Y_t is represented by an ARMA(p, max{p, $q-l$}) process, and may or may not be invertible. From Remark 4.3, when Y_t is invertible, the manufacturer can infer D_t accurately without direct access to demand information, and can further apply Zhang's approach to compute its orders to its supplier. When Y_t is not invertible, then the manufacturer cannot infer D_t

Figure 1 **Impact of the Values of Time-Series Parameters on the Need for Information Sharing for ARMA (1, 1) Demand**



Notes. A ($\rho \le -0.64$, $|\lambda| \ge 1$): D_t is neither invertible nor inferable; B ($\rho > -0.64$, $|\lambda| \ge 1$): D_t is not invertible but inferable; C (area to the right of the curve and with $|\lambda| < 1$): D_t and Y_t are both invertible; D (area to the left of the curve and with $|\lambda| < 1$): D_t is invertible but Y_t is not invertible. There is need for information sharing in areas A and D, but not in areas B and C. In this figure, the values of lead time are $l = L = 2$.

from Y_t and there is need for sharing demand information. Further, the manufacturer and all upstream decision makers in the supply chain will have to use our approach to compute their orders to their upstream partners.

When D_t is not invertible, the retailer computes Y_t using our approach. In this case, Y_t is represented by an ARMA(p, $p+q$) process. The need for information sharing in this case is determined by Theorem 2, i.e., if D_t is inferable, then there is no need for information sharing, while if D_t is not inferable, then there is need for information sharing. In addition, Y_t is itself not invertible, and the manufacturer and all upstream decision makers in the supply chain must continue to use our approach to compute their orders to their upstream partners.

We now present a numerical study to evaluate the value of demand information to the manufacturer. There are two cases. In Case 1, there is no sharing of

Table 1 **Implications of ARMA Demand for Information Sharing and Ordering Decisions**

			Implications for the manufacturer	
	Time series characterization of Y_t	Resulting properties	Information sharing	Ordering decision
D_t invertible	Based on Zhang (2004)	Y_t is invertible, and D_t is inferable	Not required	Use Zhang's approach
		Y_t is not invertible, and D_t is not inferable	Required	Use our approach
D_t not invertible	Based on our approach	Y_t is not invertible, but D_t is inferable	Not required	Use our approach
		Y_t is not invertible, and D_t is not inferable	Required	Use our approach

demand information, and further, the manufacturer does not attempt to infer D_t from Y_t. Instead, the manufacturer uses only the most recent p orders to forecast its lead-time demand. This case is analogous to the no-information-sharing case in LST. In Case 2, the manufacturer either uses shared demand information or infers the retailer's demand from historical orders to forecast its lead-time demand.

When D_t is not invertible, the conditional variance of the manufacturer's lead-time demand in the two cases is computed using our approach as follows. (The corresponding formulas for invertible demand can be obtained using Zhang's approach in a similar fashion.) In Case 1, the manufacturer uses the ARMA$(p, p+q)$ representation (7) of Y_t to forecast its lead-time demand. With some algebraic manipulation, it can be shown that the conditional variance of the manufacturer's lead-time demand at time t given $Y_t, \ldots, Y_{t-p+1}$ is a function of $\epsilon_{t-p-q+1}, \ldots, \epsilon_{t+L+1}$. In particular, if D_t is ARMA(1, 1), then we obtain

$$\mathrm{Var}\left[\sum_{k=1}^{L+1} Y_{t+k} \,\middle|\, Y_t, \ldots, Y_{t-p+1}\right] = \mathrm{Var}\left[\sum_{j=1}^{L+1} \frac{1-\rho^{L+2-j}}{1-\rho} c_1 \epsilon_{t+j} + \sum_{j=0}^{L} \frac{1-\rho^{L+1-j}}{1-\rho} c_2 \epsilon_{t+j} + \sum_{j=-1}^{L-1} \frac{1-\rho^{L-j}}{1-\rho} c_3 \epsilon_{t+j}\right], \quad (14)$$

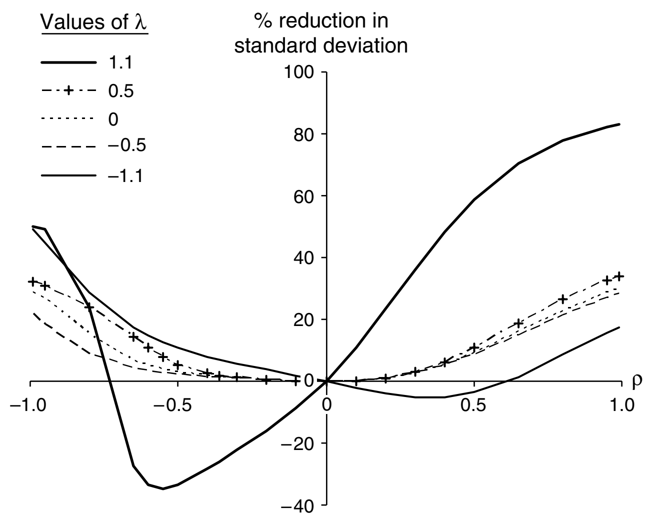
where $c_1 = 1+\alpha_1$, $c_2 = -a_1\lambda_1 - \alpha_1$, $c_3 = \lambda_1\alpha_1$.

In Case 2, first consider the situation when D_t is not inferable and the manufacturer and the retailer share demand information. The manufacturer can use (4) recursively to estimate its lead-time demand as a function of $D_t, D_{t-1}, \ldots$ and $\epsilon_{t-q+1}, \ldots, \epsilon_{t+L+1}$. From this procedure, it can be shown that the conditional variance of the manufacturer's lead-time demand at time t given $D_t, D_{t-1}, \ldots$ is a function of $\epsilon_{t-q+1}, \ldots, \epsilon_{t+L+1}$. If D_t is ARMA(1, 1), then we obtain

$$\mathrm{Var}\left[\sum_{k=1}^{L+1} Y_{t+k} \,\middle|\, D_t, D_{t-1}, \ldots\right] = \mathrm{Var}\left[\sum_{j=1}^{L+1} \frac{1-\rho^{L+2-j}}{1-\rho} c_1 \epsilon_{t+j} + \sum_{j=0}^{L} \frac{1-\rho^{L+1-j}}{1-\rho} c_2 \epsilon_{t+j} + \sum_{j=0}^{L-1} \frac{1-\rho^{L-j}}{1-\rho} c_3 \epsilon_{t+j} + \frac{1-\rho^{L+1}}{1-\rho} \alpha_1 \epsilon_t\right], \quad (15)$$

where c_1, c_2, c_3 are as defined in (14). Now consider the situation when D_t is inferable. Theorem 2 implies that, as t increases, the conditional variance of the manufacturer's lead-time demand when D_t is inferable converges to the conditional variance under full information sharing. Hence, the formula for the conditional variance of the manufacturer's lead-time demand in this situation is identical to (15).

Figure 2 **Impact of Shared or Inferred Demand Information on the Standard Deviation of Manufacturer's Lead Time for ARMA(1, 1) Demand**



Note. The values of the lead time are $l = L = 2$.

We compare Cases 1 and 2 numerically for a wide range of values of ρ, λ, l, and L. We say that demand information is valuable if the standard deviation in Case 2 is less than that in Case 1.[5] Figure 2 shows the percent change in standard deviation in Case 2 compared to Case 1 for the following values of parameters: $l = L = 2$, $\lambda \in \{-1.1, -0.5, 0, 0.5, 1.1\}$, and $\rho \in (-1, 1)$. The values of λ correspond to different cases in Table 1: $\lambda = -0.5, 0, 0.5$ correspond to the case when D_t is invertible and the retailer uses Zhang's approach; $\lambda = 1.1, -1.1$ correspond to the case when D_t is not invertible and the retailer uses our approach.

We observe that the change in standard deviation varies considerably with the values of ρ and λ. Demand information is always beneficial when ρ and λ have the same sign. Moreover, in these cases, the benefit of demand information is increasing in the magnitudes of ρ and λ. The average reduction in standard deviation is 41.1% for $\rho \geq 0.5$ and $\lambda \in [0.5, 2]$, and 12.2% for $\rho \leq -0.5$ and $\lambda \in [-2, -0.5]$. We also find that the benefit under ARMA(1, 1) demand is significantly higher than under AR(1) demand whenever both ρ and λ are positive.

We also observe that demand information may not be beneficial when demand is not invertible, and ρ and λ have opposite signs. This effect is explained by the inability of the manufacturer to decipher the

[5] We also evaluated the reduction in the manufacturer's average on-hand inventory and average cost. The conclusions using these variables are identical to those obtained by comparing the standard deviations in Cases 1 and 2.

ϵ_t terms from the demand information in these cases. Thus, demand information leads to an increase in the variance of the manufacturer's forecast of lead-time demand. As a result, the manufacturer is better off ignoring the demand information and treating the retailer's orders as an independent noninvertible ARMA time-series for planning its production.

6. Conclusions

We have shown how the value of sharing demand information in a supply chain depends on the time-series structure of the demand process. When both the demand process and the resulting order process are invertible, demand can be inferred by the manufacturer without requiring further information from the retailer. When demand is invertible but the resulting order process is not, sharing demand information is necessary. In the situation where demand is not invertible, we provide a rule to determine whether demand can be inferred from the order process. Using these results, we show how a manufacturer can determine when there is value to sharing demand information, inferring demand information, or treating the order process as an independent noninvertible ARMA time-series. Our numerical results show that the value of sharing demand information, where present, can be quite significant.

Our results are applicable to multistage supply chains because the order process at each successive level remains an ARMA process. Thus, each successive upstream player in the supply chain can use the same rules to determine the value of information sharing. Our results could be used for investigating the bullwhip effect in multistage supply chains under autoregressive demand. Our results could also form the basis for empirical investigations into the relationship between the time-series structure of demand and demand propagation, as well as value of information sharing in supply chains.

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Appendix

Proof of Theorem 2. We have

$$\widetilde{D}_t - D_t = \sum_{j=1}^{p} A_{(1j)}^{t-p+1}(\widetilde{D}_{p-j} - D_{p-j}) \le \sum_{j=1}^{p} \left|A_{(1j)}^{t-p+1}\right| \cdot \left|\widetilde{D}_{p-j} - D_{p-j}\right|. \quad (16)$$

We obtain a bound on the values of $A_{(ij)}^k$ in terms of the roots of the characteristic equation of matrix $\mathbf{A}$. Note that the characteristic equation of $\mathbf{A}$ is (11). By the Cayley-Hamilton theorem (Strang 1980, p. 235), $\mathbf{A}$ also satisfies this equation. Thus,

$$(1+\alpha_1)\mathbf{A}^p - (\alpha_1 - \alpha_2)\mathbf{A}^{p-1} - (\alpha_2 - \alpha_3)\mathbf{A}^{p-2} - \cdots - \alpha_p \mathbf{A}^0 = 0.$$

Multiplying by $\mathbf{A}^{k-p}$ where $k \ge p$, we get

$$(1+\alpha_1)\mathbf{A}^k - (\alpha_1 - \alpha_2)\mathbf{A}^{k-1} - (\alpha_2 - \alpha_3)\mathbf{A}^{k-2} - \cdots - \alpha_p \mathbf{A}^{k-p} = 0.$$

Because this equation must hold for all elements of $\mathbf{A}^k$, we have

$$(1+\alpha_1)A_{(ij)}^k - (\alpha_1 - \alpha_2)A_{(ij)}^{k-1} - (\alpha_2 - \alpha_3)A_{(ij)}^{k-2} - \cdots - \alpha_p A_{(ij)}^{k-p} = 0.$$

Let λ denote the largest of the absolute values of the roots of (11), and let $\lambda < 1$. Because $A_{(ij)}^k$ satisfies the characteristic equation of $\mathbf{A}$, there exists a c such that $|A_{ij}^k| < cM^k$, where $\lambda < M < 1$. Thus, $|A_{ij}^k|$ decays exponentially. Applying this to (16), and letting t tend to infinity for fixed p, we obtain the required result. □

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