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Monotone Forecasts

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In this paper we provide necessary and sufficient conditions for the distribution of demand in the future to be stochastically increasing in the demand that has been observed in the past. We base our analysis on the multiperiod inventory model examined by Eppen and Iyer (1997). In the process of establishing the necessary and sufficient conditions we develop a new property called the sequential monotone likelihood ratio property.

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1. Introduction

This paper reexamines the use of early demand information in a multiperiod inventory system setting following the work of Eppen and Iyer (1997). In this context, specifically, we investigate the intuitively appealing statement that "when you observe a higher demand in the past it is likely that you will observe higher demand in the future." Also, Lariviere and Porteus (1999, p. 359) state, "If a large demand is observed, the retailer assesses the underlying demand distribution as being larger, so another large demand is more likely than it was before observing the previous large demand." Formally, these statements are equivalent to the statement that the distribution of demand in the future is stochastically increasing in the demand that has been observed in the past. We name this property "Conditional Monotonicity" (CM). In this paper, we derive necessary and sufficient conditions for the CM property to hold. Eppen and Iyer (1997) showed that this property is pivotal for allowing a retailer to order more for the rest of the season if demand seen so far is higher than what was expected.

Understanding how to use early demand information is essential in today's competitive environment, in which customers are offered many innovative and short-life-cycle products (Fisher and Raman 1996, Lee 1996, Mahajan and van Ryzin 1998). Eppen and Iyer (1997) provide a model that firms can use to manage inventory based on progressive demand information. They model a multiperiod fashion retail inventory system where the problem in each period is to determine how much to increase inventory or how much to decrease inventory based on the observed demand as well as the forecasts for the remaining periods. First, they show that the decision to order more or to sell off excess inventory can be reduced to computing upper and lower bounds on the inventory. If the on-hand inventory level is below the lower bound, it must be increased up to the lower bound, and if it is above the upper bound, it should be decreased to the upper bound. These bounds are set at the beginning of the selling season. Then, as new demand information becomes available, the bounds are updated. Eppen and Iyer (1997) show that if CM holds, then the bounds increase or decrease with increase or decrease in the observed demand. The CM property is appealing to decision makers because they feel that they can order more inventory for the next period if the observed demand for the current period is high and do the opposite if the demand is low (Eppen and Iyer 1997). We show in this paper that despite its appeal, the CM property relies on certain strong necessary and sufficient conditions that must be verified to hold before the property is used.

We first present a summary of the notation used in Eppen and Iyer (1997). Then, we list the main assumptions made by Eppen and Iyer to obtain their results. After that, we describe the four theorems proved by them. It will become clear that their Theorem 3 establishes CM, which is then used to prove Theorem 4. Eppen and Iyer use the monotone likelihood ratio (MLR) property to establish CM in Theorem 3. However, this is a strong assumption. We examine how much we can weaken this assumption and still be able to prove Theorem 3. To that end, we show that a weaker property called the Sequential Monotone Likelihood Ratio (SMLR) property is both necessary and sufficient for Theorem 3 to hold. The SMLR property is intimately connected to CM, and thus it is of independent interest in itself.

2. Model

The decision maker in Eppen and Iyer (1997) is a catalog merchandiser. The buying season for this decision maker is divided into N time periods. At the beginning of each time period, she has to decide whether to buy more, sell off (liquidate) inventory, or do nothing according to the observed demand and forecast for the remaining time periods. The item sells for a constant price of \$$r$ per unit throughout the season. The decision maker incurs an initial purchase price \$$c$ per item. Holding cost of \$$h_t$ is incurred on the inventory held at the end of period t. When there is a selloff (inventory reduction) decision, \$$r_t$ is obtained per item liquidated. Apart from the initial purchase price, it costs the decision maker \$$c_t$ for each additional unit purchased at the beginning of period t. This model assumes lost sales, so there is no backorder cost. Unit penalty cost of π per unit is incurred when lost sales occur. The notation used is given below.

$N =$ number of periods in a season.

$t = 1, 2, \ldots, T =$ index for time periods.

$i = 1, 2, \ldots, I =$ index for pure demand processes.

$d_t =$ demand observed in period t.

$D_t = d_1 + d_2 + \cdots + d_t =$ cumulative demand observed through periods 1 to t.

$P_{i1} =$ prior probabilities over the set of pure demand processes at the beginning of Period 1.

$\phi_{i,(j,k)}(x) =$ probability that demand in periods j through k is equal to x for pure demand process i.

$\Phi_{i,(j,k)}(x) =$ cumulative probability distribution of demand in periods j through k for pure demand process i.

$\phi_t(x \mid D_{t-1}) =$ probability that demand in period t is x given D_{t-1}.

$\Phi_t(x \mid D_{t-1}) =$ probability that demand in period t is $\leqslant x$ given D_{t-1}.

$P_{it}(D_{t-1}) =$ probability that the demand is generated by pure demand process i given that the observed cumulative demand up to time $(t-1)$ is equal to D_{t-1}.

We now list the main assumptions used in Eppen and Iyer (1997).

- Demand is generated by one of several pure demand processes. The decision maker has a prior distribution over these demand processes. She updates this distribution at the end of each period using Bayes' formula (Box and Tiao 1973) after observing the demand for that period.
- Demand in each period for each pure demand process is independent of the demands in other periods as well as independent of the other pure demand processes. The fact that demand is independent of demands in other periods is not stated in the paper, but is used in the Bayesian updating formula.
- The demand processes are assumed to have the following four properties: (1) family consistency, (2) sum sufficiency, (3) stochastic dominance, and (4) the MLR property. We shall use labels 1 to 4 to refer to these properties.

Family consistency, or Property 1, means that for each pure demand process, the distribution of demand in adjacent periods must be members of the same family. Sum sufficiency, the second property, means that the sum of demands for periods 1 to t is sufficient to determine the posterior distribution of the processes that generated the observations. For Property 3 to hold, pure demand process $k+1$ should be stochastically larger than the pure demand process k (i.e., $\Phi_{k+1,(i,i)}(x) \leqslant \Phi_{k,(i,i)}(x)$ for $k = 1$ to $I-1$ and $1 \leqslant i \leqslant N$). Finally, for Property 4 or the MLR property, the ratio $\phi_{k+1,(i,i)}(x)/\phi_{k,(i,i)}(x)$ must be nondecreasing in x for each $k = 1$ to $I-1$ and $1 \leqslant i \leqslant N$. In this paper, we examine how to weaken Property 4 without affecting CM.

3. Analysis

Eppen and Iyer (1997) provide a dynamic programming formulation to determine the optimal solution. Four theorems are used by them to describe this optimal solution. In Theorem 1, they show that the cumulative demand, i.e., D_t, is a sufficient statistic to forecast the demand in periods $t+1$ through N. In Theorem 2, they prove the existence of a pair of lower and upper control bounds, namely $L_t(D_{t-1})$ and $U_t(D_{t-1})$. In period t, the decision maker compares the on-hand inventory level with these bounds. If on-hand inventory I_{t-1} is less than $L_t(D_{t-1})$, then she orders the difference $(L_t(D_{t-1}) - I_{t-1})$. Otherwise, if I_{t-1} is larger than the upper bound, then she sells the difference $(I_{t-1} - U_t(D_{t-1}))$. When the inventory is between the bounds, she does nothing. Observe that this theorem does not rely on Properties 1, 3, or 4. It therefore defines the inventory control policy regardless of whether CM holds or fails to hold.

After the first two theorems, Theorems 3 and 4 become the main theorems of interest. In Theorem 3, the authors establish the CM property, namely that $d\Phi_{t+1}(x \mid D_t)/dD_t \leqslant 0$, by using the stochastic dominance and MLR property. As pointed out by a referee, the assumption of stochastic dominance is redundant, because the MLR property implies stochastic dominance (see Shaked and Shanthikumar 1994, Theorem 1.C.1, p. 28). The authors use the CM property in Theorem 4 to show that the bounds behave intuitively, i.e.,

$$\text{if } D_{t-1} \leqslant D'_{t-1}, \quad \text{then } L_t(D_{t-1}) \leqslant L_t(D'_{t-1}) \text{ and } U_t(D_{t-1}) \leqslant U_t(D'_{t-1}).$$

This is the monotonicity result mentioned in the introduction. The solution is easy to implement and also to explain to managers, as described in Eppen and Iyer (1997) and Lariviere and Porteus (1999). The MLR condition used by Eppen and Iyer to prove Theorem 3 is sufficient, but not necessary. In what follows, we state the SMLR property and establish that it provides the necessary and sufficient conditions for CM.

Sequential Monotone Likelihood Ratio (SMLR) Property. Demand processes $i = 1$ to I, with density functions $\phi_{i,(1,t)}(y)$, are said to possess the SMLR $(\pi_1, \pi_2, \ldots, \pi_I)$ property if

$$\frac{\sum_{j=1}^{i} \pi_{I-i+j}\phi_{I-i+j,(1,t)}(y)}{\sum_{j=1}^{I-i} \pi_j \phi_{j,(1,t)}(y)}$$

is nondecreasing in y, for $i = 1$ to $I-1$.

Theorem 1. *Given I pure demand processes and also that Properties 1, 2, and 3 (i.e., family consistency, sum sufficiency, and stochastic dominance) hold, then $d\Phi_{t+1}(x \mid y)/dy \leqslant 0$ if and only if they are ordered according to the SMLR $(P_{11}, P_{21}, \ldots, P_{I1})$.*

Proof. The unconditional probability that the demand during periods 1 to t is equal to y is given by

$$\Pr(D_t = y) = \sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y). \tag{1}$$

By assumption, the probability that the demand in periods 1 to t is equal to y and demand in period $t+1$ is less than or equal to x is given by

$$\Pr(D_t = y \cap d_{t+1} \leqslant x) = \sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)\Phi_{i,(t+1,t+1)}(x). \tag{2}$$

Then, from (1) and (2), the conditional probability

$$\Pr(d_{t+1} \leqslant x \mid D_t = y) = \Phi_{t+1}(x \mid y) = \frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)\Phi_{i,(t+1,t+1)}(x)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)}. \tag{3}$$

The conditional probabilities that the demand in period $t+1$ is less than or equal to x given $D_t = y$ can be written as

$$\Pr(d_{t+1} \leqslant x \mid D_t = y) = \frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)\Phi_{i,(t+1,t+1)}(x)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)}. \tag{4}$$

Similarly,

$$\Pr(d_{t+1} \leqslant x \mid D_t = y+1) = \frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)\Phi_{i,(t+1,t+1)}(x)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)}. \tag{5}$$

Thus, for CM to hold, we require from (4) and (5) for any $x \in \mathbf{Z}$ and $y \in \mathbf{Z}$,

$$\frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)\Phi_{i,(t+1,t+1)}(x)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)} \geqslant \frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)\Phi_{i,(t+1,t+1)}(x)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)}. \tag{6}$$

In Lemma 3 in the Appendix, let f_i be $\Phi_{I-i+1,(t+1,t+1)}(x)$, let a_i be $P_{i1}\phi_i(y+1)$, and let b_i be $P_{i1}\phi_i(y)$. Observe that $\Phi_{I-i+1,(t+1,t+1)}(x)$ is nondecreasing in i due to Property 3 (stochastic dominance). Therefore, the f_is are nondecreasing, as required in Lemma 3. Then, using Lemma 3, we obtain

$$\frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)\Phi_{i,(t+1,t+1)}(x)} \leqslant \frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)\Phi_{i,(t+1,t+1)}(x)},$$

or, equivalently,

$$\frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)\Phi_{i,(t+1,t+1)}(x)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)\Phi_{i,(t+1,t+1)}(x)} \leqslant \frac{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y+1)}{\sum_{i=1}^{I} P_{i1}\phi_{i,(1,t)}(y)} \tag{7}$$

holds for every $\Phi_{I-i+1,(t+1,t+1)}(x)$ that is nondecreasing in i if and only if

$$\frac{\sum_{j=1}^{I-i} P_{j1}\phi_{j,(1,t)}(y+1)}{\sum_{j=1}^{I-i} P_{j1}\phi_{j,(1,t)}(y)} \leqslant \frac{\sum_{j=1}^{i} P_{(I-i+j)1}\phi_{I-i+j,(1,t)}(y+1)}{\sum_{j=1}^{i} P_{(I-i+j)1}\phi_{I-i+j,(1,t)}(y)} \tag{8}$$

for $i = 1$ to $I-1$ (see (17)). Thus,

$$\frac{\sum_{j=1}^{i} P_{(I-i+j)1}\phi_{I-i+j,(1,t)}(y)}{\sum_{j=1}^{I-i} P_{j1}\phi_{j,(1,t)}(y)}$$

has to be nondecreasing in y, or CM is obtained if and only if the SMLR $(P_{11}, P_{21}, \ldots, P_{I1})$ property is satisfied. □

Remark 1. When $I = 2$, then SMLR is MLR.

Proof. We show that (8) reduces to the MLR property for $I = 2$. Let $i = I - 1$. We can rewrite (8) as follows:

$$\frac{\sum_{j=1}^{1} P_{(1+j)1}\phi_{1+j,(1,t)}(y)}{\sum_{j=1}^{1} P_{j1}\phi_{j,(1,t)}(y)} \leqslant \frac{\sum_{j=1}^{1} P_{(1+j)1}\phi_{1+j,(1,t)}(y+1)}{\sum_{j=1}^{1} P_{j1}\phi_{j,(1,t)}(y+1)}.$$

After some manipulation, we obtain

$$\frac{\phi_{2,(1,t)}(y)}{\phi_{2,(1,t)}(y) + (d/dy)\phi_{2,(1,t)}(y)\triangle y} \leqslant \frac{\phi_{1,(1,t)}(y)}{\phi_{1,(1,t)}(y) + (d/dy)\phi_{1,(1,t)}(y)\triangle y}. \tag{9}$$

Cross multiplying, cancelling equal terms, and passing to the limit gives

$$\phi_{2,(1,t)}(y)\frac{d}{dy}\phi_{1,(1,t)}(y) \leqslant \phi_{1,(1,t)}(y)\frac{d}{dy}\phi_{2,(1,t)}(y)$$

or

$$\frac{d}{dy}\left[\frac{\phi_{2,(1,t)}(y)}{\phi_{1,(1,t)}(y)}\right] \geqslant 0. \tag{10}$$

This inequality is the MLR property. □

Corollary 1. *If the SMLR property has to hold over all possible prior distributions $(P_{11}, P_{21}, \ldots, P_{I1})$, then the demand distributions must be ordered as per MLR.*

Proof. It follows from Remark 1 by letting all but two adjacent P's be positive. □

As pointed out by a referee, Corollary 1 implies the following:

(a) If the decision maker is unsure about the prior probabilities and CM has to hold for all possible priors, then the demand distributions have to be ordered as per MLR.

(b) On the other hand, if the decision maker knows the prior probabilities and wants to gather more data as time goes on to collect additional demand information, then choosing among distributions that satisfy the SMLR property is sufficient for CM to hold. In other words, knowing the prior gives more flexibility in choosing the distributions of the pure demand processes.

Remark 2. (a) Eppen and Iyer (1997) assume that the sum sufficiency property holds in their model. We also assumed that this property holds to establish the necessary and sufficient conditions for CM. The SMLR property can be extended with some effort when sum sufficiency does not hold.

(b) We do not explicitly require family consistency to hold, but dropping this assumption will require that in most practical instances we have to drop the sum sufficiency assumption as well.

Remark 3. In a two-period setting, we can immediately extend and compare our results with other models that apply Bayesian forecasting to inventory models. For example, in most of the past work, such as Scarf (1959), Iglehart (1964), and Azoury (1985), the MLR property has been fundamental to obtaining the CM property. Scarf used the MLR property to show that CM holds for the exponential family. The sufficient statistic for this family is the sum $S=\sum_{i=1}^{n-1} D_i$. Iglehart extended the results for the range family, where the sufficient statistic is $S=\max_{1\leqslant i\leqslant n-1}\{D_i\}$. Later, Azoury showed that CM holds for the uniform, the Weibull, and the gamma distributions. In contrast, SMLR allows one to weaken some of the distributional requirements. We illustrate this weakening with a numerical example in Remark 6.

In addition to the examples (normal, Poisson, and negative binomial) given by Eppen and Iyer (1997), we note the following situation when sum sufficiency holds.

Remark 4. Let the density of pure demand process i be a truncated normal distribution with mean μ_i, standard deviation σ, and density given by

$$K_i e^{-(1/2)[(x-\mu_i)/\sigma]^2} \quad \text{where } x\geqslant 0 \quad \text{and}$$
$$K_i=\frac{\sigma}{\sqrt{2\pi}\left[\int_0^\infty e^{-(1/2)[(x-\mu_i)/\sigma]^2}\,dx\right]}.$$

Then, both sum sufficiency and SMLR (and MLR) properties hold. Please see the Appendix for the proof.

Remark 5. Let the number of periods $N=2$. Let the demand processes be as given in Remark 4, except that in the first period process i demand has standard deviation σ_i, then SMLR (and MLR) is obtained in the second period when

$$\sigma_i\leqslant\sigma_{i+1} \quad \text{and} \quad \frac{\mu_i}{\sigma_i^2}\leqslant\frac{\mu_{i+1}}{\sigma_{i+1}^2}.$$

Please see the Appendix for the proof.

When the pure demand processes have the normal distribution, this condition relaxes Eppen and Iyer's requirement that the processes have the same standard deviation.

Remark 6. With additional work, we can further weaken the conditions of Remark 5. Please see the Appendix for details. Table 1 displays four different cases for $I=3$. The prior probabilities are assumed to be equal (i.e., $P_{11}=P_{21}=P_{31}=1/3$). The condition in Eppen and Iyer (1997) holds for Case 1. MLR holds for Cases 1 and 3, yet fails in Cases 2 and 4. In Cases 2 and 3, the SMLR property allows us to relax the variance requirement of Remark 5. However, in Case 4, when large variances are chosen, both SMLR and MLR properties fail to hold.

Case 1. $\mu_1\leqslant\mu_2\leqslant\mu_3$ and $\sigma_1=\sigma_2=\sigma_3$, thus $\mu_3/\sigma_3^2\geqslant\mu_2/\sigma_2^2\geqslant\mu_1/\sigma_1^2$; see Remark 5.

Case 2. $\mu_1\leqslant\mu_2\leqslant\mu_3$ and $\sigma_1\leqslant\sigma_2\leqslant\sigma_3$, $\mu_1/\sigma_1^2\leqslant\mu_3/\sigma_3^2$, $\mu_1/\sigma_1^2\leqslant\mu_2/\sigma_2^2$; violates Remark 5 condition but satisfies weaker condition based on SMLR given in the Appendix.

Case 3. $\mu_1\leqslant\mu_2\leqslant\mu_3$ and $\sigma_1\leqslant\sigma_2\leqslant\sigma_3$, $\mu_3/\sigma_3^2=\mu_2/\sigma_2^2=\mu_1/\sigma_1^2$; Remark 5 conditions hold.

Case 4. $\mu_1\leqslant\mu_2\leqslant\mu_3$ and $\sigma_1\leqslant\sigma_2\leqslant\sigma_3$, $\mu_3/\sigma_3^2\leqslant\mu_2/\sigma_2^2\leqslant\mu_1/\sigma_1^2$; violates Remark 5 and also weaker conditions given in the Appendix.

Remark 7. It can be shown that the CM property is obtained in the general linear model of Bayesian forecasting; such models are described in West and Harrison (1989).

Table 1. Numerical example ($I=3$).

	Demand Process 1 $(\mu_1,\sigma_1,\mu_1/\sigma_1^2)$	Demand Process 2 $(\mu_2,\sigma_2,\mu_2/\sigma_2^2)$	Demand Process 3 $(\mu_3,\sigma_3,\mu_3/\sigma_3^2)$	Eppen and Iyer (1997) Condition	MLR Condition	SMLR Condition
Case 1	(1, 0.70, 2.04)	(2, 0.70, 4.08)	(3, 0.70, 6.12)	Holds	Holds	Holds
Case 2	(1, 0.70, 2.04)	(2, 0.85, 2.76)	(3, 1.10, 2.47)	Fails	Fails	Holds
Case 3	(1, 0.70, 2.04)	(1, 0.70, 2.04)	(2, 0.99, 2.04)	Fails	Holds	Holds
Case 4	(1, 1.00, 1.00)	(1, 2.00, 0.25)	(1, 3.00, 0.11)	Fails	Fails	Fails

4. Analysis of SMLR Property

Distributions that are ordered according to SMLR inherit some properties from the MLR order as described below. In our exposition, we restrict ourselves to continuous distributions with differentiable density functions. Let $X_1, X_2, \ldots, X_I$ be continuous random variables with densities $g_1, g_2, \ldots, g_I$ and distribution functions $G_1, G_2, \ldots, G_I$, respectively. Let $X_1, X_2, \ldots, X_I$ be ordered according to SMLR $(\pi_1, \pi_2, \ldots, \pi_I)$. Also, let $\pi_i > 0\ \forall i$.

Theorem 2. *Define two random variables Y_1 and Y_2 such that $Y_1 = X_k$ with probability $\pi_k / \sum_{j=1}^{i} \pi_j$, $k = 1, \ldots, i$, and $Y_2 = X_k$ with probability $\pi_k / \sum_{j=i+1}^{I} \pi_j$, $k = i+1, \ldots, I$. Then, Y_1 is smaller than Y_2 in* (a) *MLR order,* (b) *hazard rate order, and* (c) *the usual stochastic order.*

Proof. (a) Follows from the definition of SMLR.

(b) Follows from Theorem 1.C.1 (p. 28) of Shaked and Shanthikumar (1994).

(c) Follows from part (b) and Theorem 1.B.1 (p. 14) of Shaked and Shanthikumar (1994). □

Theorem 3. (a) *When $I = 3$, i.e., X_1, X_2, X_3 satisfy the SMLR (π_1, π_2, π_3) property, then X_1 is smaller than X_3 in the MLR order.*

(b) *Given that X_1, X_2, X_3 satisfy the SMLR (π_1, π_2, π_3) property, they also satisfy SMLR $(\tilde{\pi}_1, \tilde{\pi}_2, \tilde{\pi}_3)$ when the perturbed priors are given by*

(i) $$\tilde{\pi}_1 = \pi_1 + \epsilon, \quad \tilde{\pi}_2 = \pi_2\left(\frac{\pi_2 + \pi_3 - \epsilon}{\pi_2 + \pi_3}\right), \quad \tilde{\pi}_3 = \pi_3\left(\frac{\pi_2 + \pi_3 - \epsilon}{\pi_2 + \pi_3}\right), \qquad \epsilon \geqslant 0.$$

(ii) $$\tilde{\pi}_1 = \pi_1\left(\frac{\pi_1 + \pi_2 - \epsilon}{\pi_1 + \pi_2}\right), \quad \tilde{\pi}_2 = \pi_2\left(\frac{\pi_1 + \pi_2 - \epsilon}{\pi_1 + \pi_2}\right), \quad \tilde{\pi}_3 = \pi_3 + \epsilon, \qquad \epsilon \geqslant 0.$$

(iii) *No other perturbation in (π_1, π_2, π_3) preserves SMLR without additional assumptions.*

(c) *In general, when there are I pure demand processes, the allowed perturbations in the priors are*

(i) $$\pi_1 + \epsilon, \quad \pi_i\left(\frac{\sum_{j=2}^{I} \pi_j - \epsilon}{\sum_{j=2}^{I} \pi_j}\right), \qquad \epsilon \geqslant 0, i = 2, \ldots, I.$$

(ii) $$\pi_i\left(\frac{\sum_{j=1}^{I-1} \pi_j - \epsilon}{\sum_{j=1}^{I-1} \pi_j}\right), \qquad i = 1, \ldots, I-1 \text{ and } \pi_I + \epsilon,\ \epsilon \geqslant 0.$$

(d) *For $k \leqslant l$, let $Y_1 = X_i$ with probability $\pi_i / \sum_{j=1}^{k} \pi_j$ and $Y_2 = X_i$ with probability $\pi_i / \sum_{j=l}^{I} \pi_j$. Then, Y_1 is smaller than Y_2 in the MLR order.*

Proof of (a) and (b)(ii). Let

$$\tilde{\pi}_1 = \pi_1\left(\frac{\pi_1 + \pi_2 - \epsilon}{\pi_1 + \pi_2}\right),$$
$$\tilde{\pi}_2 = \pi_2\left(\frac{\pi_1 + \pi_2 - \epsilon}{\pi_1 + \pi_2}\right),$$

and $\tilde{\pi}_3 = \pi_3 + \epsilon$, $\epsilon \geqslant 0$. When $I = 3$, by the SMLR property, both

$$\frac{\pi_2 g_2(y) + \pi_3 g_3(y)}{\pi_1 g_1(y)} \quad \text{and} \quad \frac{\pi_3 g_3(y)}{\pi_1 g_1(y) + \pi_2 g_2(y)}$$

are nondecreasing in y.

Therefore, differentiating $(\pi_2 g_2(y) + \pi_3 g_3(y))/\pi_1 g_1(y)$ with respect to y yields

$$\frac{\pi_2 g_2'(y) + \pi_3 g_3'(y)}{\pi_2 g_2(y) + \pi_3 g_3(y)} \geqslant \frac{g_1'(y)}{g_1(y)}. \tag{11}$$

On the other hand, differentiating $\pi_3 g_3(y)/(\pi_1 g_1(y) + \pi_2 g_2(y))$ with respect to y yields

$$\frac{g_3'(y)}{g_3(y)} \geqslant \frac{\pi_1 g_1'(y) + \pi_2 g_2'(y)}{\pi_1 g_1(y) + \pi_2 g_2(y)}. \tag{12}$$

Observe that inequality (12) continues to hold when π_1 and π_2 are substituted with $\tilde{\pi}_1$ and $\tilde{\pi}_2$. In essence, each term of the left-hand side of the inequality is multiplied by the same term $(\pi_1 + \pi_2 - \epsilon)/(\pi_1 + \pi_2)$. Thus, cancelling this term gives us the same inequality. Second, we prove by contradiction that inequality (11) continues to hold.

Assume

$$\frac{\tilde{\pi}_2 g_2'(y) + \tilde{\pi}_3 g_3'(y)}{\tilde{\pi}_2 g_2(y) + \tilde{\pi}_3 g_3(y)} < \frac{g_1'(y)}{g_1(y)}.$$

We have two cases to consider:

Case 1. Assume $g_3'(y)/g_3(y) \geqslant g_1'(y)/g_1(y)$. Let $\rho = (\pi_1 + \pi_2 - \epsilon)/(\pi_1 + \pi_2)$. Note that $\rho \leqslant 1$. After substituting $\tilde{\pi}_2 = \pi_2 \rho$ and $\tilde{\pi}_3 = \pi_3 + \epsilon$, we obtain

$$\frac{\pi_2 \rho g_2'(y) + (\pi_3 + \epsilon) g_3'(y)}{\pi_2 \rho g_2(y) + (\pi_3 + \epsilon) g_3(y)} < \frac{g_1'(y)}{g_1(y)}.$$

Rearranging terms,

$$\frac{\pi_2 \rho g_2'(y) + \pi_3 \rho g_3'(y) + (\pi_3 + \epsilon - \pi_3 \rho) g_3'(y)}{\pi_2 \rho g_2(y) + \pi_3 \rho g_3(y) + (\pi_3 + \epsilon - \pi_3 \rho) g_3(y)} < \frac{g_1'(y)}{g_1(y)}.$$

Dividing each term by ρ, we obtain

$$\frac{\pi_2 g_2'(y) + \pi_3 g_3'(y) + (1/\rho)(\pi_3 + \epsilon - \pi_3 \rho) g_3'(y)}{\pi_2 g_2(y) + \pi_3 g_3(y) + (1/\rho)(\pi_3 + \epsilon - \pi_3 \rho) g_3(y)} < \frac{g_1'(y)}{g_1(y)}. \tag{13}$$

We know that $(\pi_3 + \epsilon - \pi_3 \rho)/\rho > 0$ and $g_3'(y)/g_3(y) \geqslant g_1'(y)/g_1(y)$. Thus, this inequality and (13) cannot both hold by Lemma 2. Therefore, $g_3'(y)/g_3(y)$ should be less than $g_1'(y)/g_1(y)$.

Case 2. Let $g_3'(y)/g_3(y) < g_1'(y)/g_1(y)$. If $g_3'(y)/g_3(y) \geqslant g_2'(y)/g_2(y)$, then by the analysis of Case 1, we have $g_1'(y)/g_1(y) > g_3'(y)/g_3(y)$ and $g_1'(y)/g_1(y) > g_2'(y)/g_2(y)$. However, then inequality (11) cannot hold. Therefore, $g_3'(y)/g_3(y) < g_2'(y)/g_2(y)$ and $g_3'(y)/g_3(y) < g_1'(y)/g_1(y)$ should simultaneously hold. In that case, inequality (12) fails to hold. Thus, $g_3'(y)/g_3(y) \geqslant g_1'(y)/g_1(y)$. Therefore, this gives us the desired contradiction; hence (11) continues to hold. Thus, the SMLR property is preserved for the perturbations of the priors shown in part (b)(ii).

Moreover, we have also shown that $g_3'(y)/g_3(y) \geqslant g_1'(y)/g_1(y)$. This implies that $g_3(y)/g_1(y)$ is nondecreasing in y. Thus, X_1 is smaller than X_3 in MLR order. This proves part (a). □

Proof of (b)(i). Observe that for the given perturbation of priors, inequality (11) continues to hold by the same argument used for (12) in part (b)(ii). To prove (12) holds, let

$$\rho = \left(\frac{\pi_2+\pi_3-\epsilon}{\pi_2+\pi_3}\right) \quad \text{and} \quad \tilde{\pi}_1 = \pi_1+\epsilon,$$

$$\tilde{\pi}_2 = \pi_2\left(\frac{\pi_2+\pi_3-\epsilon}{\pi_2+\pi_3}\right), \quad \tilde{\pi}_3 = \pi_3\left(\frac{\pi_2+\pi_3-\epsilon}{\pi_2+\pi_3}\right).$$

After rearranging, we obtain

$$\frac{\tilde{\pi}_1 g_1'(y)+\tilde{\pi}_2 g_2'(y)}{\pi_1 g_1(y)+\pi_2 g_2(y)} = \frac{(1/\rho)(\pi_1+\epsilon-\pi_1\rho)g_1'(y)+\pi_1 g_1'(y)+\pi_2 g_2'(y)}{(1/\rho)(\pi_1+\epsilon-\pi_1\rho)g_1(y)+\pi_1 g_1(y)+\pi_2 g_2(y)}.$$

By Lemma 2 and the fact that $g_3'(y)/g_3(y) \geqslant g_1'(y)/g_1(y)$ from part (a),

$$\frac{(1/\rho)(\pi_1+\epsilon-\pi_1\rho)g_1'(y)+\pi_1 g_1'(y)+\pi_2 g_2'(y)}{(1/\rho)(\pi_1+\epsilon-\pi_1\rho)g_1(y)+\pi_1 g_1(y)+\pi_2 g_2(y)} \leqslant \frac{g_3'(y)}{g_3(y)}.$$

Thus, (12) continues to hold. □

Proof of (b)(iii). Without additional assumptions, we cannot definitely say whether $g_1(y)/g_2(y)$ or $g_2(y)/g_3(y)$ is nondecreasing in y. Thus, due to part (a), perturbations that are of the form

$$\text{(i)} \quad \pi_1-\epsilon, \quad \pi_2\left(\frac{\pi_2+\pi_3+\epsilon}{\pi_2+\pi_3}\right), \quad \pi_3\left(\frac{\pi_2+\pi_3+\epsilon}{\pi_2+\pi_3}\right), \quad \epsilon \geqslant 0,$$

$$\text{(ii)} \quad \pi_1\left(\frac{\pi_1+\pi_2+\epsilon}{\pi_1+\pi_2}\right), \quad \pi_2\left(\frac{\pi_1+\pi_2+\epsilon}{\pi_1+\pi_2}\right), \quad \pi_3-\epsilon, \quad \epsilon \geqslant 0,$$

$$\text{(iii)} \quad \pi_2\pm\epsilon, \quad \pi_1\left(\frac{\pi_1+\pi_3\pm\epsilon}{\pi_1+\pi_3}\right), \quad \pi_3\left(\frac{\pi_1+\pi_3\pm\epsilon}{\pi_1+\pi_3}\right), \quad \epsilon \geqslant 0,$$

could violate (11) or (12). We can also show that any perturbation in priors can be represented as a sum of perturbations of the form in parts (b)(i) or (b)(ii) and perturbations which are of the above three forms. Thus, without additional assumptions, no other perturbations can be assumed to preserve the SMLR property. □

Proof of (c). Similar to part (b). □

Proof of (d). After grouping the pure demand distributions into three subsets $(X_1, \ldots, X_k)$, $(X_{k+1}, \ldots, X_{l-1})$, and $(X_l, \ldots, X_I)$, the desired result follows by using part (a). □

Remark 8. Unlike MLR, we are unable to show that the SMLR property implies that Property 3, i.e., stochastic dominance, holds. However, we conjecture that Property 3 is not redundant when SMLR is assumed instead of MLR.

It is worth contrasting Theorem 1, Theorem 3, and Corollary 1. Even though all of them pertain to whether CM is obtained when the prior distribution gets perturbed, they differ significantly. Theorem 1 implies that even though the priors at the beginning of each period are updated as more information becomes available, the conditional distribution of the demand in each period still satisfies CM. Thus, CM is always obtained regardless of the actual evolution of demand. In contrast, Theorem 3 states that if we were to perturb the original priors, then only certain perturbations will preserve the SMLR property. The two are different statements: The first pertains to updating the priors over time; the second to changing the priors before demand information is obtained. Finally, Corollary 1 states that if CM is required to hold for any initial priors, then the MLR property is necessary. Like Theorem 3, this is a statement about perturbing priors before any demand has been observed.

5. Conclusions

CM is a useful property to have when interpreting early demand information. It justifies the prediction of higher demand in the future based on larger sales in the past. However, CM cannot be assumed to hold automatically. In this paper, we derived the necessary and sufficient conditions for CM to hold in the Eppen and Iyer (1997) model. There are stronger conditions given in the literature under which CM holds. The interested reader is referred to the work of Cohen and Sacrowitz (1993, 1995) as well as the chapter on multivariate stochastic orders in Shaked and Shanthikumar (1994), for further details.

Appendix

Lemma 1. *If $a>0$, $b>0$, $c>0$, $d>0$, $\alpha>0$, then*

$$\frac{\alpha a+c}{\alpha b+d} \leqslant (\geqslant) \frac{a}{b} \tag{14}$$

if and only if

$$\frac{c}{d} \leqslant (\geqslant) \frac{a}{b}. \tag{15}$$

Proof. If (14) holds, then (15) is obtained by cross-multiplying terms, canceling αab from both sides and rearranging the remaining terms. If (15) holds, then there exists a $\delta > (<)1$ such that $\delta c/d = a/b$. Thus, we obtain

$$\frac{\alpha a + c}{\alpha b + d} \leqslant (\geqslant) \frac{\alpha a + \delta c}{\alpha b + d} = \frac{a}{b}. \quad \square$$

Lemma 2. *If $a/b \geqslant (\leqslant) c/d$, $a/b \geqslant (\leqslant) e/f$, and $b > 0$, $d > 0$, $f > 0$, then $a/b \geqslant (\leqslant)(e+\alpha c)/(f+\alpha d)$ for $\alpha \geqslant 0$.*

Proof. Cross-multiplying $a/b \geqslant (\leqslant)(e+\alpha c)/(f+\alpha d)$, we obtain $af + \alpha ad \geqslant (\leqslant) be + \alpha bc$. This inequality holds when $af \geqslant (\leqslant) be$ and $ad \geqslant (\leqslant) bc$ both hold, which can be obtained by cross-multiplying $a/b \geqslant (\leqslant) c/d$ and $a/b \geqslant (\leqslant) e/f$. $\square$

Lemma 3. *Let a_i, b_i, $i = 1, 2, \ldots, n$ be greater than zero. Then,*

$$\frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} \geqslant \frac{\sum_{i=1}^{n} f_i a_i}{\sum_{i=1}^{n} f_i b_i} \tag{16}$$

for all $0 \leqslant f_1 \leqslant f_2 \leqslant \cdots \leqslant f_n$ if and only if

$$\frac{\sum_{j=1}^{i} a_j}{\sum_{j=1}^{i} b_j} \geqslant \frac{\sum_{j=i+1}^{n} a_j}{\sum_{j=i+1}^{n} b_j} \tag{17}$$

for every $i = 1$ to n.

Proof. We shall establish this lemma by use of induction. Let $n = 2$. Then,

$$\frac{f_1 a_1 + f_2 a_2}{f_1 b_1 + f_2 b_2} = \frac{f_1(a_1+a_2) + (f_2 - f_1)a_2}{f_1(b_1+b_2) + (f_2 - f_1)b_2}.$$

Then, either $f_2 - f_1 > 0$ or $f_2 - f_1 = 0$. If the latter case holds, we have equality in (16). If the former case holds, then applying Lemma 1 we see that (16) is equivalent to (17). Thus, for (16) to be true it is necessary and sufficient that (17) holds.

Let the lemma be true for n. Assume that (17) holds for $i = 1$ to n. Then,

$$\frac{\sum_{i=1}^{n+1} f_i a_i}{\sum_{i=1}^{n+1} f_i b_i} = \frac{f_1(a_1+a_2) + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)a_i + (f_2 - f_1)\sum_{i=2}^{n+1} a_i}{f_1(b_1+b_2) + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)b_i + (f_2 - f_1)\sum_{i=2}^{n+1} b_i}. \tag{18}$$

In (17), let $i = 1$. Then,

$$\frac{a_1}{b_1} \geqslant \frac{\sum_{j=2}^{n+1} a_j}{\sum_{j=2}^{n+1} b_j}, \tag{19}$$

which implies by Lemma 1,

$$\frac{\sum_{j=1}^{n+1} a_j}{\sum_{j=1}^{n+1} b_j} = \frac{a_1 + \sum_{j=2}^{n+1} a_j}{b_1 + \sum_{j=2}^{n+1} b_j} \geqslant \frac{\sum_{j=2}^{n+1} a_j}{\sum_{j=2}^{n+1} b_j}. \tag{20}$$

Let $i = 2$. Then, by (17),

$$\frac{a_1 + a_2}{b_1 + b_2} \geqslant \frac{\sum_{j=3}^{n+1} a_j}{\sum_{j=3}^{n+1} b_j}. \tag{21}$$

Let $e = a_1 + a_2$ and $f = b_1 + b_2$. Then, (17) is satisfied by

$$\frac{e + \sum_{j=3}^{i} a_j}{f + \sum_{j=3}^{i} b_j} \geqslant \frac{e + \sum_{j=i+1}^{n+1} a_j}{f + \sum_{j=i+1}^{n+1} b_j} \tag{22}$$

for every $i = 2$ to $n+1$.

Moreover, using (22) and the fact that Lemma 3 holds for n,

$$\frac{e + \sum_{i=3}^{n+1} a_i}{f + \sum_{i=3}^{n+1} b_i} \geqslant \frac{f_1 e + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)a_i}{f_1 f + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)b_i}, \tag{23}$$

because $0 \leqslant f_1 \leqslant f_3 - f_2 + f_1 \leqslant f_4 - f_2 + f_1 \leqslant \cdots \leqslant f_{n+1} - f_2 + f_1$.

We know by (20) that

$$\frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} \geqslant \frac{\sum_{i=2}^{n+1} a_i}{\sum_{i=2}^{n+1} b_i} \tag{24}$$

or

$$\frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} = \frac{\beta(f_2 - f_1)\sum_{i=2}^{n+1} a_i}{(f_2 - f_1)\sum_{i=2}^{n+1} b_i}, \quad \beta \geqslant 1. \tag{25}$$

Then, by Lemma 1 and (23),

$$\begin{aligned}
&\frac{\sum_{i=1}^{n+1} a_i}{\sum_{i=1}^{n+1} b_i} \\
&\geqslant \frac{\beta(f_2 - f_1)\sum_{i=2}^{n+1} a_i + f_1(a_1+a_2) + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)a_i}{(f_2 - f_1)\sum_{i=2}^{n+1} b_i + f_1(b_1+b_2) + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)b_i} \\
&\geqslant \frac{f_1(a_1+a_2) + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)a_i + (f_2 - f_1)\sum_{i=2}^{n+1} a_i}{f_1(b_1+b_2) + \sum_{i=3}^{n+1}(f_i - f_2 + f_1)b_i + (f_2 - f_1)\sum_{i=2}^{n+1} b_i} \\
&= \frac{\sum_{i=1}^{n+1} f_i a_i}{\sum_{i=1}^{n+1} f_i b_i}.
\end{aligned}$$

This shows the sufficiency of (17) for $n+1$.

To establish the necessity of (17) we have only to produce a single $0 \leqslant f_1 \leqslant f_2 \leqslant f_3 \leqslant \cdots \leqslant f_{n+1}$ for which (16) does not hold when (17) does not hold. Let the inequality in (17) fail to hold for some i. In this case, set $f_1 = f_2 = f_3 = \cdots = f_i = 0$, and the rest of the fs equal to one. Then, due to the fact that (17) fails to hold for i,

$$\frac{\sum_{j=1}^{i} a_j}{\sum_{j=1}^{i} b_j} < \frac{\sum_{j=i+1}^{n+1} a_j}{\sum_{j=i+1}^{n+1} b_j}.$$

Therefore, by Lemma 1,

$$\frac{\sum_{j=1}^{i} a_j + \sum_{j=i+1}^{n+1} a_j}{\sum_{j=1}^{i} b_j + \sum_{j=i+1}^{n+1} b_j} < \frac{\sum_{j=i+1}^{n+1} a_j}{\sum_{j=i+1}^{n+1} b_j} = \frac{\sum_{j=1}^{n+1} f_i a_i}{\sum_{j=1}^{n+1} f_i b_i},$$

which is the desired contradiction. $\square$

Condition for Remark 4. Let the demand observed in periods 1 to n be $x_1, x_2, \ldots, x_n$. Let $\bar{x} = \sum_{i=1}^{n} x_i/n$. Then,

$$\begin{aligned}\Pr(d_1 = x_1, d_2 = x_2, \ldots, d_n = x_n \mid \text{distribution is } i)\\ &= K_i \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{x_j-\mu_i}{\sigma}\right)^2\right]\\ &= K_i \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{x_j-\bar{x}}{\sigma}\right)^2 - \frac{n}{2}\left(\frac{\bar{x}-\mu_i}{\sigma}\right)^2\right]\\ &= MK_i \exp\left[-\frac{n}{2}\left(\frac{\bar{x}-\mu_i}{\sigma}\right)^2\right], \end{aligned} \tag{26}$$

where

$$M = \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\left(\frac{x_j-\bar{x}}{\sigma}\right)^2\right].$$

The value of M is independent of distribution i. Hence,

$$\Pr(\text{distribution is } i) = \frac{K_i \exp\left[-(n/2)\left((\bar{x}-\mu_i)/\sigma\right)^2\right]}{\sum_{i=1}^{I} K_i \exp\left[-(n/2)\left((\bar{x}-\mu_i)/\sigma\right)^2\right]}. \tag{27}$$

Thus, $\bar{x}$ is the sufficient statistic. Similarly, it can be verified that SMLR (and MLR) hold.

Condition for Remark 5. Consider the case when $I = 2$. Let the densities of the demand in the first two periods be

$$A(x) = K_1 e^{-(1/2)[(x-\mu_1)/\sigma_1]^2} \text{ and } B(x) = K_2 e^{-(1/2)[(x-\mu_2)/\sigma_2]^2}$$

respectively. If SMLR (π_1, π_2) holds, then $(\pi_2 B)/(\pi_1 A)$ should be increasing in x. We differentiate $(\pi_2 B)/(\pi_1 A)$ with respect to x to obtain

$$\frac{d}{dx}\left[\frac{\pi_2 K_2 e^{-(1/2)[(x-\mu_2)/\sigma_2]^2}}{\pi_1 K_1 e^{-(1/2)[(x-\mu_1)/\sigma_1]^2}}\right] = \left(\frac{\pi_2}{\pi_1}\right)\left[\frac{B'}{A} - \frac{BA'}{A^2}\right] = \left(\frac{\pi_2}{\pi_1}\right)\frac{B}{A}\left[\frac{x-\mu_1}{\sigma_1^2} - \frac{x-\mu_2}{\sigma_2^2}\right]. \tag{28}$$

Observe that

1. $[(x-\mu_1)/\sigma_1^2 - (x-\mu_2)/\sigma_2^2]$ is increasing in x if $1/\sigma_1^2 \geqslant 1/\sigma_2^2$.
2. Thus, at $x = 0$, $[(x-\mu_1)/\sigma_1^2 - (x-\mu_2)/\sigma_2^2] = \mu_2/\sigma_2^2 - \mu_1/\sigma_1^2 \geqslant 0$ is necessary and sufficient for SMLR (π_1, π_2) to hold.
3. It can be shown by induction that when $I = n$ we need

$$\frac{1}{\sigma_1^2} \geqslant \frac{1}{\sigma_2^2} \geqslant \cdots \geqslant \frac{1}{\sigma_n^2} \quad \text{and} \quad \frac{\mu_n}{\sigma_n^2} \geqslant \frac{\mu_{n-1}}{\sigma_{n-1}^2} \geqslant \cdots \geqslant \frac{\mu_2}{\sigma_2^2} \geqslant \frac{\mu_1}{\sigma_1^2}.$$

Condition for Remark 6. Let

$$I = 3 \quad \text{and} \quad C(x) = K_3 e^{-(1/2)[(x-\mu_3)/\sigma_3]^2}.$$

Let $\sigma_1 \leqslant \sigma_2$, $\sigma_1 \leqslant \sigma_3$, and $\mu_1/\sigma_1^2 \leqslant \mu_3/\sigma_3^2$, $\mu_1/\sigma_1^2 \leqslant \mu_2/\sigma_2^2$.

Observe that for SMLR to hold ($I = 3$, $i = 2$ in (8)), we need $(\pi_2 B + \pi_3 C)/(\pi_1 A)$ to be increasing in x. This is guaranteed by the conditions stated in the previous remark. We also need ($I = 3$, $i = 1$ in (8)) $\pi_3 C/(\pi_2 A + \pi_1 B)$ to be increasing in x. After differentiation with respect to x, this simplifies to requiring

$$\pi_2 B\left[\frac{x-\mu_2}{\sigma_2^2} - \frac{x-\mu_3}{\sigma_3^2}\right] + \pi_1 A\left[\frac{x-\mu_1}{\sigma_1^2} - \frac{x-\mu_3}{\sigma_3^2}\right] \geqslant 0$$

or

$$\frac{\pi_2 B}{\pi_1 A}\left[\frac{x-\mu_2}{\sigma_2^2} - \frac{x-\mu_3}{\sigma_3^2}\right] + \left[\frac{x-\mu_1}{\sigma_1^2} - \frac{x-\mu_3}{\sigma_3^2}\right] \geqslant 0. \tag{29}$$

Therefore, if

$$\frac{x-\mu_2}{\sigma_2^2} \geqslant \frac{x-\mu_3}{\sigma_3^2} \quad \text{as} \quad \left[\frac{x-\mu_1}{\sigma_1^2} \geqslant \frac{x-\mu_3}{\sigma_3^2}\right]$$

by assumption for all x, the result is trivial. But we know

$$\frac{\pi_2 B}{\pi_1 A}\left[\frac{x-\mu_2}{\sigma_2^2} - \frac{x-\mu_3}{\sigma_3^2}\right] + \left[\frac{x-\mu_1}{\sigma_1^2} - \frac{x-\mu_3}{\sigma_3^2}\right] \geqslant 0, \quad \text{when } x \geqslant \frac{\mu_2\sigma_3^2 - \mu_3\sigma_2^2}{\sigma_3^2 - \sigma_2^2}. \tag{30}$$

Therefore, as $(\pi_2 B)/(\pi_1 A)$ is increasing, it is sufficient if

$$\sup_{x \in \left[0,\, (\mu_2\sigma_3^2 - \mu_3\sigma_2^2)/(\sigma_3^2 - \sigma_2^2)\right]} \left(\frac{\pi_2 B}{\pi_1 A}\right)\left[\frac{x-\mu_2}{\sigma_2^2} - \frac{x-\mu_3}{\sigma_3^2}\right]_{x=0} + \left[\frac{x-\mu_1}{\sigma_1^2} - \frac{x-\mu_3}{\sigma_3^2}\right]_{x=0} \geqslant 0. \tag{31}$$

This condition is weaker than MLR. Table 1 was constructed using this inequality. □

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