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Information Processing Letters 57 (1996) 273–277

Information Processing Letters

The two headed disk: Stochastic dominance of the greedy policy

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Received 1 March 1995; revised 1 July 1995
Communicated by S.C. Akl

Abstract

In his paper "Should the two-headed disk be greedy? – Yes, it should" Hofri defined a "greedy policy" as follows. Assuming that the range of disk addresses is [0,1], a request at location x is served by the closest arm while the other arm jockeys to a new position, z, where $z=(1/3)x$ or $z=2/3+x/3$ depending on whether x is larger or smaller than $1/2$. Hofri proved that this policy minimizes the expected seek distance for uniform request probabilities and conjectured that it stochastically dominates every other policy. Stochastic dominance is of practical importance in this context as it guarantees that a policy that optimizes expected seek distance also guarantees optimal seek time. The main result of this paper is a proof of Hofri's conjecture. The paper contains two proofs, the first establishes the conjecture, and the second shows that if the seek distance is stochastically minimized under a repositioning policy, then the policy must be Hofri's greedy policy and the request distribution must be uniform.

Keywords: Two headed disk; Seek time; Stochastic optimality; Analysis of algorithms

1. Introduction

Due to the mismatch between the speed of processors and the I/O subsystem, there are many research papers dealing with methods of minimizing disk seek times. One common approach is to use multiple disk arms, or one arm with multiple heads attached to it. A suitable policy needs to be devised as to which arm (or head) will service the next request. Cases where one arm has two heads attached to it with a fixed distance between them was analyzed in [1]. The case where the disk arms move independently was analyzed in [2], where it was also suggested that while one arm services the next request, the other one can jockey to a new position to anticipate the next request. In [3] the author suggested moving the disk head to an anticipatory position between requests to minimize the seek distance. In [4], an analysis of optimal anticipation points for reading large objects is given. In what follows we use the term strategy to refer to the idea of allowing a head to reposition in anticipation of the next request and the term policy to denote where to reposition the head.

In most of the above work the quantity analyzed is the *expected seek distance* (ESD) rather

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SSDI 0020-0190(95)00210-3

than the *expected seek time* (EST). Although seek time is an increasing function of seek distance, it does not necessarily follow that a policy that minimizes the ESD also minimizes the EST. However, if policies that minimize the expected seek distance also minimize the seek distance in a stochastic sense (to be defined below), the expected seek time will also be minimized. Assuming that the range of disk addresses is [0,1], we prove in Theorem 1 that Hofri's greedy policy is optimal in the stochastic sense when the requests are uniformly distributed on [0,1]. However, as shown in Theorem 2, the existence of an optimal policy in the stochastic sense in this case implies that the request distribution is uniform. We interpret this result as indicative of the fact that policies that are optimal in the stochastic sense need not exist in general, and that in practice the user is better off seeking to minimize the expected value of the travel distance. In preparation for the proofs, we give below the definition of usual stochastic order and provide an example to illustrate its application in our context.

Definition 1 (see [5]). A random variable X is said to be smaller in the usual stochastic order (sense) compared to another random variable Y, denoted by $X \leqslant_{\mathrm{st}} Y$, if $\mathrm{P}(X > t) \leqslant \mathrm{P}(Y > t)$, $\forall t$.

Definition 2. Let D_X be the random variable representing the seek distance under policy X for some known distribution of requests. A policy S stochastically dominates a policy T if $D_S \leqslant_{\mathrm{st}} D_T$.

Lemma 3 (see [5]). $X \leqslant_{\mathrm{st}} Y \Leftrightarrow \mathrm{E}[f(X)] \leqslant \mathrm{E}[f(Y)]$ *for all non-decreasing functions* $f(\,)$ *and when the expectations exist, where* E[ ] *denotes the expectation of a random variable.*

By Lemma 3, we also have the implication, $X \leqslant_{\mathrm{st}} Y \Rightarrow \mathrm{E}[X] \leqslant \mathrm{E}[Y]$. Thus minimization in the usual stochastic order implies minimization of the ETD (and EST too).

The notion of being smaller in the usual stochastic order can be quite useful in the present context. To illustrate this fact we consider a disk with four cylinders and *one* head and look at policies which minimize the seek time to service the next request by repositioning the disk head between requests as in [3]. The request probabilities for the four cylinders are 0.21, 0.30, 0.45 and 0.04 (see Fig. 1). Denote the travel distance to service the next request measured in cylinders as D. We consider two policies S and S' for positioning the disk head between requests. Under S we position it at cylinder #2 and under S' at cylinder #3. It is easy to verify that under policy S, the expected distance to service the next request is 0.74 which is smaller than the expected distance of 0.76 under S'. However, the travel distance is not minimized stochastically under this policy. This is because when the head is at cylinder #2, the probability that D is equal to zero is 0.3 (written as $\mathrm{P}(D=0)=0.3$) whereas by repositioning the head at cylinder #3, $\mathrm{P}(D=0) = 0.45$. As a consequence, if the seek *time* is given by $D^{0.5}$ (which is a reasonable choice as the time to travel is often a concave function of the distance traveled), then the expected seek time under S is $(0.21 + 0.45 + 0.04 \times 2^{0.5}) = 0.7166$, whereas under S' it is $(0.21 \times 2^{0.5} + 0.30 + 0.04) = 0.637$. This reversal cannot happen if the repositioning strategy minimizes the distance in the usual stochastic order, see Lemma 3.

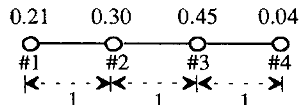

Fig. 1.

2. The disk model

We first describe Hofri's model of a two headed disk. In this model, each head can move independently but there is only one data path, i.e., only one head can read at a time. As one of the heads services the current request, the other head can jockey to any position at zero cost. One head has to go to attend to this request, and the nearest head should go to x (this is the reason why we say that the policy is greedy). The other head is allowed to jockey in anticipation of the next request. Let it be *re*positioned at y. We allow cross over of the two heads, so y can take any value in [0, 1]. Let $D(x, y)$ be the random distance to be traveled to attend to the next request. Hofri [2] showed that the jockeying policy

$$y = \begin{cases} \frac{2}{3} + \frac{x}{3} & \text{if } x \leqslant \frac{1}{2}, \\ \frac{x}{3} & \text{if } x > \frac{1}{2} \end{cases}$$

minimizes the expected travel distance and conjectured it also minimizes the travel distance in distribution (which is equivalent to saying that the travel distance is the smallest in the usual stochastic order). We prove this conjecture in Theorem 4. Denote the probability that $D(x, y)$ is greater than a as $P(D(x, y) > a)$, $a \in [0, 1]$. Due to symmetry we need only analyze the requests which occur at x, $x < 1/2$.

Theorem 4. *$P(D(x, y) > a)$ is minimized by Hofri's policy for a two headed disk, when the head not attending the current request jockeys to position y, when requests do not interfere with one another, and the distribution of requests is i.i.d. and uniform on* [0, 1].

Proof. As discussed above, assume without loss of generality that $x < 1/2$. First we prove that repositioning should be done at $y > x$. We show this through contradiction. Let $y \leqslant x$. Call this policy I. Consider an alternate policy II, that sets $y' = 1 - y$. Under the alternate policy the two heads are at locations $(x, 1-y)$. For II, consider the *suboptimal* policy of attending the next request as given below:

(i) Head at $(1-y)$ attends to all requests in the region $[1-(x+y)/2, 1]$.

(ii) Head at x attends to all requests in the region $[0, 1-(x+y)/2]$.

Under policy I, use the nearest head policy for attending the next request. Thus we have:

(iii) Head at y attends to all requests in the region $[0, (x+y)/2]$.

(iv) Head at x attends to requests in the region $[(x+y)/2, 1]$.

These policies are depicted in Fig. 2.

Let Z be the location of the next request. Denote equality in distribution by $\overset{d}{=}$. From (i), (iii), and symmetry (see Fig. 2), the conditional distributions:

$$(D(x, y) \mid Z \in [0, (x+y)/2]) \overset{d}{=} (D(x, 1-y) \mid Z \in [1-(x+y)/2, 1]).$$

Thus (i) and (iii) can be set off against one another. In (ii) and (iv) the common region is $[(x+y)/2, 1-(x+y)/2]$, see Fig. 2. This implies

$$(D(x, y) \mid Z \in [(x+y)/2, 1-(x+y)/2]) \overset{d}{=} (D(x, 1-y) \mid Z \in [(x+y)/2, 1-(x+y)/2]).$$

So we are left to compare attending requests in $[0, (x+y)/2]$ (using the head at x) under II

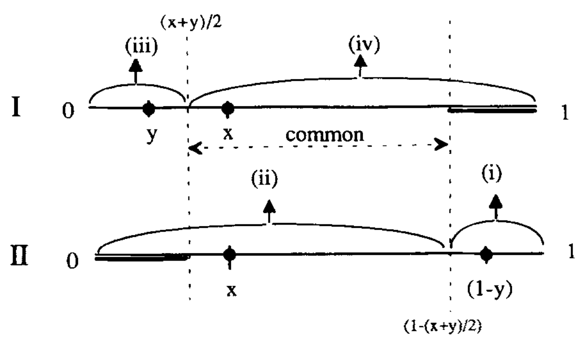

Fig. 2.

versus attending requests in $[1-(x+y)/2, 1]$ using the head at x under I, see the heavy lines in Fig. 2. From the assumptions follows

$$x-(x+y)/2=(x-y)/2<1-(x+y)/2-x \quad \text{as } 2x<1$$

and

$$x<1-x \quad \text{as } 2x<1.$$

This implies that the point x is closer to each point in the strip $[0, (x+y)/2]$ than to the corresponding point in the strip $[1-(x+y)/2, 1]$. (The correspondence is that a point t in $[0, (x+y)/2]$ gets mapped to the point $(1-(x+y)/2+t)$ in the other strip.) So policy II is better than I (in the usual stochastic order). Now consider $y>x$. We obtain

$$\begin{aligned} &\mathrm{P}(D(x, y)>a) \\ &\quad =(x-a)^{+}+2((y-x)/2-a)^{+} \\ &\qquad +(1-y-a)^{+}, \qquad (1) \\ &\mathrm{P}(D(x, 2/3+x/3)>a) \\ &\quad =(x-a)^{+}+3(1/3-x/3-a)^{+}, \qquad (2) \end{aligned}$$

where $(z)^{+}$ is the positive part of z. Whenever a is greater than or equal to $(1/3-x/3)$, (2) is not larger than (1). So assume that a is less than $(1/3-x/3)$. We need to consider several cases:

Case (i) $(y-x)/2<a$: $(1-y-a)-(1-x-3a)=(2a-(y-x))>0$ by hypothesis for this case. This implies, $(1-y-a)^{+}\geqslant(1-x-3a)^{+}\Rightarrow 2((y-x)/2-a)^{+}+(1-y-a)^{+}\geqslant 3(1/3-x/3-a)^{+}\Rightarrow(1)\geqslant(2)$.

Case (ii) (a): $(y-x)/2\geqslant a$ and $(1-y)\geqslant a$. In this case it can be verified that (1) = (2), as

$$2((y-x)/2-a)+(1-y-a)=1-x-3a.$$

Case (ii) (b): $(y-x)/2\geqslant a$ and $(1-y)<a$. (1) $-(2)=y-x-2a-(1-x-3a)=(-1+y+a)>0$ by hypothesis for this case. □

We next show that the "only" continuous distribution on [0, 1] that has this property (of minimizing the travel distance for a two headed disk in the usual stochastic order) is the uniform distribution, and the only policy that has this property is the greedy policy. The qualification on "only" is made clear below.

Theorem 5. *Let $f(x)\,\mathrm{d}x$ be the probability that a request for a record will be made in the interval $[x, x+\mathrm{d}x]$. Let it be given that $f(x)$ is continuous and $f(x)>0$, $\forall x\in[0, 1]$. Assume that the nearest head attends the next request and the other head will be repositioned to minimize the expected travel distance for the subsequent request. If this policy stochastically minimizes the travel distance, then $f(x)$ must be constant on* [0, 1], *and the policy must be the greedy policy.*

Proof. Assume without loss of generality that the current request is made at location $x\geqslant\frac{1}{2}$. Denote the travel distance for the subsequent request as D. Then we first claim that the repositioning of the free head has to be done at location $x/3$. To prove this first assume that the head is repositioned at $y>x$. But this can not happen, because $\mathrm{P}(D>1/2)>0$ under this policy, whereas this probability is equal to zero when $y=x/3$. Therefore $y<x$. Next consider the nondecreasing function $g(d)=[d-x/3]^{+}$. The repositioning policy stochastically minimizes the travel for the subsequent request, therefore $\mathrm{E}[g(D)]$

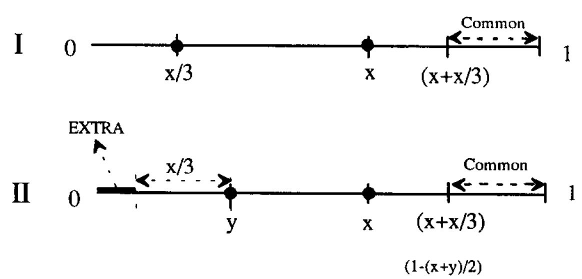

Fig. 3.

must be minimized under it. Let $y=x/3$ for policy I, and $x/3 \neq y < x$ for policy II. Denote the position of the next request as the random variable Z, $I\{A\}$ to be the indicator of the set A, and the expectations under the two policies by $\mathrm{E_I}[\ ]$ and $\mathrm{E_{II}}[\ ]$. Then, referring to Fig. 3, we must have

$$\mathrm{E_I}[g(D)] = \mathrm{E}\left[((Z-x)-x/3)^+ I\{Z \geqslant x\}\right] < \mathrm{E_{II}}[g(D)].$$

This leads to a contradiction that policy II stochastically minimizes the travel distance, and shows that the optimal policy must be Hofri's greedy policy.

We now show that the request distribution must be uniform. If the repositioning is done at y, then the expected travel distance is given by

$$\mathrm{E}[D] = \int_0^y (y-z) f(z)\,\mathrm{d}z + \int_y^{(x+y)/2} (z-y) f(z)\,\mathrm{d}z + \int_{(x+y)/2}^{x} (x-z) f(z)\,\mathrm{d}z + \int_x^1 (z-x) f(z)\,\mathrm{d}z. \tag{3}$$

As $\mathrm{E}[D]$ is minimized under the optimal policy, the first derivative of the expression in (3) with respect to y must be zero at $y=x/3$. Using the rule for differentiation under the integral sign:

$$\frac{\mathrm{dE}[D]}{\mathrm{d}y} \equiv \int_0^y f(z)\,\mathrm{d}z + [(y-z)f(z)]_{z=y} - \int_y^{(x+y)/2} f(z)\,\mathrm{d}z + \frac{1}{2}[(z-y)f(z)]_{z=(x+y)/2} - \frac{1}{2}[(x-z)f(z)]_{z=(x+y)/2} + 0 = 0. \tag{4}$$

Leading to:

$$[F(y) - (F((x+y)/2) - F(y))]_{y \doteq x/3} = 0 \Leftrightarrow F(2x/3) = 2F(x/3). \tag{5}$$

The last relation given in (5) leads to the conclusion that $F(x)$ is linear in x. □

Acknowledgments

We are grateful for the detailed comments provided by Professor Micha Hofri of Rice University on an earlier version of this paper. His comment, that (5) implies $F(x)$ is linear in x, resulted in simplifying the proof of Theorem 5. We also thank two anonymous referees for their comments and suggestions that improved the presentation of this paper.

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