Instructor: Victor F. Araman February 18, 2005

Stochastic Simulation

Homework 2

due on Monday, 28 of February, 2005

Exercise 1 The infamous RANDU number generator is given by

$$X_n = (2^{16} + 3)X_{n-1} \mod 2^{31}$$

(i) Prove that $(X_{n+2} - 6X_{n+1} + 9X_n) \mod 2^{31} = 0$

(ii) $U_n = X_n/2^{31}$. Prove that $-6 < U_{n+2} - 6U_{n+1} + 9U_n < 10$

(iii) Conclude from (i) and (ii) that (U_n, U_{n+1}, U_{n+2}) lie on at most 15 hyperplanes.

Exercise 2 Let X be a random variable with density function

$$f_X(x) = \begin{cases} 0, & x < 0, \\ 1/4, & 0 \le x \le 1, \\ x - 1, & 1 < x \le 2, \\ \exp(-4(x-2)), & 2 < x. \end{cases}$$

(i) Write the density of X as a linear combination of three different densities.

(ii) Give an algorithm for generating X which uses only one uniform random variable, U

(iii) Give an algorithm for generating X which uses a blend of acceptance/rejection method and inverse distribution method

(iv) Compute the expected number of uniform random variables required for your algorithm in (iii)

Exercise 3 Shuttle train with finite waiting room

Consider a train that shuttles back and forth between two stations and has seats for $K \geq 1$ passengers. Passengers that wish to board the train and ride to the other station arrive at station i according to a renewal process, queue, and subsequently board the train in the order in which they arrive at the station. Assume that passengers board the train instantaneously and all passengers that board at station i disembark instantaneously when the train arrives at the other station. At each station passengers disembark before any passengers board the train. The waiting room at station i has finite capacity $B_i \geq 1$. A passenger that wishes to board the train at station i and arrives when there are already B_i passengers waiting to board is turned away. The time required for the train to travel from one station to the other is distributed as a positive random variable L with finite mean. Travel times are independent of interarrival times, and the arrival process of passengers for boarding at station 1 is independent of the arrival process of passengers for boarding at station 2. Assume that distribution of interarrival times of passengers at station i admits a density function. What continuous time stochastic process $\{X(t) : t \geq 0\}$ would you simulate to obtain estimates for the number of passengers waiting to board the train at station i, the number of occupied seats on the train, and the waiting times (before boarding) experienced by passengers? Specify the state space $\mathcal{S}$ of the process $\{X(t) : t \geq 0\}$ and the set $\mathcal{E}$ of events associated with simulation of the process. Also specify the process $\{X(t) : t \geq 0\}$ as a GSMP with state space $\mathcal{S}$ and event set $\mathcal{E}$. Show that the state transitions and the state transition times determine the occurrence of trips from station 1 to station 2 in which all seats are occupied and the waiting times experienced by passengers that board the train at station i.

Exercise 4 Multidimensional Geometric Brownian motion

A multidimensional geometric Brownian motion can be specified through a system of SDEs of the form

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dX_i(t), \quad i = 1, ..., d,$$

where each X_i is a standard one-dimensional Brownian motion and $X_i(t)$ and $X_j(t)$ have correlation ρ_{ij}. If we define a $d \times d$ matrix Σ by setting $\Sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$. In a convenient abuse of terminology, we refer to the vector μ as the drift vector of S and Σ as its covariance matrix.

(i) Write the integral solution of the SDEs and compute $\text{Cov}[S_i(t), S_j(t)]$

(ii) Give an algorithm to simulate the d-dimensional gBm based on a random walk construction where your primitives are independent normal random variables.