The Mixed Logit Model: The State of Practice and Warnings for the Unwary

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28 November 2001

Abstract

The mixed logit model is considered to be the most promising state of the art discrete choice model currently available. Increasingly researchers and practitioners are estimating mixed logit models of various degrees of sophistication with mixtures of revealed preference and stated preference data. It is timely to review progress in model estimation since the learning curve is steep and the unwary are likely to fall into a chasm if not careful. These chasms are very deep indeed given the complexity of the mixed logit model. Although the theory is relatively clear, estimation and data issues are far from clear. Indeed there is a great deal of potential mis-inference consequent on trying to extract increased behavioural realism from data that are often not able to comply with the demands of mixed logit models. Possibly for the first time we now have an estimation method that requires extremely high quality data *if* the analyst wishes to take advantage of the extended behavioural capabilities of such models. This paper focuses on the new opportunities offered by mixed logit models and some issues to be aware of to avoid misuse of such advanced discrete choice methods by the practitioner[1].

Key Words: Mixed logit, Random Parameters, Estimation, Simulation, Data Quality, Model Specification, Distributions

1. Introduction

The logit family of models is recognised as the essential toolkit for studying discrete choices. Starting with the simple binary logit model we have progressed to the multinomial logit model (MNL) and the nested

[1] We are indebted to Ken Train for the many hours we have discussed the challenges facing modellers estimating mixed (or random parameter) logit models. Jordan Louviere, David Brownstone and David Bunch also provided a platform to test ideas. Chandra Bhat provided informative comments on an earlier draft.

logit (NL) model, the latter becoming the main modelling tool for sophisticated practitioners (see Koppelman and Sethi 2000 for an overview). This progress occurred primarily between the mid 1960's through to the late 1970's. Although more advanced choice models such as the Generalised Extreme Value (GEV) and multinomial probit (MNP) models existed in conceptual and analytical form in the late 1970s, parameter estimation was seen as a pratctical barrier to their empirical usefulness. During the 1980's we saw a primary focus on refinements in MNL and NL models as well as a greater understanding of their behavioural and empirical strengths and limitations (including the data requirements to assist in minimising violation of the underlying behavioural properties of the random component of the utility expression for each alternative)[2]. Software such as Limdep/Nlogit and Alogit offered a relatively user-friendly capability to estimate MNL and NL models[3].

The breakthrough in the ability to estimate more advanced choice models came with the development of simulation methods (eg simulated maximum likelihood estimation) that enabled the open-form[4] models such as multinomial probit and mixed logit to be estimated with relative ease. Papers by McFadden (1985), Boersch-Supan and Hajvassiliou (1990), Geweke et al (1994), McFadden and Ruud (1994), to name a few, all reviewed in Stern (1997), established methods to simulate the choice probabilities and estimating all parameters, by drawing pseudo-random realisations from the underlying error process (Boersch-Supan and Hajivassiliou 1990).

The method is one initially introduced by Geweke (and improved by Keane, McFadden, Boersch-Supan and Hajivassiliou - see Geweke et al 1994, McFadden and Ruud 1994) of computing random variates from a multivariate truncated normal distribution. Although it fails to deliver unbiased multivariate truncated normal variates (as initially suggested by Ruud and detailed by Boersch-Supan and Hajivassiliou (1990)), it does produce unbiased estimates of the choice probabilities. The cumulative distribution function in their research is assumed to be multivariate normal and characterised by the covariance matrix M. The approach is quick and generated draws and simulated probabilities depend continuously on the parameters β and M. This latter dependence enables one to use conventional numerical methods such as quadratic hill climbing to solve the first order conditions for maximising the simulated likelihood function (equation 1) across a sample of q=1,…,Q individuals; hence the term maximum simulated likelihood (MSL) (Stern 1997).

$$\bar{L}(\beta, M) = \prod_{r=1}^{R} \prod_{q=1}^{Q} \bar{P}_r(\{i_q\}) \qquad (1)$$

[2] Regardless of what is said about advanced discrete choice models, the MNL model should always be the starting point for empirical investigation. It remains a major input into the modelling process, helping to ensure that the data are clean and that sensible results (eg parameter signs and significance) can be obtained from models that are not 'cluttered' with complex relationships (see Louviere et al 2000).

[3] Although there were a number of software tools available prior to the late 1980s, the majority of analysts used Limdep (Econometric Software), Alogit (Hague Consulting Group, now Rand Europe) and Blogit (Hensher and Johnson 1981). Today Limdep/Nlogit and Alogit continue to be the main software packages for MNL and NL estimation with SSP also relatively popular although its development is limited. Hlogit (Boersch-Supan) and Hielow (Brierle) are used by a small number of researchers.

[4] In contrast to the closed form models such as MNL and NL whose probabilities can be evaluated after estimation without further analytical or numerical integration.

Boersch-Supan and Hajivassiliou (1990) have shown that the choice probabilities are well approximated by formula (2), even for a small number of replications (r=1,…,R).

$$\bar{P}(\{i_q\}) = \frac{1}{R}\sum_{r=1}^{R} \bar{P}_r(\{i_{qr}\}) \qquad (2)$$

Discrete choice models are described by a likelihood function which is a product of the choice probabilities (equation 3), given i=1,...,I alternatives and t=1,....T profiles per observation.

$$L(\beta, M) = \prod_{q=1}^{Q} P(\{i_{tq}\} \mid \{X_{itq}\}; \beta, M \qquad (3)$$

Computation of the choice probabilities in equation (3) typically requires Monte Carlo integration. The computation involves the generation of *pseudo-random sequences* intended to mimic independent draws from a uniform distribution on the unit interval. Although these pseudo-random sequences cannot be distinguished from draws from a uniform distribution, they are not spread uniformly over the unit interval.

Bhat (2000, 2001) however has shown that an alternative quasi-random maximum simulated likelihood method (known as Halton Sequences) which uses non-random more uniformly distributed sequences instead of pseudo-random points provides greatly improved accuracy with far fewer draws and computational time. These sequences yield more accurate approximations in Monte Carlo integration relative to standard pseudo-random sequences (Brownstone 2001). The reason for the superior performance of these sequences is shown in Figure 1 (from Bhat (2001)). Even with 1,000 draws, the pseudo-random sequences leave noticeable holes in the unit square, while the Halton sequence used by Bhat gives very uniform coverage.

Bhat (2001) gives results from a Monte Carlo study of simulated maximum mixed logit models to compare the performance of the Halton sequence and the standard pseudo-random sequence. For four and five dimension integrals the Halton sequence methods required 125 draws to achieve the same accuracy as 2,000 draws with the standard pseudo-random number sequences. As a result, the computation time required to estimate the mixed logit model using Halton sequences was 10% of the time required for the standard methods. Train (1999), Revelt and Train (1999) and Hensher (2001a) have also reported similar large reductions in computation time using Halton sequences for mixed logit estimation. These results clearly demonstrate the promise of these new numerical methods for estimating mixed logit models to which we now turn.

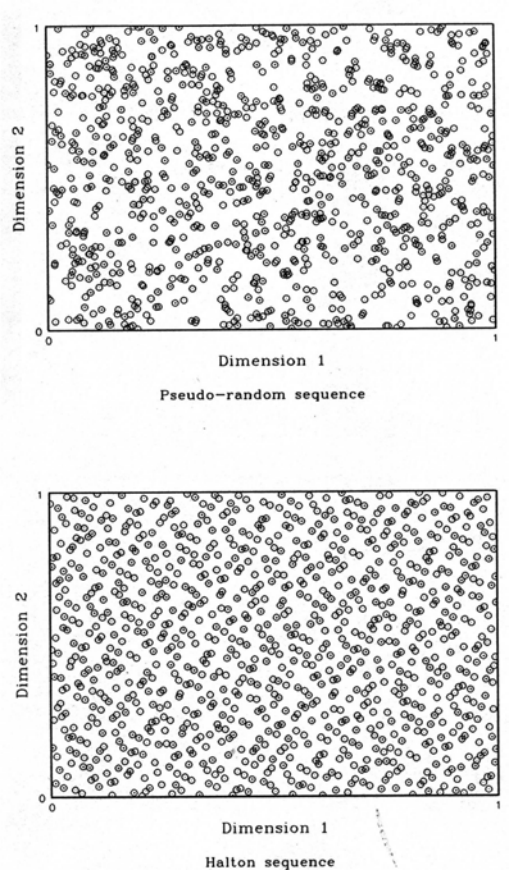

Figure 1. 1000 Draws on the Unit Square (from Bhat (2001))

2. An Intuitive Description of Mixed Logit[5]

Like any random utility model of the discrete choice family of models, we assume that a sampled individual (q=1,…,Q) faces a choice amongst *I* alternatives in each of *T* choice situations[6]. An individual *q* is assumed to consider the full set of offered alternatives in choice situation *t* and to choose the alternative with the highest utility. The (relative) utility associated with each alternative *i* as evaluated by each individual *q* in choice situation *t* is represented in a discrete choice model by a utility expression of the general form in (4).

$$U_{qit} = ß_q X_{qit} + e_{qit} \quad (4)$$

X_{qit} is a vector of (non-stochastic) explanatory variables that are observed by the analyst (from any source) and include attributes of the alternatives, socio-economic characteristics of the respondent and descriptors of the decision context and choice task itself (eg task complexity in stated choice experiments) in choice situation t. $ß_q$ and e_{qit} are not observed by the analyst and are treated as stochastic influences. Within a logit context we impose the condition that e_{qit} is independent and identically distributed (iid) extreme value type

[5] Also referred to in various literatures as random parameter logit (RPL), mixed multinomial logit (MMNL), kernel logit and hybrid logit.

[6] A single choice situation refers to a set of alternatives (or choice set) from which an individual chooses one alternative. They could also rank the alternatives but we focus on first preference choice. An individual who faces a choice set on more than one occasion (eg in a longitudinal panel) or a number of choice sets, one after the other as in stated choice experiments, is described as facing a number of choice situations.

1. However we want to allow for the possibility that the information relevant to making a choice that is unobserved may indeed be sufficiently rich in reality to induce correlation across the alternatives in each choice situation and indeed across choice situations. We would want to be able to take this into account in some way. One way to do this is to partition the stochastic component into two additive (ie uncorrelated) parts. One part is correlated over alternatives and heteroskedastic, and another part is independently, identically distributed over alternatives and individuals as shown in equation (5) (ignoring the t subscript).

$$U_{iq} = \beta' x_{iq} + [\eta_{iq} + \varepsilon_{iq}] \quad (5)$$

where η_{iq} is a random term with zero mean whose distribution over individuals and alternatives depends in general on underlying parameters and observed data relating to alternative i and individual q; and ε_{iq} is a random term with zero mean that is iid over alternatives and does not depend on underlying parameters or data. For any specific modelling context, the variance of ε_{iq} may not be identified separately from β, so it is normalised to set the scale of utility.

The Mixed Logit class of models assumes a general distribution for η and an iid extreme value distribution for ε[7]. That is, η can be normal, lognormal, triangular etc (see below). Denote the density of η by $f(\eta|\Omega)$ where Ω are the fixed parameters of the distribution. For a given value of η, the conditional choice probability is logit, since the remaining error term is iid extreme value:

$$L_i(\eta) = \exp(\beta' x_i + \eta_i) / \sum_j \exp(\beta' x_j + \eta_j). \quad (6)$$

Since η is not given, the (unconditional) choice probability is this logit formula integrated over all values of η weighted by the density of η is as shown in equation (7).

$$P_i = \int L_i(\eta) f(\eta|\Omega) d\eta \quad (7)$$

Models of this form are called *mixed logit* because the choice probability is a mixture of logits with *f* as the mixing distribution. The probabilities do not exhibit IIA, and different substitution patterns are obtained by appropriate specification of *f*. The mixed logit model recognises the role of such information and handles it in two ways (both leading to the same model only when the random effects model has a non-zero mean). The first way, known as random parameter specification, involves specifying each β_q associated with an attribute of an alternative as having both a mean and a standard deviation (ie it is treated as a random parameter instead of a fixed parameter[8]). The second way, known as the error components approach, treats the unobserved information as a separate error component in the random component. Since the standard deviation of a random parameter is essentially an additional error component, the estimation outcome is identical.

The presence of a standard deviation of a beta parameter accommodates the presence of preference heterogeneity in the sampled population. This is often referred to as unobserved heterogeneity. While one might handle this heterogeneity through data segmentation (eg a different model for each trip length range)

[7] The proof in McFadden and Train (2001) that mixed logit can approximate any choice model, including any multinomial probit model is an important message. The reverse cannot be said: a multinomial probit model cannot approximate any mixed logit model, since multinomial probit relies critically on normal distributions. If a random term in utility is not normal, then mixed logit can handle it and multinomial probit cannot. Apart from this point, the difference between the models is a matter of which is easier to use in a given situation.

[8] A fixed parameter essentially treats the standard deviation as zero such that all the behavioural information is captured by the mean).

and/or attribute segmentation (eg separate betas for different trip length ranges), the challenge of these segmentation strategies is in picking the right segmentation criteria and range cut-offs and indeed being confident that one has accounted for the unobserved heterogeneity by observed effects. A random parameter representation of preference heterogeneity is more general; however such a specification carries a challenge in that these parameters have a distribution that is unknown. Selecting such a distribution has plenty of empirical challenges (see below). As shown below the concern that one might not know the location of each individual's preferences on the distribution can be accommodated by retrieving individual-specific preferences by deriving the individual's conditional distribution based (within-sample) on their choices (ie prior knowledge). Using Bayes Rule we can define the conditional distribution as equation (8).

$$H_q(ß|\theta) = L_q(ß)g(ß|\theta)/P_q(\theta) \quad (8)$$

$L_q(ß)$ is the likelihood of an individual's choice if they had this specific ß; $g(ß|\theta)$ is the distribution in the population of ßs (or the probability of a ß being in the population), and $P_q(\theta)$ is the choice probability function defined in open-form as:

$$P_q(\theta) = \int L_q(ß)g(ß|\theta)\, dß \quad (9)$$

Another attractive feature of mixed logit is the ability to re-parameterise the mean estimates of random parameters to establish heterogeneity associated with observable influences. For example we can make the mean ß of travel time a linear function of one or more attributes (such as trip length in the examples below). This is one way of 'removing' some of the unobserved heterogeneity from the parameter distribution by 'segmenting' the mean with continuous or discrete variation (depending on how one defines the observed influences).

The choice probability cannot be calculated exactly because the integral does not have a closed form in general. The integral is approximated through simulation (using the ideas developed above in Section 1). For a given value of the parameters, a value of η is drawn from its distribution. Using this draw, the logit formula $L_i(\eta)$ is calculated. This process is repeated for many draws, and the mean of the resulting $L_i(\eta)$'s is taken as the approximate choice probability giving equation (10) or (2).

$$SP_i = (1/R) \Sigma_{r=1,\ldots,R} L_i(\eta^r) \quad (10)$$

R is the number of replications (i.e., draws of η), η^r is the r^{th} draw, and SP_i is the simulated probability that an individual chooses alternative i.[9]

After model estimation, there are many outputs for interpretation. An early warning – parameter estimates typically obtained from a random parameter or error components specification should not be interpreted as stand-alone parameters but must be assessed jointly with other linked parameter estimates. For example, the mean parameter estimate for travel time, its associated heterogeneity in mean parameter (eg. for trip length) and the standard deviation parameter estimate for travel time represent the marginal utility of travel time associated with a specific alternative *and* individual. The most general formula will be written out

[9] By construction, SP_i is an unbiased estimate of P_i for any R; its variance decreases as R increases. It is strictly positive for any R, so that ln (SP_i) is always defined in a log-likelihood function. It is smooth (i.e., twice differentiable) in parameters and variables, which helps in the numerical search for the maximum of the likelihood function. The simulated probabilities sum to one over alternatives (Brownstone 2001).

with due allowance for the distributional assumption on the random parameter (see Section 4.10 for more details). The four most popular distributions can be defined in equations 11a-11d using a travel time function (noting that *rnn* is a normal distribution, *u* is a uniform distribution and *t* is a triangular distribution):

Lognormal : $\text{Exp}(ß_{mean} + ß_{trip\ length} * \text{trip length} + ß_{standard\ deviation} * rnn(0,1))$ (11a)

Normal: $ß_{mean} + ß_{trip\ length} * \text{trip length} + ß_{standard\ deviation} * rnn(0,1)$ (11b)

Uniform: $ß_{mean} + ß_{trip\ length} * \text{trip length} + ß_{spread} * u$ (11c)

Triangular: $ß_{mean} + ß_{trip\ length} * \text{trip length} + ß_{spread} * t$ (11d)

This particular formula assumes that the attributes of alternatives are independent. If we allow for attribute (ie alternative) correlation, then the standard deviation beta would be replaced with the diagonal and off-diagonal elements of the Cholesky matrix in the row referencing that attribute (see below for more details).

In the late 1990s we started seeing an increasing number of applications of mixed logit models and an accumulating knowledge base of experiences in estimating such models with available and new data sets. A close reading of this literature often fails to warn the analyst of many of the underlying (often not revealed) challenges that modellers experienced in arriving at a preferred model. The balance of this paper focuses on some of the most recent experiences of a number of active researchers estimating mixed logit models. Sufficient knowledge has been acquired in the last few years to be able to share some of the early practical lessons.

We draw on three data sets to illustrate a number of issues although like any evidence it has to be conditional on the particular data set until we establish some common trends. The main data sets used herein are drawn from two stated choice experiments undertaken in New Zealand in 1999 and 2000 and a revealed preference data set from Australia (1987).

3. The Data Sources Used to Illustrate Specific Issues

Three data sets have been selected to illustrate the range of specification, estimation and application issues. We briefly summarise their informational contact and cross-reference to other sources for further details.

3.1 A Stated choice experiment for long distance car travel (Data Set 1)

A survey of long-distance road travel was undertaken in 2000, sampling residents of six cities/regional centres in New Zealand[10]. The main survey was executed as a laptop-based face to face interview in which each respondent was asked to complete the survey in the presence of an interviewer. Each sampled respondent evaluated 16 stated choice profiles, making two choices: the first involving choosing amongst three labelled SC alternatives and the current RP alternative, and the second choosing amongst the three

[10] Auckland, Hamilton, Palmerston North, Wellington, Christchurch, and Dunedin on both the North and South Islands

SC alternatives[11]. A total of 274 effective interviews[12] with car drivers were undertaken producing 4,384 car driver cases for model estimation (ie 274*16 treatments).

The choice experiment presents four alternatives to a respondent:

A. The current road the respondent is/has been using
B. A hypothetical 2 lane road
C. A hypothetical 4 lane road with no median
D. A hypothetical 4 lane road with a wide grass median

There are two choice responses, one including all four alternatives and the other excluding the current road option. All alternatives are described by six attributes except alternative A, which does not have toll cost. Toll cost is set to zero for alternative A since there are currently no toll roads in New Zealand. The attributes in the stated choice experiment are:

1. Time on the open road which is free flow (in minutes)
2. Time on the open road which is slowed by other traffic (in minutes)
3. Percentage of total time on open road spent with other vehicles close behind (ie tailgating) (%)
4. Curviness of the road (A four-level attribute - almost straight, slight, moderate, winding)
5. Running costs (in dollars)
6. Toll cost (in dollars)

The experimental design is a 4^6 profile in 32 runs. That is, there are two versions of 16 runs each. The design has been chosen to minimise the number of dominants in the choice sets. Within each version the order of the runs has been randomised to control for order effect. For example, the levels proposed for alternative B should always be different from those of alternatives C and D.

In 32 runs it is straightforward to construct the following main effects plan: $4^9\ 2^4$. No interactions can be estimated without imposing some correlation. To obtain the 4^6 design, six columns in four levels were extracted from the nine columns available in the plan. This formed the base and the levels were manipulated to eliminate dominant alternatives in the choice sets. This is achieved, for example, by changing 0,1,2,3 to 2,1,0,3. Given that there are four levels and six attributes, a lot of designs can be produced. It is not difficult to produce a few of them and keep the one with the minimum number of dominant alternatives. In the present case the result of this procedure yielded a design with only one choice set presenting a dominant alternative. The dominant alternative has been used in a two-lane road. Therefore all respondents who prefer driving on a four lane road might not see it as being a dominant alternative, because although all attributes of the two lane road are better, they may still be willing to trade them off for a four lane road. This produces a design that should conform well with the specifications of the study[13]. One of the two-level variables has been used to create the versions.

[11] The development of the survey instrument occurred over the period March to October 2000. Many variations of the instrument were developed and evaluated through a series of skirmishes, pre-pilots and pilot tests.

[12] We also interviewed truck drivers but they are excluded from the current empirical illustrations (See Hensher and Sullivan (2001) for the truck models).

[13] The SC design is generic. The mean, range and standard deviation across 2-lane, 4 lane no median and 4 lane with median are identical. Although the attribute levels seen across the alternatives on each screen are different the design levels overall are identical. An alternative-specific design would be more complex since one can have different ranges across alts and would really require more choices or loss of explanatory capability on 16 sets from full 64. This generic structure has produced a generic specification for the design attributes that are treated in estimation as having random parameters.

These six attributes have four levels which, were chosen as follows

- Free Flow Travel Time: -20%, -10%, +10%, +20%
- Time Slowed Down: -20%, -10%, +10%, +20%
- Percent of time with vehicles close behind: -50%, -25%, +25%, +50%
- Curviness: almost straight, slight, moderate, winding
- Running Costs: -10%, -5%, +5%, +10%
- Toll cost for car and double for truck if trip duration is:
  - 1 hours or less 0, 0.5, 1.5, 3
  - between 1 hour and 2 hours 30 minutes 0, 1.5, 4.5, 9
  - more than 2 and a half hours 0, 2.5, 7.5, 15

The design attributes together with the choice responses and contextual data provide the information base for model estimation. An example of a stated choice screen is shown in Figure 2. Further details are given in Hensher and Sullivan (2001). Herein we focus only on models where individuals choose amongst the three SC alternatives.

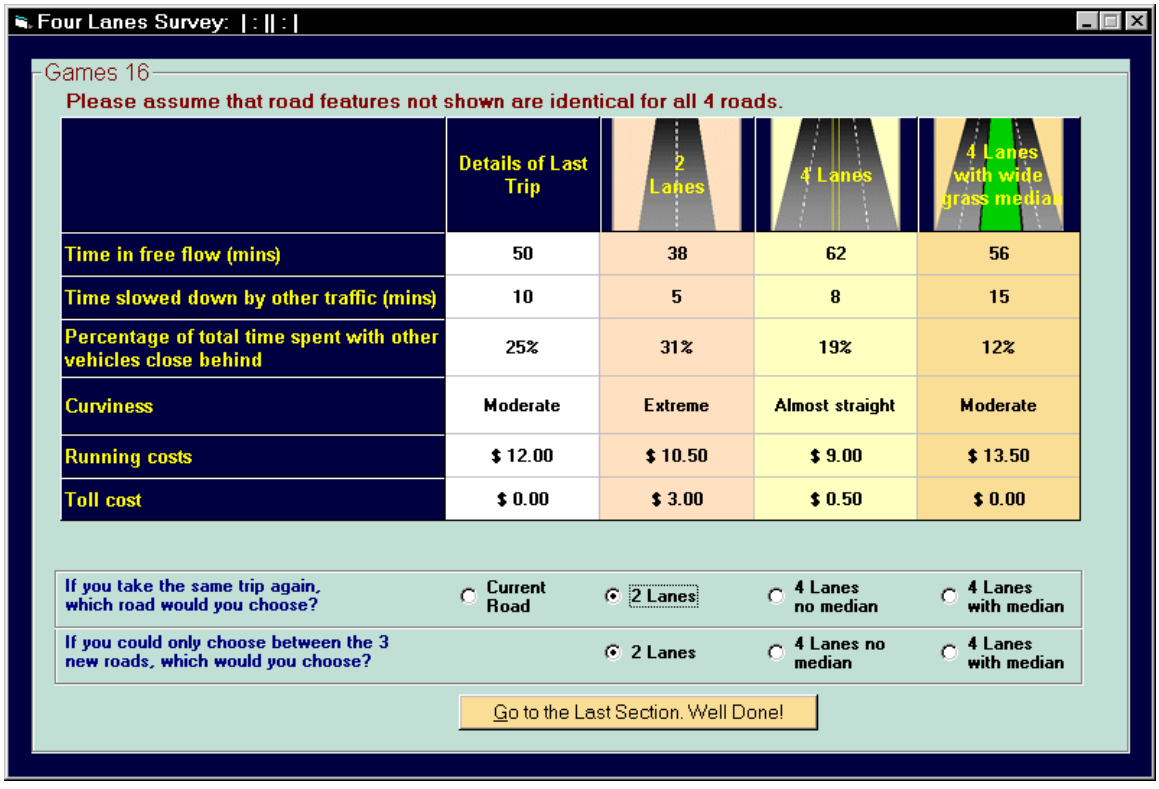

Figure 2. An example of a stated choice screen for data set 1

3.2 A Stated choice experiment for urban commuting (Data Set 2)

A survey of a sample of 143 commuters was undertaken in late June and early July 1999 in urban New Zealand sampling residents of seven cities/regional centres[14]. The main survey was executed as a laptop-based face to face interview in which each respondent was asked to complete the survey in the presence of an interviewer. Each sampled respondent evaluated 16 choice profiles, choosing amongst two SC alternatives and the current RP alternative. The 143 interviews represent 2,288 cases for model estimation (ie 143*16 treatments).

The stated choice experimental design is based on two unlabelled alternatives (A and B) each defined by six attributes each of four levels (ie 4^{12}): free flow travel time, slowed down travel time, stop/start travel time, uncertainty of travel time, running cost and toll charges. Except for toll charges, the levels are *proportions* relative to those associated with a current trip identified prior to the application of the SC experiment:

Free flow travel time:	-0.25, -0.125, 0.125, 0.25
Slowed down travel time:	-0.5, -0.25, 0.25, 0.5
Stop/Start travel time:	-0.5, -0.25, 0.25, 0.5
Uncertainty of travel time:	-0.5, -0.25, 0.25, 0.5
Car running cost:	-0.25, -0.125, 0.125, 0.25
Toll charges ($):	0, 2, 4, 6

The levels of the attributes for both SC alternatives were rotated to ensure that neither A nor B would dominate the RP trip, and to ensure that A and B would not dominate each other. For example, if free flow travel time for alternative A was better than free flow travel time for the RP trip, then we structured the design so that at least one among the five remaining attributes would be worse for alternative A relative to the RP trip; and likewise for the other potential situations of domination. The fractional factorial design has 64 rows. We allocated four blocks of 16 "randomly" to each respondent, defining block 1 as the first 16 rows of the design, block 2 the second set of 16 etc. The assignment of levels to each SC attribute conditional on the RP levels is straightforward. An SC screen is shown in Figure 2. Further details are provided in Hensher (2001a, 2001b).

3.3 A revealed preference study of long distance non-commuting modal choice (Data Set 3)

The data, collected as part of a 1987 intercity mode choice study, is a sub-sample of 210 non-business trips between Sydney, Canberra and Melbourne in which the traveller chooses a mode from four alternatives (plane, car, bus and train). The sample is choice-based with over-sampling of the less popular modes (plane, train and bus) and under-sampling of the more popular mode, car. The level of service data was derived from highway and transport networks in Sydney, Melbourne, non-metropolitan N.S.W. and Victoria, including the Australian Capital Territory. The data file contains the following information:

Mode	Equal 1 for the mode chosen and 0 otherwise
Ttme	Terminal waiting time for plane, train and bus (minutes)
Invc	In-vehicle cost for all stages (dollars)
Invt	In-vehicle time for all stages (minutes)
Gc	Generalised cost = Invc + (Invc*value of travel time savings) (dollars)
Hinc	Household income ($'000s)
Psize	Travelling group size (number)

Further information is given in Louviere et al (2000).

[14] Auckland, Wellington, Christchurch, Palmerston North, Napier/Hastings, Nelson and Ashburton on both the North and South Islands

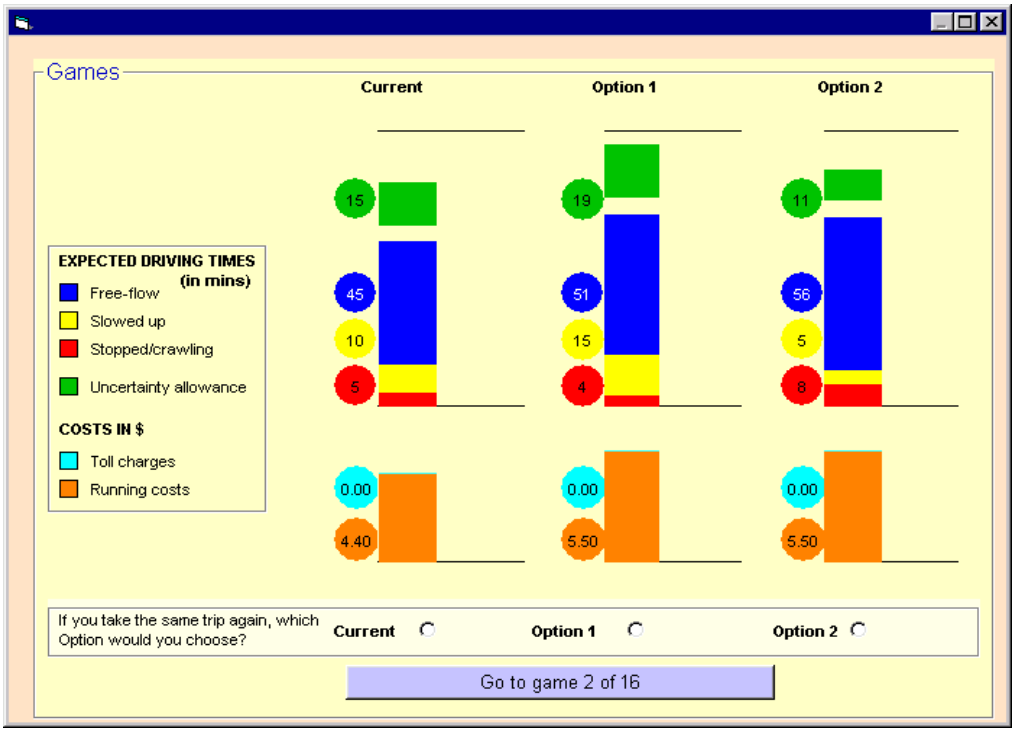

Figure 3. An example of a stated choice screen for data set 2

4. The Main Model Specification Issues

There are at least ten key empirical issues to consider in specifying, estimating and applying a mixed logit model:

1. Selecting the parameters that are to be random parameters
2. Selecting the distribution of the random parameters
3. Specifying the way random parameters enter the model
4. Selecting the number of points on the distributions and parameter stability
5. Decomposing mean parameters to reflect covariate heterogeneity
6. Empirical distributions
7. Accounting for observations drawn from the same individual
8. Accounting for correlation between attributes
9. Taking advantage of priors in estimation and posteriors in application
10. Willingness to pay challenges

4.1 Selecting the parameters that are to be random parameters

The random parameters are the basis for accommodating correlation across alternatives (via their attributes) and across choice sets. They also define the degree of unobserved heterogeneity (via the standard deviation of the parameters) and preference heterogeneity around the mean (equivalent to an interaction between the attribute specified with a random parameter) and another attribute of an alternative, an individual, a survey method and/or choice context.

It is important to allocate a good proportion of time estimating models in which many of the attributes of alternatives are considered as having random parameters. The possibility of different distributional assumptions (see section 4.2) for each attribute should also be investigated, especially where sign is important. A warning: the findings will not necessarily be independent of the number of random or intelligent draws and so establishing the appropriate set of random parameters requires taking into account the number of draws, the distributional assumptions and, in the case of multiple choice sets per individual, whether correlated choice sets are accounted for. These interdependencies make for a lengthy estimation process. Starting values from multinomial logit models, while helpful, cannot help in the selection of random parameterised attributes (unless extensive segmentation on each attribute within an MNL model occurs).

The Lagrange Multiplier tests proposed in McFadden and Train (2000) for testing the presence of random components provides one statistical basis for accepting/rejecting the preservation of fixed-point estimates. Brownstone (2001) provides a succinct summary of the test. These tests work by constructing artificial variables as in (12).

$$z_{in} = \left(x_{in} - \bar{x}_i\right)^2, \text{ with } \bar{x}_i = \sum_j x_{jn} P_{jn} \tag{12}$$

and P_{jn} is the conditional logit choice probability. The conditional logit model is then re-estimated including these artificial variables, and the null hypothesis of no random coefficients on attributes x is rejected if the coefficients of the artificial variables are significantly different from zero. The actual test for the joint significance of the z variables can be carried out using either a Wald or Likelihood Ratio test statistic. These Lagrange Multiplier tests can be easily carried out in any software package that estimates the conditional logit model. Brownstone suggests that these tests are easy to calculate and appear to be quite powerful omnibus tests; however, they are not as good for identifying which error components to include in a more general mixed logit specification.

4.2 Selecting the distribution of the random parameters (eg normal, lognormal, triangular, uniform)

If there is one single issue that can cause much concern it is the influence of the distributional assumptions of random parameters. The layering of selected random parameters can take a number of predefined functional forms, the most popular being normal, triangular, uniform and lognormal. The lognormal form is often used if the response parameter needs to be a specific (non-negative) sign. A uniform distribution with a (0,1) bound is sensible when we have dummy variables.

Distributions are essentially arbitrary approximations to the real behavioural profile. We select specific distributions because we have a sense that the 'empirical truth' is somewhere in their domain. All distributions in common practice unfortunately have at least one major deficiency – typically with respect to sign and length of the tail(s). Truncated or constrained distributions appear to be the most promising direction in the future given recent concerns (see Section4.2.4). For example, we might propose the generalised constrained triangular in which the spread =Z*mean where Z lies in the range 0.1 to 1.0.

4.2.1 Uniform distribution

The spread of the uniform distribution (ie the distance up and down from the mean) and the standard deviation are different and the former needs to be used in representing the uniform distribution. Suppose SPD is the spread, such that the time coefficient is uniformly distributed from (mean-SPD) to (mean+ SPD). Then the correct formula for the distribution is (mean parameter estimate + SPD*(2*rnu-1)). Since

rnu is uniform from 0 to 1, 2*rnu-1 is uniform from -1 to +1; then multiplying by SPD gives a uniform +/- SPD from the mean. The spread can be derived from the standard deviation by multiplying the standard deviation by the square root of 3.

4.2.2 Triangular distribution

For the triangular distribution, the density function looks like a tent: a peak in the centre and dropping off linearly on both sides of the centre. Let c be the centre and s the spread. The density starts at c-s, rises linearly to c, and then drops linearly to c+s. It is zero below c-s and above c+s. The mean and mode are c. The standard deviation is the spread/(square root of 6) and hence the spread is the standard deviation * square root of 6. The height of the tent at c is 1/s (such that each side of the tent has area s*(1/s)*(1/2)=1/2, and both sides have area 1/2+1/2=1, as required for a density)[15]. The slope is $1/s^2$. The complete density (f(x)) and cumulative distribution (F(x)) are[16]:

- for x<c-s: f(x)=F(x)=0
- for c-s<=x<=c: f(x)=(x-(c-s))/ $s^2$and F(x)=(x-(c-s))^2/ s^2
- for c<=x<=c+s: f(x)=((c+s)-x)/s^2 and F(x)=((c+s)-x)^2/ s^2
- for x>c+s: f(x)=0 and F(x)=1 >

4.2.3 Lognormal distribution

The lognormal distribution is very popular for the following reasoning. The central limit theorems explain the genesis of a normal curve. If a large number of random shocks, some positive, some negative, change the size of a particular attribute, x, in an additive fashion, the distribution of that attribute will tend to become normal as the number of shocks increases. But if these shocks act multiplicatively, changing the value of x by randomly distributed proportions instead of absolute amounts, the central limit theorems applied to Y=lnx. (where ln is to base e) tend to produce a normal distribution. Hence x has a lognormal distribution. The substitution of multiplicative for additive random shocks generates a positively skewed, leptokurtic, lognormal distribution instead of a symmetric, mesokurtic normal distribution. The degree of skewness and kurtosis of the two-parameter lognormal distribution depends only on the variance, and so if this is low enough, the lognormal approximates the normal distribution. Lognormals are appealing in that they are limited to the non-negative domain; however they typically have a very long right-hand tail which is a disadvantage (especially for willingness-to-pay calculations – see Section 4.10)[17].

Given the (transform) link with the normal distribution, lognormals are best estimated with starting values from normals. However experience suggests that they iterate many times looking for the maximum, and often get stuck along the way. The unbounded upper tail which is often behaviourally unrealistic and often quite fat does not help. Individuals typically do not have an unbounded willingness to pay for any attribute, as lognormals imply. In contrast other distributions such as the triangular and uniform are bounded on both sides, making it relatively easy to check whether the estimated bounds make sense. We will say more about the lognormal's behavioural implications in later sections.

4.2.4 Imposing constraints on a distribution

In practice we often find that any one distribution has strengths and weaknesses. The weakness is usually associated with the spread or standard deviation of the distribution at its extremes including behaviourally

[15] In Limdep, for example, one transforms a uniform(0,1) variable, as such: CREATE ; V = RNU(0,1) ; IF(V <= .5)T=SQR(2*V)-1 ; (ELSE) T=1-SQR(2*(1-V)) $

[16] Proof: Without loss of generality, let c=0. Find E[x|x>0] = s/3 and E[x|x<0] = -s/3. By integration - the conditional density is 2*unconditional density in either left or right half. In the same way, get E[x^2|x>0] = s^2/6 = E[x^2|x<0]. This gives you the conditional variances by the expected square - squared mean. Now, the unconditional variance is the Variance of the conditional mean plus the expected value of the conditional variance. A little algebra produces the unconditional variance = s^2/6.

[17] Although the ratio of two lognormals is also lognormal which is convenient result for WTP calculations despite the long tail.

unacceptable sign changes for the symmetrical distributions. The lognormal has a long upper tail. The normal, uniform and triangular give the wrong sign to some share.

One appealing 'solution' is to make the spread or standard deviation of each random parameter a function of the mean. For example, the usual specification in terms of a normal distribution (which uses the standard deviation rather than the spread) is to define ß(i) = ß + s*v(i) where v(i) is the random variable. The constrained specification would be ß (i) = ß + ß*v(i) when the standard deviation equals the mean or ß (i) = ß + z*ß*v(i) when z is a scalar taking any positive value. We would generally expect z to lie in the 0-1 range since a standard deviation (or spread) greater than the mean estimate *typically*[18] results in behaviourally unacceptable parameter estimates.

This constraint specification can be applied to any distribution. For example, for a triangular with mean=spread, the density starts at zero, rises linearly to the mean, and then declines to zero again at twice the mean. It is peaked, like one would expect. It is bounded below at zero, bounded above at a reasonable value that is estimated, and is symmetric such that the mean is easy to interpret. It is appealing for handling willingness to pay parameters. Also with ß (i)= ß + ß v(i), where v(i) has support from -1 to +1, it does not matter if ß is negative or positive. A negative coefficient on v(i) simply reverses all the signs of the draws, but does not change the interpretation[19].

4.2.5 Discrete distributions[20]

The set of continuous distributions presented above impose a priori restrictions. An alternative is a discrete distribution. Such a distribution may be viewed as a nonparametric estimator of the random distribution. Using a discrete distribution that is identical across individuals is equivalent to a latent segmentation model with the probability of belonging to a segment being only a function of constants (See Ch 10 of Louviere et al (2000) for a discussion on such models). However allowing this probability to be a function of individual attributes is equivalent to allowing the points characterising the nonparametric distribution to vary across individuals. In this paper, we focus on a continuous distribution for the random components.

4.2.6 An Empirical comparison of the distributions

In most empirical studies, one tends to get similar means and comparable measures of spread (or standard deviation) for normal, uniform and triangular distributions[21]. With the lognormal, however, the evidence tends to shift around a lot, but the mean of a normal, uniform or triangular, typically existing between the mode and mean of the lognormal. This does not suggest however that we have picked the best analytical distribution to represent the true empirical distribution. This topic is investigated in some detail in Section

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[18] We say typically but this is not always the case. One has to judge the findings on their own merits.

[19] One could specify the relationship as ß(i)= ß+|ß|v(i), but that would create numerical problems in the optimisation routine.

[20] Discussions with Chandra Bhat on this theme are gratefully acknowledged.

[21] One can however use different distributions on each attribute. The reason you can do this is that you are not using the distributional information in constructing the estimator. The variance estimator is based on the method of moments. Essentially, you are estimating the variance parameters just by computing sums of squares and cross products. In more detail (in response to a student inquiry) Ken Train comments that it is possible to have underlying parameters jointly normal with full covariance and then transform these underlying parameters to get the parameters that enter the utility function. For example, suppose V= $\alpha_1 x_1 + \alpha_2 x_2$. We can say that β_1 and β_2 are jointly normal with correlation and that $\alpha_2 = \exp(\beta_2)$ and $\alpha_1 = \beta_1$. That gives you a lognormal and a normal with correlation between them. The correlation between α_2 and α_2 can be calculated from the estimated correlation between β_1 and β_2 if you know the formula. Alternatively one can calculate it by simulating many α_1 and α_2's from many draws of β_1 and β_2's from their estimated distribution and then calculate the correlation between the α_1 and α_2's. This can be applied for any distributions. Let α_2 have density $g(\alpha_2)$ with cum dist $G(\alpha_2)$, and let α_1 be normal.. $F(\beta_2 | \beta_2)$ is the normal cum dist for β_2 given β_1. Then α_2 is calculated as $\alpha_2 = G^{-1}(F(\beta_2|\beta_1))$. For some G's there must be limits on the correlation that can be attained between α_1 and α_2 using this procedure.

4.6. This sub-section presents some typical findings (Table 1, Data Set 1), noting that the standard deviation is used in the normal and lognormal distributions and the spread in the uniform and triangular distributions. The values of travel time savings (VTTS) are derived using the formulae in (11a-11d), dividing by the parameter estimate for travel cost and multiplying by 60 to convert from dollars per minute to dollars per hour.

Value of travel time savings: lognormal
rnla=rnn(0,1)
mlvotl=-60*(exp(-5.40506-.0075148*tripl+2.36613*rnla))/-.1048
Value of travel time savings: normal
rnnb=rnn(0,1)
mlvotn=60*(-.012575+.00002840*tripl+.00881228*rnnb)/-.10355
Value of travel time savings: triangular
V=rnu(0,1)
if(V<=.5)T=sqr(2*V)-1;(else) T=1-sqr(2*(1-V))
mlvott=60*(-.0125428+.000028117*tripl+.0203768*t)/-.103448
Value of travel time savings: uniform
rnuc=rnu(0,1)
mlvotu=60*(-.0120956+.0000258667*tripl+.0128616*(2*rnuc-1))/-.1032216
(notte: tripl = trip length in minutes).

Table 1. A comparison of values of travel time savings (Data Set 1)

Value of Travel Time Savings ($ per person hour)			
		Mean	Standard Deviation
Lognormal	mlvotl	14.759	165.4
Normal	mlvotn	4.665	5.361
Triangular	mlvott	4.629	5.107*
Uniform	mlvotu	4.628	4.592*
Using Standard Deviation instead of Spread:			
Triangular	mlvotts	4.636	2.588**
Uniform	mlvotus	2.451	2.001**

Note: stdt=.0203768/sqr(6), stdu=.0128616/sqr(3). mlvotts=60*(-.0125428+.000028117*tripl+stdt*t)/-.103448
mlvotus=60*(-.0120956+.0000258667*tripl+stdu*rnuc)/-.1032216. * indicates that we have calculated the standard deviation for the descriptive statistics based on the application of the spread formula (and the application of the standard deviation formula for **).

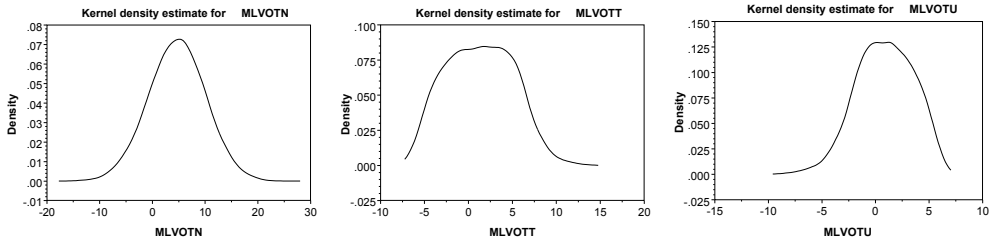


Figure 4 VTTS distributions for normal, triangular and uniform (to illustrate incidence of negative VTTS)

As expected, the normal, triangular and uniform are quite similar and the lognormal is noticeably different with an unacceptably large standard deviation. The lognormal however guarantees non-negative VTTS whereas the other three (unconstrained distributions) almost certainly guarantee some negative VTTS (Figure 4). In this application, the percentage of VTTS that are negative for normal, triangular and uniform are respectively 19.21%, 39.33% and 37.92%.[22]

[22] Note what happens if you accidentally use the standard deviation instead of the spread for the uniform and triangular distributions (Table 1). The mean and standard deviation for VTTS across the sample changes quite markedly (except in this case the mean for the triangular is very similar by coincidence).

4.3 Specifying the way random parameters enter the model under a lognormal distribution

Entering an attribute specified with a random parameter that is lognormally distributed with a positive sign typically causes the model to either not converge or converge with unacceptably large mean estimates (see Section 4.10). The trick to overcome this is to reverse the sign of the attribute prior to model estimation (ie define the negative of the attribute instead of imposing a sign change on the estimated parameter). The logic is as follows. The lognormal has a nonzero density only for positive numbers. So to ensure that an attribute has a negative parameter for all sampled individuals, one has to enter the negative of the attribute. A positive lognormal parameter for the negative of the attribute is the same as a negative lognormal parameter on the attribute itself.

4.4 Selecting the number of points on the distributions: parameter stability

The number of draws required to secure a stable set of parameter estimates varies enormously. In general, it appears that as the model specification becomes more complex in terms of the number of random parameters and the treatment of heterogeneity around the mean, correlation of attributes and alternatives, the number of required draws increases. There is no magical number but experience suggests that a choice model with three alternatives and one or two random parameters (with no correlation between the attributes and no decomposition of heterogeneity around the mean) can produce stability with as low as 25 *intelligent* draws, although 100 appears to be a 'good' number. The best test however is to always estimate models over a range of draws (eg 25, 50, 100, 250, 500 and 1000 draws). Confirmation of stability/precision for each and every model is very important. Table 2 provides a series of runs from 25 to 2000 intelligent draws (car drivers in Data Set 1). The results stabilise after 250 draws, which is more than are necessary, especially given only one dimension of integration.

Bhat (2001) and Train (1999) found that the simulation variance in the estimated parameters was lower using 100 Halton numbers than 1,000 random numbers. With 125 Halton draws, they both found the simulation error to be half as large as with 1,000 random draws and smaller than with 2,000 random draws[23]. The estimation procedure is much faster (often 10 time faster). Hensher (2000) investigated Halton sequences involving draws of 10, 25, 50, 100, 150 and 200 (with three random generic parameters) and compared the findings in the context of value of travel time savings with random draws. In all models investigated Hensher concluded that a small number of draws (as low as 25) produces model fits and mean values of travel time savings that are almost indistinguishable. This is a phenomenal development in the estimation of complex choice models. However before we can confirm that we have found the 'best' draw strategy, researchers are finding that other possibilities may be even better. For example, ongoing research by Train and Sandor investigating random, Halton, Niederreiter and orthogonal array latin hypercube draws finds the results 'often perplexing' (in the words of Ken Train), with purely random draws sometimes doing much better than they should and sometimes all the various types of draws doing much worse than they should. What are we missing in simulation variance of the estimates? Perhaps the differences in estimates with different draws is due to the optimisation algorithm?[24] Recent research by

[23] The distinction between intelligent draws and random draws is very important given recent papers circulating by Joan Walker of MIT about the need to use 5,000 to 10,000 draws. Walker is referring to random draws.

[24] Train and Sandor identify draws where one never gets to the maximum of the likelihood function, with a wide area where the algorithms converge indicating a close enough solution. Depending on the path by which this area is approached (which will differ with different draws), the convergence point differs. As a result, there is a greater difference in the convergence points than there is in the actual maximum.

Bhat (in press) on the type of draws vis-a-vis the dimensionality of integration suggests that the uniformity of the standard Halton sequence breaks down in high dimensions because of the correlation in sequences of high dimension. Bhat proposes a scrambled version to break these correlations, and a randomised version to compute variance estimates. These examples of recent research demonstrate the need for ongoing inquiry into simulated draws, especially as the number of attributes with imposed distributions increases.

Table 2 Mixed Logit Models. All travel times are in minutes and costs are in dollars. T-values in brackets Source: Data Set 1.

Attributes	Alternative	Mixed Logit (lognormal random parameter) Number of Intelligent Draws						
(i) With Heterogeneity in Mean		***25***	***50***	***100***	***250***	***500***	***1000***	***2000***
Random Parameters:								
Total time	All	-4.9174 (-7.0)	-5.2996 (-5.6)	-5.416 (-5.9)	-5.232 (-6.1)	-5.158 (-6.1)	-5.1043 (-6.1)	-5.1279 (6.2)
Fixed Parameters:								
Total cost	All	-.1294 (-48)	-.1294 (-48)	-.1295 (-48)	-.1296 (-48)	-.1296 (-48)	-.1296 (-48)	.1296 (-48)
Tailgate percentage	All	-.01138 (-9.4)	-.01134 (-9.3)	-.01135 (-9.4)	-.01125 (-9.4)	-.01134 (-9.4)	-.01134 (-9.4)	-.01135 (-9.4)
Non-winding vs winding curviness	2-lane	0.3036 (2.3)	.3085 (2.3)	.3080 (2.3)	.3073 (2.3)	.3066 (2.3)	.3056 (2.3)	.3068 (2.3)
Non-winding vs winding curviness	4-lane	0.5652 (10)	.5707 (10.3)	.5703 (10.3)	.5712 (10.3)	.5711 (10.2)	.5709 (10.3)	.5708 (10.3)
Constant	4 no median	0.1179 (1.0)	.1200 (.94)	.1196 (0.9)	.1175 (0.9)	.1169 (0.9)	.1168 (0.9)	.1173 (0.9)
Constant	4 with median	0.7569 (5.9)	.7491 (5.9)	..7484 (5.9)	.7461 (5.9)	.7453 (5.9)	.7453 (5.9)	.7458 (5.9)
Heterogeneity in mean:								
Travel time: Trip length	All	-.006375 (-2.3)	-.00913 (-2.0)	-.00757 -1.85)(	-.00799 (-1.97)	-.00789 (-2.0)	-.00796 (-2.1)	-.00788 (-2.0)
Std Deviation. of parameter distribution								
Total time	All	1.6085 (7.2)	2.2926 (5.9)	2.0946 (6.1)	1.9103 (5.5)	.1.8348 (5.0)	1.7969 (4.8)	1.8097 (5.2)
Pseudo-r^2 adjusted		0.167	.1669	.1675	.1679	.1679	.1679	.1679
Log-likelihood		-4009.9	-4008.1	-4005.3	-4003.4	-4003.4	-4003.3	-4003.4
Ratio RP mean/sd (*)		3.057	2.31	2.59	2.74	2.81	2.84	2.83
Run time		12m,34s	26m8s	50m,36s	126m,20s	242m	500m	1075m
(ii) Without Heterogeneity in Mean		***25***	***50***	***100***	***250***	***500***	***1000***	***2000***
Random Parameters:								
Total time	All	-6.519 (-17.3)	-6.637 (-14.6)	-7.5291 (-9.1)	-7.2502 (10.0)	-7.251 (-9.6)	-7.2154 (-9.5)	-7.1899 (-9.6)
Fixed Parameters:								
Total cost	All	-.1297 (-48)	-.1296 (-48)	-.1297 (-48)	-.1298 (-48)	-.1298 (-48)	-.1299 (-48)	-.1299 (-48)
Tailgate percentage	All	-.01136 (-9.4)	-.01134 (9.4)	-.01137 (-9.4)	-.01136 (-9.4)	-.01136 (-9.4)	-.01136 (-9.4)	-.01136 (9.4)
Non-winding vs winding curviness	2-lane	.3037 (2.3)	.3045 (2.3)	.3078 (2.3)	.3071 (2.3)	.3067 (2.3)	.3065 (2.3)	.3064 (2.3)
Non-winding vs winding curviness	4-lane	.566 (10.3)	.5667 (10.3)	.5714 (10.3)	.5714 (10.3)	.5718 (10.3)	.5717 (10.3)	.5717 (10.3)
Constant	4 no median	.1195 (0.9)	.1196 (0.9)	.1185 (0.9)	.1170 (0.9)	.1163 (0.9)	.1161 (0.9)	.1160 (0.9)
Constant	4 with median	.7473 (6.0)	.7475 (6.0)	.7466 (5.9)	.7452 (6.0)	.7444 (5.9)	.7441 (5.9)	.7434 (5.9)
Std Deviation. of parameter distribution								
Total time	All	2.057 (10.9)	2.0763 (9.0)	2.6516 (5.9)	2.321 (5.9)	2.310 (5.2)	2.279 (5.0)	2.248 (4.9)
Pseudo-r^2 adjusted		.1658	.1661	.1669	.1673	.1673	.1673	.1673
Log-likelihood		-4013.9	-4012.5	-4008.7	-4007.0	-4007.1	-4007.1	-4007.1
Ratio RP mean/sd		3.168	3.212	2.839	3.124	3.139	3.166	3.198

* This ratio does not account for the trip length effect around the mean but is useful in gauging how the ratios varies.

4.5 Heterogeneity around the mean of a random parameter

Except for the lognormal, adding in a set of covariates around the mean (for uniform, triangular, normal and any other distribution that does not require some non-linear transformation) is equivalent to interacting a covariate with the random parameter attribute and adding it in as a fixed parameter. It simplifies model estimation[25]. However, one cannot do it this way with the lognormal because of the exponential form. As summarised in Table 2 above, we ran a set of models for 25, 50, 100, 250, 500, 1000 and 2000 intelligent draws with and without heterogeneity around the mean defined by trip length for the lognormal distribution[26]. As might be expected the presence of the interacting covariate reduces the role of the 'residual' mean estimate for travel time but when combined with this mean estimate produces a relativity between the overall mean and the parameter estimate of the standard deviation that is very similar[27]. The interest in this relativity is attributed to the desire to reduce the standard deviation of the parameter estimate in order to establish sensible estimates across the entire distribution (which is not always possible with unconstrained distributions). What we find here is that the sources of unobserved heterogeneity (or unobserved variance) are not represented to some extent by the decomposition of the mean. This highlights the growing need to focus research on the variability in the random component (Louviere et al 2001) and a recognition that potential sources of variability are associated with many sources (such as the study design) often not captured by the attributes of alternatives and characteristics of respondents.

As an important diversion, what many researchers call "unobserved heterogeneity" might be better termed "unobserved variability" because equations (4) and (5) strictly tell us that there are many potential sources of unobserved variability, of which differences in individuals is only one (Louviere et al 2001). Thus, research would benefit from a significant switch in focus away from heterogeneity and towards *all* relevant sources of unobserved variability. In data sources that involve individuals, one tends to think that individual differences explain differences in behavioural response outcomes. However, equation (4) suggests that this is only one aspect of unobserved variability, hence it is likely that heterogeneity observed in any one data source is conditional on other sources of variability on the right-hand side of (4).

Put another way, despite great progress in developing ever more powerful and complex models that can capture many aspects of choice behaviour, it nonetheless is the case that such models are only as good as the data from which they are estimated. Many results are potentially context-dependent in so far as behavioural outcomes depend not only on attributes of alternatives and characteristics of individuals, but also on particular factorial combinations of conditions, contexts, circumstances or situations; geographical, spatial or environmental, characteristics that are relatively constant in one place but may vary from place to place; and particular time slices or periods in which they are embedded (Louviere and Hensher 2001). Failure to take all these sources into account in complex models calls generaliseability into question, and suggests the need to give serious thought to the real meaning or interpretation of effects observed/captured/modelled in complex statistical models such as mixed logit.

4.6 Revealing empirical distributions

[25] The standard multinomial logit model (as part of a lognormal run) does not have this term, and so it is hard to compare the multinomial logit model with the mixed logit model. In building up a mixed logit, we have found it preferable to exclude this part of the specification until a stable result is obtained using a range of distributions.

[26] We also ran the triangular distribution and the stability findings are the same as the lognormal.

[27] One can define a parameter in a mixed logit model as a function of other things, but not have a variance. The simpler one is just specify a constant distribution. This makes the parameter 'constant' instead of, say, 'N' for normal.

Selecting a distribution that has desirable behavioural properties is not an easy task as already indicated. Indeed the real distribution may be bi-modal or multi-modal with the consequence that none of the popular distributions are suitable. Given the uncertainty in picking an appropriate distribution for the random parameters, an empirical perspective can be useful. This involves establishing unique (mean) parameter estimates for each sampled observation and then plotting the distribution (simply calculating a standard deviation fails to reveal the shape of the distribution[28]). To illustrate this, given a sufficiently rich data set (such as Data Set 2) in which we have multiple observations on each sampled individual (common in stated choice experiments), we might estimate a multinomial logit model for each sampled individual using a 16 choice set stated preference data set. The derived individual-specific parameter estimates can be plotted non-parametrically using kernel densities to reveal information about their distribution across the sampled population. Examining the empirical distribution gives clues about structure and ways that this structure might be incorporated back into a more general model such as mixed logit.[29]

Establishing the true distribution empirically is however a challenge because of the biases that can exist in real data, be it revealed or stated preference data. When individual-specific models are to be estimated, the variability in attribute levels across the choice sets becomes even more crucial. Stated choice designs with limited variability (especially if the variability is a fixed range relative to a current alternative) create problems in achieving asymptotically efficient estimates. It is not uncommon to find very large t-values and incorrect signs. For example, on a 16 choice set application, using data set 2, less than 20% of the sampled individuals had statistically significant parameter estimates with the correct signs; in contrast many of the remaining 80% had one parameter with the wrong sign (with mixtures of statistical significance and insignificance) and a lot had very large t-values (in the 100 to 300 range). This is largely we suspect the product of limited variability in the attribute levels offered in the stated choice experiments across the choice sets at the individual respondent level[30]. There is a big difference between degrees of variability in attribute levels and the variance of the attribute levels. Variability is as important as variance. This can be achieved by a number of strategies such as increasing the number of levels in a wide range, sampling across alternative attribute ranges for a given attribute with a common number of levels (eg 4 levels) across the choice sets. It could also be accommodated by pooling specific respondents provided one can establish agreed segmentation criteria (eg trip length, personal income). The selection of an appropriate strategy is complex and is under-researched.

Our proposed approach involves estimating a separate model for all but one respondent, each time removing an individual and re-estimating the model[31]. A comparison of the parameter estimates for a model based on the full sample and the model based on the Q-1 individuals provides the contribution of the

[28] Especially if the Spread is the correct measure of distribution around the mean.

[29] Through a richer non-linear specification of the observed influences on choice response (including spline representation and polynomial expansions) there may be more scope with simpler models (such as MNL and NL) to capture much of what mixed logits attempt to represent. It is early days yet, but the undoing of mixed logits may well be the unsatisfactory nature of analytical distributions that behaviourally fail (or are extremely difficult) to replicate the choice process within a heterogeneous sample of decision makers.

[30] This is in itself an important finding, suggesting that a wider range is generally preferred to a narrower range (within limits of meaningfulness to the respondent). Although one usually pools data across the sample, the analysis at the individual level should reveal important behavioural properties of the design configuration. What we have discovered is that the pivoting of the attribute levels around the current levels, which is intuitively appealing, has a potential downside of limiting the variability profile of the attributes across the alternatives in a choice set and across choice sets for each respondent. A way around this is to have a range of attribute ranges (eg for a 4 level attribute we might have +25%, +10%, -10%, +25% and +59%, +20%, -20% and –50%). Current research by Louviere, Hensher, Street and Anderson is developing a template for a generic design that provides precision of estimates for each and every sampled individual.

[31] This idea has been around for sometime and has been mentioned in various contexts by David Hensher, Pierre Uldry and Jordan Louviere.

single individual to the overall role of each mean parameter estimate and hence the profile of individual unobserved heterogeneity. Data Set 2 is used to illustrate this procedure. A useful command structure is provided below (in Limdep/Nlogit format) to automate this process of repeated estimation of Q-1 models.

```
Sample; all$
MATRIX; Beta=Init(143,6,0.0)$  Calc;J=1$ Create; i=Trn(1,1)$ Calc; i1=1$
Proc
Sample; all$ Calc; i2=i1+47$ Create; Omit=Ind(i1,i2)$ Reject; Omit=1$
Nlogit ; lhs=choice  ;choices=curr,alta,altb ;model:u(curr,alta,altb)
=fflow*fflow+slowt*slowt+ststop*ststop+uncert*uncert+cost*Cost+toll*toll$
Calc; i1=i1+48$ Matrix; Betai=B(1:6);Betai=Betai'$ Matrix; Beta(j,*)=Betai $
Calc; j=j+1$ MATRIX; list;beta$
EndProc
Exec;N=143$
Write; beta; file=c:\vtts\nz-vttsdata99\commpara.dat$
```

The stored matrix of parameter estimates for Q-1 models have to be plotted in order to establish the empirical profile for each attributes marginal utility (ie preference heterogeneity). The kernel density estimator is a useful device since it can describe the distribution of an attribute non-parametrically, that is, without any assumption of the underlying distribution. The kernel density function for a single attribute is computed using formula (13)[32].

$$f(z_j) = \frac{1}{n}\sum_{i=1}^{n} \frac{K\left[(z_j - x_i)/h\right]}{h}, j = 1,...,M. \qquad (13)$$

The function is computed for a specified set of values z_j, j = 1,...,*M*. *zj* is a partition of the range of the attribute. Each value requires a sum over the full sample of n values. The primary component of the computation is the kernel function, $K[.]$ which take a number of forms. For example, the logit is $K[z]= \Lambda(z)[1-\Lambda(z)]$, the normal is $K[z]= \phi(z)$ (normal density), and the uniform is $K[z]= .5$ if $|z| < 1$, 0 1 else.

The other essential part of the computation is the smoothing (bandwidth) parameter, *h*. Large values of *h* stabilise the function, but tend to flatten it and reduce the resolution. Small values of *h* produce greater detail, but also cause the estimator to become less stable. An example of a bandwidth is given in formula (14).

$$h = .9Q/n^{0.2} \text{ where } Q = \min(\text{standard deviation, range}/1.5) \qquad (14)$$

A number of points have to be specified. The set of points z_j is (for any number of points) defined by formula (15).

$$z_j = z_L + j*[(z_U - z_L)/M], j = 1,...,M \; z_L = \min(x)\text{-}h \text{ to } z_U = \max(x)+h \qquad (15)$$

The procedure produces an $M\times2$ matrix in which the first column contains z_j and the second column contains the values of $f(z_j)$ and plot of the second column against the first – this is the estimated density function. Using the kernel density to graphically describe the empirical distributions for three attributes – free flow time, slowed down time and toll cost. (Figure 5), we can establish the empirical shape of each distribution. A close inspection of the properties of each distribution (ie kurtosis and skewness) suggest

[32] This formula is embedded in the 2002 version of Limdep.

approximate analytical distributions. For example, the toll cost attribute looks very much like a lognormal, in contrast the free flow parameter is normal.

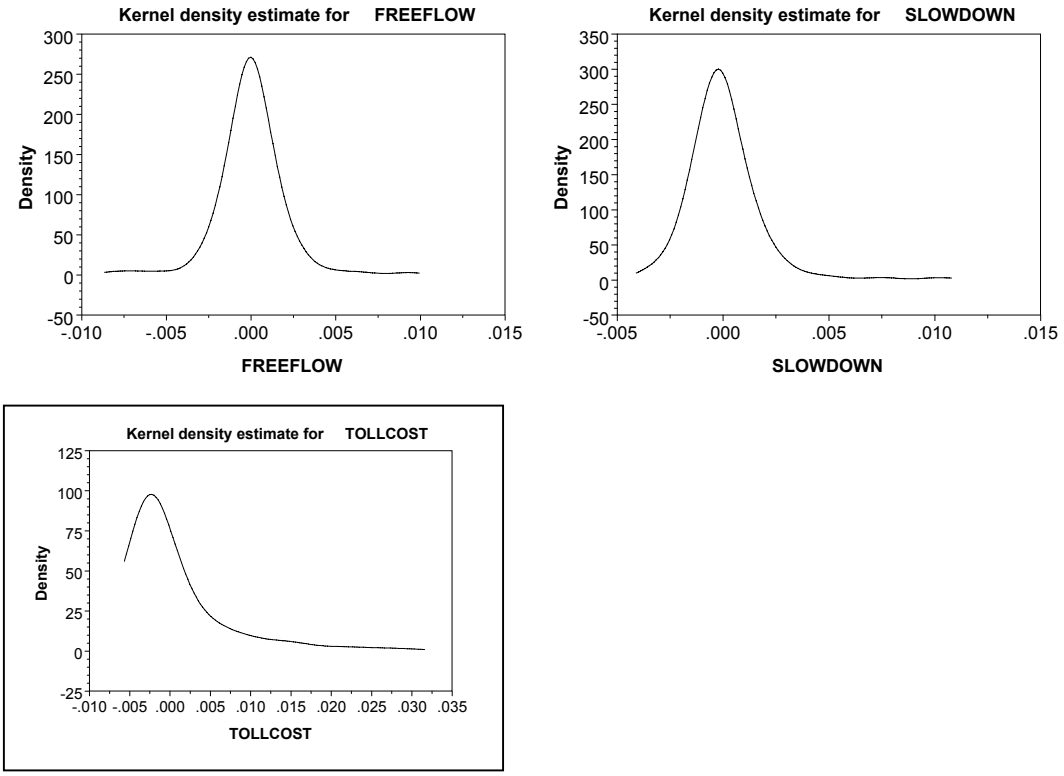


Figure 5. Empirical distributions (Data Set 2) derived non-parametrically for three parameters

4.7 Accounting for observations drawn from the same individual (eg stated choice data): correlated choice sets

Observations drawn from the same individual, as in stated choice (SC) experiments, are a common source of data for mixed logit estimation. In part this link is the result of recognition that SC data are usually much richer than revealed preference (RP) data (even when treated as a cross section) and hence opens up real opportunities to benefit by the increased richness of the mixed logit model's behavioural capability.

There is however one feature of SC data commonly available that is missing in RP data (except panel data); namely the presence of multiple observations on choice responses for each sampled individual. This means that the potential for correlated responses across observations is a violation of the independence of observations assumption in classical choice model estimation. This correlation can be the product of many sources including the commonality of socio-economic descriptors that do not vary across the choice sets for a given sampled individual[33] and the sequencing of offered choice sets that results in mixtures of learning and inertia effects[34], amongst other possible influences on choice response.

Mixed logit models, through the relaxation of the IIA property enable the model to be specified in such a way that the choice sets can be correlated across each individual. To motivate this point and show in

[33] This hints at a link between unobserved heterogeneity and correlation.

[34] The latter can in part be controlled for by randomisation of order and also including an order effect for each choice set (except one) in model estimation.

particular that correlation and unobserved heterogeneity are related and hence a key as to how mixed logits handle correlation across choice sets, think of the unobserved effects and how they might be treated. Consider a simple random utility model, in which there are heterogeneous preferences for observed and unobserved attributes of offered alternatives:

$$U_{qjt} = \alpha_{qj} + p_{qjt}\gamma_q + x_{qjt}\beta_q + \varepsilon_{qjt} \tag{16}$$

U_{qjt} is the utility that individual q receives given a choice of alternative j on occasion t. In an SC experiment, t would index choice sets. p_{qjt} denotes price, and x_{qjt} denotes another observed attribute of j (which for complete generality varies across individuals and choice sets). α_{qj} denotes the individual specific intercept for alternative j, arising from q's preferences for unobserved attributes of j. γ_q and β_q are individual specific utility parameters that are intrinsic to the individual and hence invariant over choice sets. The ε_{qjt} can be interpreted as task-specific shocks to q's tastes, which for convenience are assumed to be independent over choice sets, alternatives and individuals.

Suppose we estimate an MNL model, incorrectly assuming that the intercept and slope parameters are homogeneous in the population. The random component in this model will be

$$w_{qjt} = \hat{\alpha}_{qj} + p_{qjt}\,\hat{\gamma}_q + x_{qjt}\,\hat{\beta}_q + \varepsilon_{qjt} \tag{17}$$

where $\hat{}$ denotes the individual specific deviation from the population mean. Observe that (from the analyst's perspective) the variance of this error term for individual q on choice set t is

$$\mathrm{var}(w_{qjt}) = \sigma_\alpha^2 + p_{qjt}^2\,\sigma_\gamma^2 + x_{qjt}^2\,\sigma_\beta^2 + \sigma_\varepsilon^2 \tag{18}$$

and the covariance between choice sets t and t–1 is

$$\mathrm{cov}(w_{qjt}, w_{qj,t-1}) = \sigma_\alpha^2 + p_{qjt}\,p_{qj,t-1}\,\sigma_\gamma^2 + x_{qjt}\,x_{qj,t-1}\sigma_\beta^2 \tag{19}$$

Equations (18) and (19) reveal two interesting consequences of ignoring heterogeneity in preferences. First, the error variance will differ across choice sets as the price p and attribute x are varied. If one estimates an MNL model with a constant error variance, this will show up as variation in the intercept and slope parameters across choice sets. In an SC experiment context, this could lead to a false conclusion that there are order effects in the process generating responses[35].

Second, equation (19) shows how preference heterogeneity leads to correlated errors across choice sets. That heterogeneity is a special type of choice set correlation is not well understood. To obtain efficient estimates of choice model parameters one should include a specification of the heterogeneity structure in the model. Daniels and Hensher (2000) and Bhat and Castelar (forthcoming) indicate that the inter-alternative error correlation could be corrupted by ignoring unobserved individual heterogeneity. One such way is to specify the parameters associated with each attribute (including price) as random[36], exactly what

[35] Order effects are due to the order of the choice sets offered to a respondent. Randomising the order across the sample should remove the potential for significant order effects

[36] Some empirical evidence (eg Daniels and Hensher 2000) suggests that once unobserved heterogeneity is taken into account via a random effects specification such as ML or RPL, serial correlation may be negligible or absent. That is, serial correlation may be spurious due to the failure to account for unobserved heterogeneity.

mixed logit permits.[37] As long as one recognises that the unobserved heterogeneity must treat all alternatives across all choice sets defining an individual's choice responses (ie 16 in data sets 1 and 2) then correlation is automatically accommodated through the explicit modelling of unobserved heterogeneity present across all choice sets as defined by the underlying covariance matrix for the random parameters. This correlation is not likely to be autoregressive for 'instantaneous' stated choices since it is not the product of a long period of accumulated experience commonly attributed to state dependence. Rather it is recognition in a very short time span of the sharing of unobserved heterogeneity between choice sets that is evaluated by the same individual. The discussion herein assumes that each attribute specified with a random parameter is independent of other such specified attributes in a given choice set (within and between alternatives). This restriction, discussed in the next section, can be relaxed and tested[38].

4.8 Accounting for correlation between attributes (and alternatives)

All data sets, regardless of the number of choice sets per sampled individual, may have unobserved effects that are correlated amongst alternatives in a given choice set. One way to recognise this is to permit correlation of attributes that are common across alternatives. This engenders a covariance matrix with off-diagonal estimates identifying the dependency of one attribute on another within and between alternatives (depending on whether the attribute parameters are generic or alternative-specific). It also has interesting ramifications for the correlated choice set issue in the previous section.

Let us define the utility expression for each alternative as before (see equation (4)): $U_{qit}=ß_qX_{qit}+\varepsilon_{qit}$. Since $ß_q$ is random, it can be rewritten as $ß_q=ß+u_q$ where ß is fixed (ie the mean) and u_q is the deviation from the mean. Then $U_{qit}=ß_qX_{qit}+ (u_q X_{qit} +\varepsilon_{qit})$. There is correlation over alternatives because U_q is the same for all alternatives. That is, each individual's preferences are used in the evaluation of the alternatives. This indicates that $Cov[(u_q X_{qit} +\varepsilon_{qit}), (u_q X_{qis} +\varepsilon_{qis})]$ equals[39] $sigma(u_q)*X_{qit}*X_{qis}$ where $sigma(u_q)$ is the variance of u_q. In addition, however, there is also correlation over choice sets (or time) for each alternative because u_q is the same in each choice set (or time period). Again another way of stating this is that each individual uses the same preferences to evaluate (relative) utilities in each choice set (or time period). Thus $Cov[(u_q X_{qit} +\varepsilon_{qit}), (u_q X_{qis} +\varepsilon_{qis})]$ equals $sigma(u_q)*X_{qit}*X_{qis}$. The behavioural implication is that random preferences induce correlation over alternatives *and* choice sets (or time).

Thus both correlated alternatives and choice sets usually go hand in hand (assuming that one identifies the set of choice sets associated with each individual)[40]. Correlation over alternatives and not over choice sets (or time period) could however be established by specifying utility as $U_{qit}=ß_{qt}X_{qit}+\varepsilon_{qit}$ where $ß_{qt}$ represents preferences instead of $ß_q$. Thus preferences vary over individuals *and* over choice sets (or time period) with $ß_{qt}$ independent over choice sets (or time period) for each individual. This is likely to be an unreasonable

[37] But more importantly, if preference heterogeneity is present it is not merely a statistical nuisance requiring correction. Rather, one should model the heterogeneity in order to obtain accurate choice model predictions, because the presence of heterogeneity will impact on the marginal rates of substitution between attributes, and lead to IIA violations.

[38] The software package Limdep handles correlated choice sets by a simple command in Nlogit: pds=C, where C is the number of choice sets. If the number is fixed across the sample, then one can simply define a value in the command (eg pds=16); if the number of choice sets varies across the sample then one has to define a data item that represents the unique choice set size for each sampled individual (eg pds=csetsize). To allow for correlation between the attributes and hence alternatives in a given choice set one invokes the additional command ;cor.

[39] Sigma is the standard deviation for the normal and lognormal and the spread for the uniform and triangular distributions.

[40] The only circumstance in which you can distinguish correlated choice sets from correlated alternatives is by ignoring the dependency between choice sets or assume that it does not exist.

assumption for most situations. In particular, preferences might vary over choice sets (or time period) for each individual, but it is doubtful that they are independent over choice sets (or time) for each sampled individual. If there is some correlation in preferences over choice sets (or time) for each individual, then random parameters means correlation over choice set and over alternatives. In general, the mixed logit model can accommodate (i) correlation over alternatives and not over choice sets by assuming $ß_{qt}$ is IID over choice sets, or (ii) correlation across choice sets but not over alternatives by fixing all of the parameters except those representing the alternative-specific constants (ASC's) and assuming that ASC parameters are IID over alternatives but the same for each individual across the choice sets.

Table 3 illustrates (using Data Set 1) the presences of correlated alternatives due to correlated random parameters. This is a single cross-section observation per sampled individual and thus correlated choice sets is not an issue. All three terms in the Cholesky matrix are statistically significant supporting the hypothesis that the three alternatives are statistically correlated.

Table 3 An example of evidence of correlated alternatives.

Attributes	Alternative	Mixed Logit (lognormal random parameter) t-values in brackets.
Total time (mins)	All	-3.84218 (-15.9)
Total cost ($)	All	-.756144 (-6.5)
Tailgate percentage	All	-.012579 (9.9)
Constant	4 no median	.42995 (17.4)
Constant	4 with median	1.1500 (63.1)
Heterogeneity in mean:		
Travel time: Trip length	All	-.010399 (-7.8)
Travel cost: Trip length	All	-.006197 (-11.5)
Cholesky matrix of random parameter:		
Total time	All	1.8812 (12.8)*
Total cost	All	.882167 (18.6)*
Time:Cost	All	1.0989 (12.3)*
Pseudo-r^2 adjusted		.2245
Log-likelihood		-3730.8

Finally a word is required on ASC's. For data sets with one observation per sampled individual, random parameters on ASCs can capture heteroscedastity by omitting the ASC whose variance is smallest, since an ASC with a random parameter necessarily has greater variance than the omitted one. This requires some experimentation to establish the minimum variance, unlike the usual MNL specification of ASC's in which the selection of the ASC to exclude in the J-1 set (ie setting it to zero) is arbitrary. This treatment of ASC's as *correlated random parameters* (with no random parameters on the observable attributes) on a single choice set per sampled individual is equivalent in a mixed logit context to a multinomial probit model with a fixed covariance matrix. This is a useful check on failures of IIA. However a warning is called for. Unless there exists substantial heterogeneity and/or ASC correlations are taken into account, the ASC's will be unidentified empirically. This is because both correlations are induced by the random parameters. Serially correlated error terms and serially correlated random parameters for ASC's are exactly the same thing. In practice however random parameters are assigned attributes other than (or in addition to) ASC's, and are usually not specified as an autoregressive (eg AR1) specification[41].

4.9 Taking advantage of priors in estimation and posteriors in application to reveal individual-specific parameter estimates

[41] An AR(1) specification is questionable for stated choice data as indicated in the previous section. It is however valid for over time panel data.

Bayesian methods are often promoted as behaviourally different and better than classical estimation methods currently used in estimation of advanced discrete choice models such as mixed logit. Huber and Train (2001) have explored the empirical similarities and differences between Hierarchical Bayes and Classical estimates in the context of estimating reliable individual-level parameters from sampled population data as a basis of market segmentation. The ability to combine information about the aggregate distributions of tastes with individual's choices to derive conditional estimates of the individual's parameters is very attractive. They conclude that the empirical results are virtually equivalent conditional estimates of marginal utilities of attributes for individuals. However what this debate has achieved in particular is to show classical estimation choice modellers that there is indeed more information in their estimation procedure that enables one to improve on the behavioural explanation within sample[42]. We discuss this herein, but begin with a summary of the Bayesian view since it provides the language we need (ie priors and posteriors). Brownstone (2001) provides a useful overview as do Chen, Shao, and Ibrahim (2000), Geweke (1999) and Train (2001). Use of information on priors (as structural parameters) and posterior individual-specific parameters estimates from conditional utility functions are included as information to captured sources of heterogeneity[43].

The key difference between Bayesian and classical statistics is that Bayesians treat parameters as random variables. Bayesians summarise their prior knowledge about parameters θ by a *prior* distribution, $\pi(\theta)$. The sampling distribution, or likelihood function, is given by $f(x \mid \theta)$. After observing some data, the information about θ is given by the *posterior* distribution:

$$p(\theta \mid x) = f(x \mid \theta)\pi(\theta) \Big/ \int f(x \mid \theta)\pi(\theta)d(\theta) \tag{20}$$

All inference is based on this posterior distribution. The optimal Bayes estimator is the mean of the posterior distribution, and Bayesian confidence bands are typically given by the smallest region of the posterior distribution with the specified coverage probability. Bayesian confidence regions are interpreted as fixed regions containing the random parameter θ with the specified coverage probability. This is different from the classical confidence region, which is a region with random endpoints that contain the true value θ with the specified probability over independent repeated realisations of the data (Brownstone 2001). Classical inference therefore depends on the distribution of unobserved realisations of the data, whereas Bayesian inference *conditions on* the observed data. Bayesian inference is also exact and does not rely on asymptotic approximations.

The Bayesian approach also requires the *a priori* specification of a prior distribution for all of the model parameters. In cases where this prior is summarising the results of previous empirical research, specifying the prior distribution is a useful exercise for quantifying previous knowledge (such as the alternative currently chosen). There are many circumstances where the prior distribution cannot be fully based on previous empirical work, and the resulting specification of prior distributions based on the analyst's subjective beliefs is the most controversial part of Bayesian methodology. Poirier (1988) argues that the subjective Bayesian approach is the only approach consistent with the usual rational actor model to explain individuals' choices under uncertainty. More importantly, the requirement to specify a prior distribution enforces intellectual honesty on Bayesian practitioners. All empirical work is guided by prior knowledge and the subjective reasons for excluding some variables and observations are usually only implicit in the classical framework. Bayesians are therefore forced to carry out sensitivity analysis across other reasonable

[42] Within-sample priors such as the actual choice can help a great deal. When applying a model out-of-sample then Bayesians need some subjective priors.

[43] We capture within the classical estimation framework the same information that Hierarchical Bayes modellers capture.

prior distributions to convince others that their empirical results are not just reflections of their prior beliefs (Brownstone 2001). The simplicity of the formula defining the posterior distribution hides some difficult computational problems, explained in Brownstone (2001)[44].

In terms of the application of models, the posterior information accounts for the parameter variation across the sampled population, with the standard deviation (or spread) of each random beta and the correlated inclusion for alternatives and choice sets being taken into account. This information is ignored in the priors. The procedure to distinguish prior and posterior information within sample is set out below and applied to a mode choice data set involving a choice amongst four modes (car, plane, train, coach) for long-distance leisure travel between Sydney, Canberra and Melbourne (Data Set 3). The sequence of calculations is as follows:

1. A mixed logit model is estimated with a set of random and fixed parameters.
2. In the selected model the means of the random parameters are a constant plus a parameter times household income (hinc)
3. Two 210 by 5 matrices of parameters are computed (defined by dimensions of sample size by number of parameters).
4. The PRIOR is the set of structural parameters of the model based on the unconditional distribution (equation 7), where the slopes on the random parameters are built up from the heterogeneity around the mean criterion (ie hinc).
5. The POSTERIOR uses the individual specific parameter estimates based on the conditional distribution (equation 8).
6. The estimated probabilities for each of the choices using the two sets of parameters are computed.
7. Finally, the average probability that this method predicted for the choice actually made by each individual is computed.

We have implemented this procedure for four distributions (normal, triangular, uniform, lognormal). The results are summarised below (Table 5) for each distribution. Table 4 lists the unconditional and conditional parameter estimates for random parameters (generalised cost and transfer time) for the first 40 individuals (treated as generic over 4 alternatives in a fixed choice set) to illustrate the differences in marginal utilities. These are plotted in Figure 5. The prior parameter estimates produced close to 0.60 prediction to the choice actually made on average for the first three distributions and 0.56 for the lognormal. In contrast the posterior increases this up to about 0.86 for the first three distributions and 0.76 for the lognormal. This is an impressive increase in overall precision. Importantly, these improvements are only possible for observations whose past choices are observed. As expected, the posterior probabilities are much closer to the actual sample shares than are the prior probabilities. A close inspection of Table 4 (and Figure 6) suggests that the conditional distribution is much closer to the aggregate actual modal shares than the unconditional distribution. As we move away from the MNL model which can guarantee reproducing the within-sample choice shares, the ability to reproduce the actual shares is no longer guaranteed (this being a property of the non-IIA condition). The fact that the conditional distribution is able to come very close to the within-sample share is impressive.

[44] Computing the posterior distribution typically requires integrating over θ, and this can be difficult for the number of parameters frequently encountered in choice modelling. Until recently Bayesians solved this problem by working with *conjugate families*. These are a family of prior distributions linked to a family of likelihood functions where the posterior distribution is in the same family as the prior distribution. For example, the Beta family is a conjugate prior for the binomial with fixed number of trials. Koop and Poirier (1993) have developed and applied a conjugate prior for the conditional (and multinomial) logit model, but there do not appear to be tractable conjugate priors for other GEV discrete choice models.

Figure 7 plots the relationship between the choice probability distributions for the unconditional and conditional choice predictions for the lognormal distribution for each mode. Interestingly the plots of individual observations show some very strong one-to-one mapping (ie the diagonal) for a large number of observations for train and bus; in contrast car predictions appear to be clustered into two mappings – those in which we have a very high unconditional choice probability (top horizontal profile) in range of 0.78-1.0 which has a conditional spread from 0.05 to 0.9; and those with a very low unconditional choice probability in the range 0 to 0.2 with an equivalent conditional choice probability spread from 0 to 0.85. In the latter case, there is a greater cluster around 0.0 to 0.2 where we have clearly mapped very well (close to the diagonal). These graphs are useful indicators of what information we might seek in an unconditional choice probability in order to move it towards the conditional distribution, given that choice priors are not available for out-of-sample applications (without resort to subjective priors). The graphs suggest that we need to include some additional attributes in the air alternative utility expression to move the horizontal cluster around zero on the unconditional distribution so as to pivot these data points upwards towards the diagonal. By drilling down in the data for these observations compared to the rest of the data one might identify possible additional attributes. The same logic would be applied to the observations on the other alternatives.

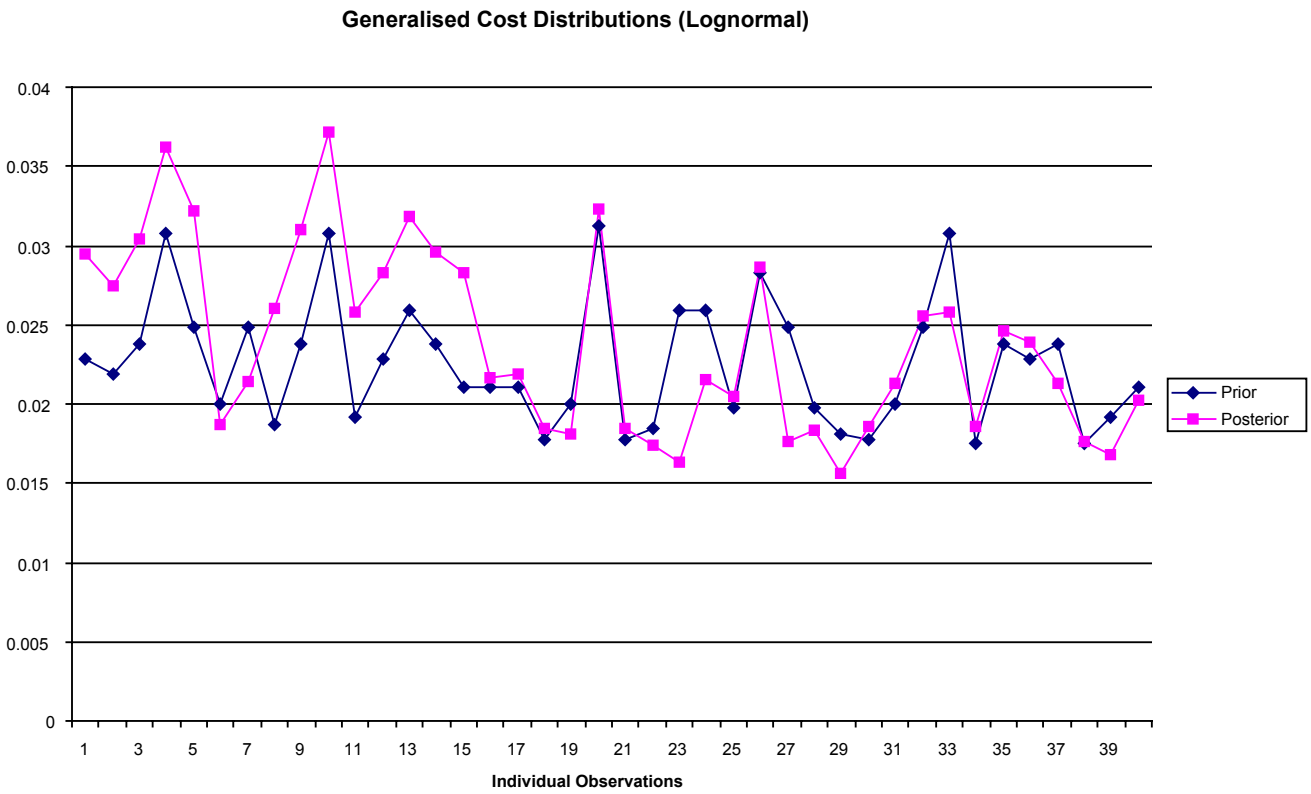
An example of a Limdep/Nlogit Input File for normal, triangular, uniform and lognormal distributions[45]

```
calc; ran(455555)$
NLOGIT; Lhs= mode; Choices = air, train, bus, car; Rhs  = gc, ttme; Rh2 = one
; RPL   = hinc; Fcn = gc(n),  ttme(n) ? note change to U, T and L
;halten; pts=50; Correlation; Parameters $
Crea;indiv=trn(4,0)$
Matr;hinci=gxbr(hinc,indiv)$
Matr;i1=init(210,1,1)
;b1=b(1)*i1+b(6)*hinci;b2=b(2)*i1+b(7)*hinci
;b3=b(3)*i1;b4=b(4)*i1;b5=b(5)*i1$
Matrix;prior=[b1,b2,b3,b4,b5]$
Matrix;postrior=beta_i$
? Parameters are b(gc), b(ttme), a_air, a_train, a_bus
Create ; nonrandm=prior(indiv,3)*aasc+prior(indiv,4)*tasc+prior(indiv,5)*basc$
Create ; UPrior = prior(indiv,1)*gc+prior(indiv,2)*ttme+nonrandm $
Create ; Upostr = postrior(indiv,1)*gc+postrior(indiv,2)*ttme+nonrandm$
Create ; ExpUpr = Exp(Uprior); ExpUpo = Exp(Upostr) $
Matrix ; sumpr  = 4*Gxbr(ExpUpr,indiv) ; sumpo  = 4*Gxbr(ExpUpo,indiv) $
Create ; ProbPr = ExpUpr / Sumpr(indiv) ; ProbPo = ExpUpo / Sumpo(indiv) $
Create ; j_po = mode*probpo ; j_pr=mode*probpr $
Calc;list;4*xbr(j_po) ; 4*xbr(j_pr) $
```

[45] The lognormal is derived using the exact same procedure as for the other three distributions except that the attributes with random parameters (ie generalised cost (gc) and transfer time (ttme) are sign reversed and the beta function has to be exponentiated (defining b1=expn(b1);b2=expn(b2)).

Table 4. List of first 50 conditional and unconditional parameter estimates for generalized cost and transfer time

Lognormal				Triangular			
GC Prior	GC Posterior	ttme Prior	ttme Postrior	GC Prior	GC Posterior	ttme Prior	ttme Postrior
0.022794	0.029517	0.129818	0.21678	-0.04093	-0.06942	-0.21598	-0.34837
0.02184	0.027484	0.126588	0.200674	-0.0389	-0.06227	-0.21272	-0.32167
0.023789	0.030458	0.133131	0.221171	-0.04296	-0.06868	-0.21924	-0.34037
0.030745	0.036185	0.154859	0.216953	-0.05514	-0.07307	-0.23883	-0.32256
0.024828	0.032143	0.136528	0.231865	-0.04499	-0.07181	-0.22251	-0.35014
0.02005	0.018678	0.120367	0.107638	-0.03485	-0.02516	-0.20619	-0.15824
0.024828	0.021422	0.136528	0.106915	-0.04499	-0.02238	-0.22251	-0.11666
0.018725	0.025999	0.115611	0.223161	-0.0316	-0.0664	-0.20097	-0.36758
0.023789	0.031042	0.133131	0.225935	-0.04296	-0.07227	-0.21924	-0.35661
0.030745	0.037159	0.154859	0.226897	-0.05514	-0.07844	-0.23883	-0.34736
0.019211	0.025819	0.117372	0.216344	-0.03282	-0.06069	-0.20293	-0.33396
0.022794	0.028236	0.129818	0.200382	-0.04093	-0.06286	-0.21598	-0.31859
0.025913	0.03187	0.140012	0.213312	-0.04702	-0.06892	-0.22577	-0.32725
0.023789	0.029617	0.133131	0.208405	-0.04296	-0.06564	-0.21924	-0.32802
0.021106	0.028264	0.124062	0.219494	-0.03728	-0.06746	-0.21011	-0.35257
0.021106	0.021615	0.124062	0.135449	-0.03728	-0.043	-0.21011	-0.23749
0.021106	0.021898	0.124062	0.139572	-0.03728	-0.043	-0.21011	-0.23678
0.017788	0.018432	0.112167	0.125359	-0.02916	-0.03592	-0.19705	-0.22795
0.02005	0.018137	0.120367	0.102836	-0.03485	-0.02339	-0.20619	-0.15203
0.031276	0.03234	0.156427	0.174549	-0.05595	-0.06029	-0.24013	-0.26071
0.017788	0.018469	0.112167	0.126231	-0.02916	-0.03598	-0.19705	-0.22916
0.018407	0.017379	0.114451	0.105134	-0.03079	-0.02076	-0.19966	-0.15278
0.025913	0.016308	0.140012	0.060105	-0.04702	0.005206	-0.22577	0.020603
0.025913	0.02156	0.140012	0.10154	-0.04702	-0.01372	-0.22577	-0.06874
0.01971	0.020435	0.11916	0.134772	-0.03403	-0.03432	-0.20489	-0.20605
0.028226	0.02863	0.147248	0.15885	-0.05108	-0.05245	-0.2323	-0.23896
0.024828	0.017578	0.136528	0.07239	-0.04499	0.001354	-0.22251	0.002496
0.01971	0.018402	0.11916	0.108326	-0.03403	-0.02212	-0.20489	-0.14879
0.018095	0.015602	0.113303	0.08779	-0.02998	-0.00909	-0.19836	-0.10106
0.017788	0.018612	0.112167	0.128989	-0.02916	-0.0361	-0.19705	-0.22922
0.02005	0.02127	0.120367	0.143487	-0.03485	-0.04145	-0.20619	-0.23589
0.024828	0.025555	0.136528	0.151345	-0.04499	-0.05014	-0.22251	-0.24774
0.030745	0.025856	0.154859	0.114096	-0.05514	-0.03739	-0.23883	-0.15728
0.017487	0.018631	0.111042	0.133189	-0.02835	-0.03564	-0.19575	-0.23051
0.023789	0.024626	0.133131	0.150239	-0.04296	-0.04953	-0.21924	-0.24977
0.022794	0.023938	0.129818	0.15047	-0.04093	-0.04685	-0.21598	-0.24367
0.023789	0.021273	0.133131	0.110608	-0.04296	-0.0308	-0.21924	-0.16378
0.017487	0.017632	0.111042	0.116544	-0.02835	-0.02722	-0.19575	-0.1911
0.019211	0.016772	0.117372	0.093467	-0.03282	-0.01421	-0.20293	-0.1164
0.021106	0.020208	0.124062	0.118668	-0.03728	-0.02556	-0.21011	-0.15466



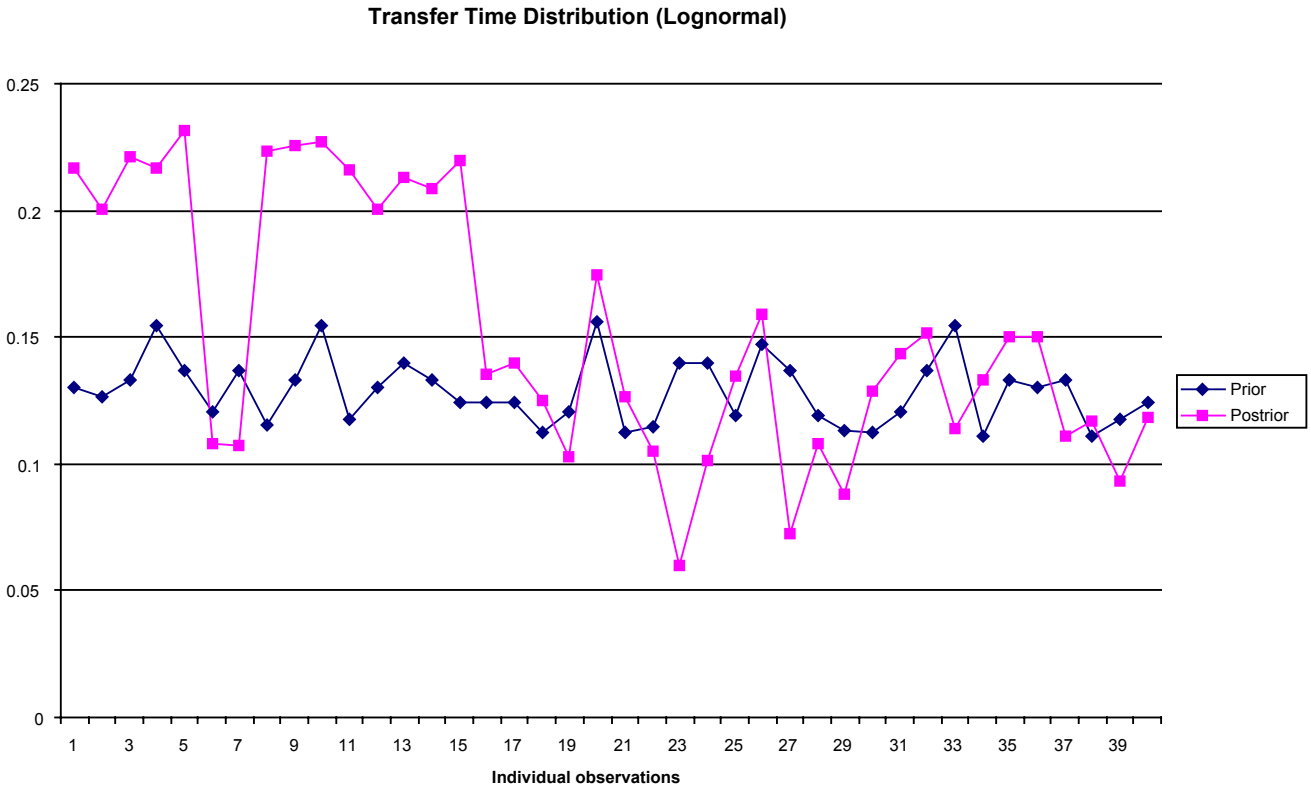


Figure 6. Comparison of the Unconditional (prior) and conditional (posterior) parameter estimates for random parameters

Table 5 Average Choice predictions for prior and posterior specifications under alternative distributional assumptions (Sample choice shares: Air = 0.276, Car = 0.281, Bus = 0.143, Train = 0.300)

Overall	*Triangular*	*Uniform*	*Normal*	*Lognormal*
Prior	0.6062	0.6051	0.6094	0.5633
Posterior	0.8565	0.8709	0.8580	0.7572
Plane	*Triangular*	*Uniform*	*Normal*	*Lognormal*
Prior	0.1948 (.329)	.207 (.337)	.195 (.328)	0.2614 (.314)
Posterior	0.2712 (.271)	.272 (.385)	.274 (.384)	0.2465 (.339)
Car	Triangular	Uniform	Normal	Lognormal
Prior	0.2116 (.294)	.202 (.287)	.223 (.299)	0.1999 (.227)
Posterior	0.2799 (.441)	.280 (.441)	.280 (.439)	0.2974 (.150)
Coach	*Triangular*	*Uniform*	*Normal*	*Lognormal*
Prior	0.2054 (.314)	.208 (.321)	.202 (.311)	0.1742 (.249)
Posterior	0.1472 (.299)	.148 (.303)	.146 (.299)	0.1503 (.255)
Train	*Triangular*	*Uniform*	*Normal*	*Lognormal*
Prior	0.3881 (.374)	.383 (.380)	.380 (.373)	0.3645 (.325)
Posterior	0.3017 (.387)	.299 (.390)	.299 (.387)	0.3058 (.348)

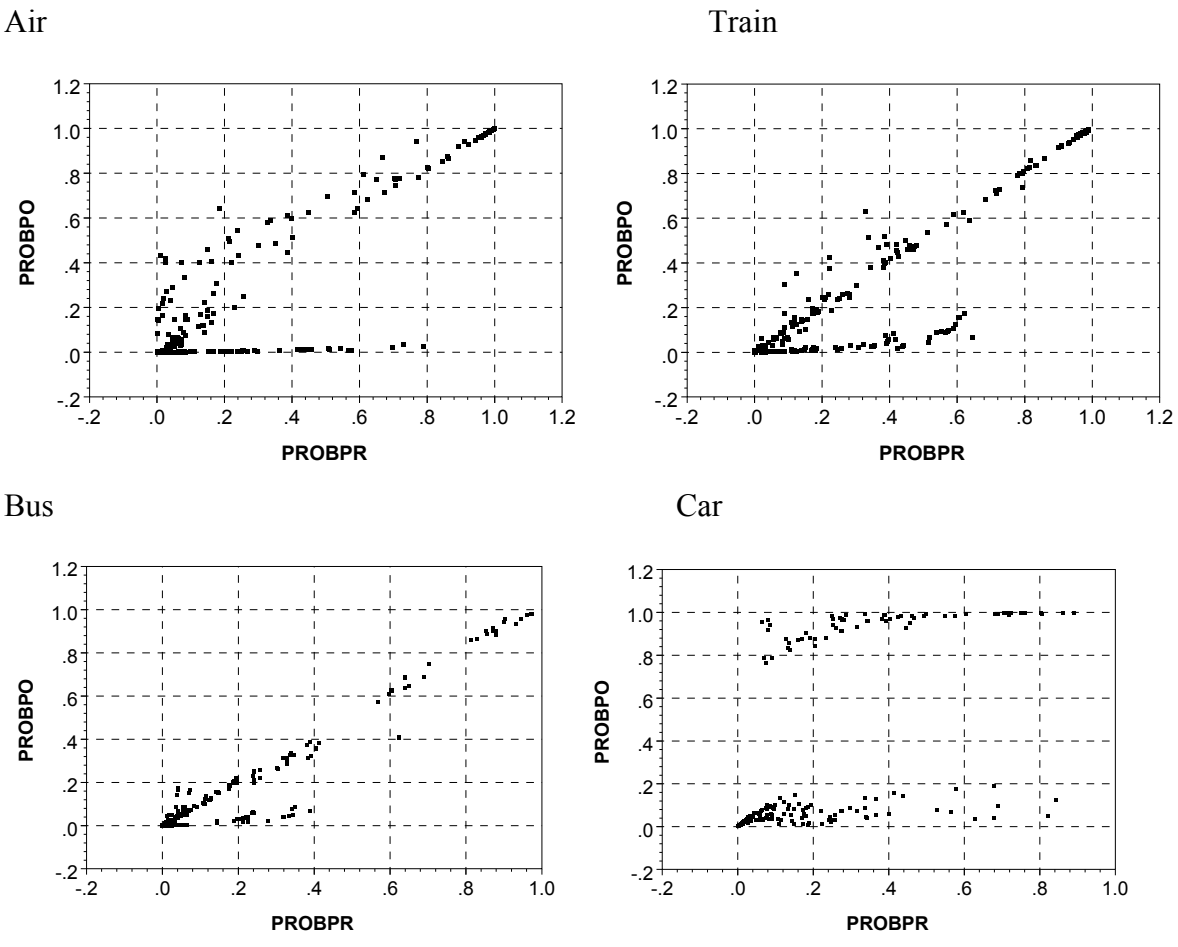


Figure 7 Choice probability distributions for the lognormal distribution (vertical axis is unconditional probability, horizontal axis is conditional probability)

4.10 Willingness to Pay (WTP) Challenges

Although selecting distributions for individual parameters is challenge enough, it is compounded when interest focuses on ratios of parameters, as in the derivation of estimates of willingness to pay. For example, the ratio of two triangular parameters has a discontinuous distribution with either distribution having a singularity unless the range is forced to exclude zero. Infinite mean and variance occurs in both cases however. The ratio of two normals has the same problem with the singularity at zero for the denominator.

In deriving WTP estimates based on random parameters one can use all the information in the distribution or just the mean and standard deviation. The former is preferred but is more complicated. Simulation is used in the former case, drawing from the estimated covariance matrix for the parameters (as in equation 11 above).

To explain the two approaches, suppose you have a model with a fixed cost parameter ß$_1$, and an attribute whose parameter is normally distributed with mean ß$_2$ and standard deviation ß$_3$. Then the willingness to pay for the attribute is distributed normally with mean ß$_2$/ ß$_1$ and standard deviation ß$_3$/ ß$_1$. You can use the point estimates of ß$_1$, ß$_2$, and ß$_3$ to calculate these ratios. This approach takes the point estimates as given and ignores the sampling variance in these point estimates. To incorporate the sampling variance let ß be the vector with elements ß$_1$, ß$_2$, and ß$_3$. The estimation process yields a covariance matrix for all the estimated parameters. One extracts the part for ß (call it M), which is a 3 by 3 symmetric matrix. Take the Cholesky factor of M and call it L. Then generate draws of ß$_1$, ß$_2$, and ß$_3$as $\hat{\beta}$ +Lu where u is a 3 by 1 vector of IID standard normal deviates drawn from a random number generator and $\hat{\beta}$is the point estimate of ß. For each draw, calculate ß$_2$/ ß$_1$ and ß$_3$/ ß$_1$, which are the mean WTP and the standard deviation in WTP implied by those draws. Do this for many draws. Then calculate the mean and standard deviation of ß$_2$/ ß$_1$ over these draws. That gives you the estimated mean WTP and the standard error in this estimated mean. Also calculate the mean and standard deviation of ß$_3$/ ß$_1$ over these draws to get the estimated standard deviation of WTP and the standard error of this estimate.

To accommodate the entire distribution of WTP (rather than just the mean and standard deviation), take a draw of ß$_1$, ß$_2$, and ß$_3$ as described above (as $\hat{\beta}$ +Lu). For this draw, one takes numerous draws of WTP, with each draw constructed as (ß$_2$ + ß$_3$*u)/ ß$_1$ where u is a standard normal deviate from a random number generator. Repeat for many draws of ß$_1$, ß$_2$, and ß$_3$, to get many sets of draws of WTP. Then, you can calculate whatever you want to know about WTP from the combined set of WTP draws; e.g., you can calculate the probability that WTP exceeds some amount. Equation (11) above uses this method which we implement below.

Using the four lane data for car drivers (Data Set 1) we estimated three models (Table 6). The first model treats travel time as a random parameter, the second model treats travel cost as a random parameter, and the third model allows both travel time and cost to have random parameters. The random parameters are assumed to be correlated in the third model and hence the Cholesky decomposition is used to identify the standard deviations (or spread)[46]. A lognormal distribution was imposed on all random parameters with

[46] In order to allow for correlation between the parameters, we would write the entire vector of correlated parameters $\boldsymbol{\beta}_i = \boldsymbol{\beta}_{mean} + \boldsymbol{\Gamma\Sigma v}_i$ where $\mathbf{v}_i$ is the set of random draws from the assigned distribution (note, these need not be the same for all parameters), $\boldsymbol{\Sigma}$ is the diagonal matrix of scale (or "spread") factors that appears above, and $\boldsymbol{\Gamma}$ is the lower triangular Cholesky factor of the correlation matrix of the parameters. (Thus, the diagonal elements of $\boldsymbol{\Gamma}$ are equal to one.) Then, the actual covariance matrix of the random terms that enters the parameters is $\boldsymbol{\Gamma\Sigma S\Sigma\Gamma'}$ where $\mathbf{S}$ is diagonal with diagonal elements equal to 1.0 for normally distributed parameters, $1/\sqrt{3}$ for uniformly distributed parameters, and $1/\sqrt{6}$ for the triangularly distributed parameters. (See

sign reversal of the attributes associated with these parameters. The value of travel time savings (VTTS) distributions produce quite different means and standard deviations. For the full sample they range from a mean of $4.773 for travel cost as a random parameter to $5.762 for travel time as a random parameter and $23.4 when both time and cost are random parameters. Since the lognormal distribution has a very long tail (see Table 7), it is often suggested that the last few percentiles could be removed to at least ensure that the mean is a better representation of the majority of the individuals (recognising this unfortunate feature of a lognormal). When we remove the highest two percentile, the mean and standard deviation change significantly, especially for the model with two random parameters (compare Figures 8 and 9). The authors' experience with a number of data sets suggests a phenomenon that is not unique to a specific data set but widespread. It is not until one investigates the WTP outputs that the actual distributions really matter[47].

Table 6 Three Mixed Logit Models with Alternative Random Parameters for WTP calculation.
T-values in brackets. *: parameters are from the Cholesky matrix. (Data Set 1).

Attributes	Alternative	Mixed Logit (lognormal random parameter)		
		M1 (time =RP)	***M2 (cost = RP)***	***M3 (time and cost = RP)***
Total time (mins)	All	-4.7136 (-7.67)	-.004874 (-6.4)	-3.84218 (-15.9)
Total cost ($)	All	-.12743 (-49.1)	-.9469 (-7.3)	-.756144 (-6.5)
Tailgate percentage	All	-.010856 (-9.0).	-.012145 (-9.7)	-.012579 (9.9)
Constant	4 no median	.3199 (13.2)	.4117 (16.7)	.42995 (17.4)
Constant	4 with median	.9418 (53.2)	1.1213 (61.3)	1.1500 (63.1)
Heterogeneity in mean:				
Travel time: Trip length	All	-.0061335 (-2.4)	-	-.010399 (-7.8)
Travel cost: Trip length	All	-	-.054599 (-9.9)	-.006197 (-11.5)
Std Deviation. of parameter distribution				
Total time	All	1.4876 (7.35)	-	1.8812 (12.8)*
Total cost	All	-	1.2578 (15.5)	.882167 (18.6)*
Time:Cost	All	-	-	1.0989 (12.3)*
Pseudo-r^2 adjusted		.1526	.2152	.2245
Log-likelihood		-4078.3	-3776.9	-3730.8
Value of Travel Time Savings ($ per person hour) based on full sample of 4384 observations				
MLVOTT	All	5.762 (20.7)		
MLVOTC	All		4.773 (11.542)	
MLVOTTC	All			23.463 (320.0)
Value of Travel Time Savings ($ per person hour) based on truncated sample post-estimation (removing last 2 percentile):				
MLVOTT	All	4.066 (6.72)		
MLVOTC	All		4.0247 (6.25)	
MLVOTTC	All			9.087 (22.4)

Formulae for calculating VTTS distribution:

rna=rnn(0,1),rnb=rnn(0,1),rnc=rnn(0,1)

mlvott=-60*(exp(-4.7136-.0061335*tripl+1.487627*rna))/-.127439

mlvotc=-60*(-.00487488)/(exp(-.946935-.0054599*tripl+1.2578*rnb))

mlvottc=60*(exp(-3.84218-.01039988*tripl+1.8811648*rna))/(exp(-.7561443-.00619*tripl+1.0989*rnb+.8822*rnc)

---

footnote 16.) In this case, it can be seen that each coefficient is equal to its mean plus a mixture of the random terms which enter some or all of the other parameters. (Since Γ is triangular and Σ is diagonal, β_{i1} is a function of v_1 only, β_{i2} is a function of v_{i1} and v_{i2} and so on.) This allows the parameters to be freely correlated and have an unrestricted scale as well while insuring that the covariance matrix that we estimate is positive definite at all times.

[47] This is not to suggest that it is an unimportant issue for prediction.

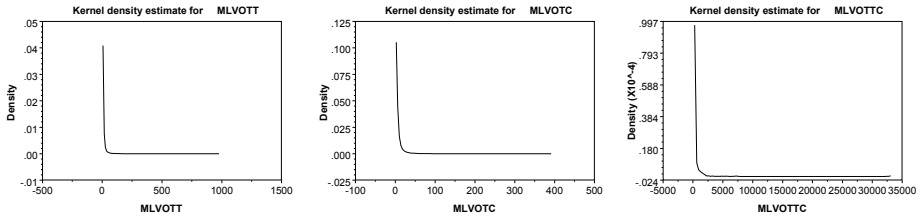


Figure 8. Full sample distribution of VTTS

Table 7 Full Sample cumulative distribution of value of travel time savings

Sample Size: 16 choice sets per individual* 274 individuals * 3 alternatives

Range ($/person hour)	MLVOTT	MLVOTC	MLVOTTC
0-1	.3897	.3352	.4450
1.01-2	.5605	.5300	.5601
2.01-3	.6577	.6461	.6247
3.01-4	.7237	.7165	.6706
4.01-5	.7673	.7687	.7007
5.01-6	.8006	.8082	.7288
6.01-7	.8225	.8390	.7505
7.01-8	.8429	.8593	.7679
8.01-9	.8597	.8761	.7834
9.01-10	.8733	.8910	.7967
10.01-15	.9196	.9356	.8410
15.01-20	.9441	.9583	.8681
20.01-50	.9840	.9916	.9338
50.01-100	.9949	.9980	.9637
100.01-200	.9989	.9995	.9803
200.02-300	.9993	.9998	.9879
300.01-400	.9995	1.000	.9916
400.01-500	.9997		.9942
500.01-1000	1.000		.9973
Over 1000			1.000

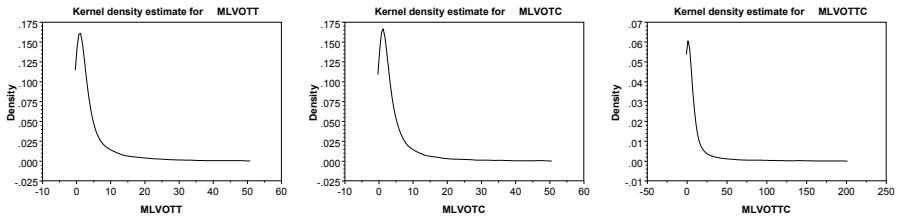


Figure 9 Removing last two percentile

The concern with arbitrarily removing part of a distribution for whatever reason[48] suggests a serious consideration of constrained distributions. To illustrate the implications of imposing a constraint on the lognormal distribution we have estimated a series of models using a fourth data set associated with light commercial vehicles collected in 2001 in Sydney[49]. We have constrained the standard deviation to be 0.25,

[48] For the normal, uniform and triangular, the negative region for VTTS is often quite substantial, indeed often exceeding 2 percentiles.

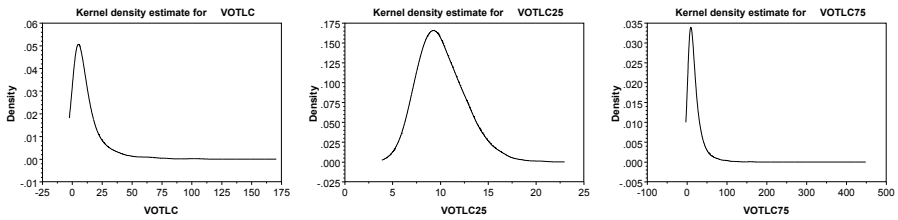
[49] This data set is currently commercial in confidence.

0.75, 1.0, 1.5 and 2.0 of the mean[50]. The results are summarised in Table 8 and Figure 10. Except for the constraint where standard deviation =0.25 of the mean, the distributions still have a long tail with mean estimates of VTTS declining (from 19.09 to 15.08) as we move from 0.75 of the mean to twice the mean. This initially seems odd given the position that a more constrained standard deviation should reduce the mean; however since the calculation of VTTS herein uses the cost variable as the random parameter (ie the denominator), the result might be expected. The evidence for a very small standard deviation however is contrary with a mean of $10.05 and a very small standard deviation of 2.53. The overall goodness of fit of the model with standard deviation equal to 0.25 of the mean is significantly worse than the other constrained distributions, despite the range being much more appealing. What is particularly noticeable is that the mean estimate of the random parameter increases substantially to almost compensate for the constrained standard deviation, resulting in far less of an impact on the overall average VTTS (with the exception of a 0.25 scale). While we are capable of imposing such a series of constraints, there appears to be no strong theoretical basis for doing so. Distributions are analytical constructs however and hence the imposition of such constraints is no better or worse than an unconstrained distribution unless there is a theoretical/behavioural rationale. Except for the sign of the WTP, we appear to have no theoretical arguments to support one distribution over another. Practitioners are likely to remain sceptical of WTP measures based on such long tails as typified by the lognormal. The alternative may well be greater consideration of segmentation of attributes in order to establish a discrete set of fixed parameters along a line (essentially points on an undefined distribution). The disadvantage of this is that one might select the set of thresholds and segment criteria that are inadequate representations of the heterogeneity in the variance structure of the unobserved effects.

Table 8 Implications of constrained lognormal distributions on value of travel time savings ($ per driver hour) (2,976 observations)

VTTS distribution	Mean	Standard distribution	minimum	maximum	Goodness of Fit
unconstrained	12.60	19.67	0.18	462.7	-740.61
SD=0.25 of mean	10.06	2.53	4.23	25.50	-885.10
SD=0.75 of mean	19.09	22.09	0.61	420.59	-761.54
SD=mean	18.68	25.13	0.40	533.58	-750.59
SD=1.5 of mean	15.57	23.10	0.26	525.14	-745.51
SD=2.0 of mean	15.08	25.31	0.18	623.99	-742.79

No scale: votlc=-60*(-.0835)/exp(-.35697+1.05369*rna)
Scale=0.25: votlc25=-60*(-.06209)/exp(-.96702+.241756*rna)
Scale = 0.75: votlc75=-60*(-.0666)/exp(-1.1737+0.88031*rna)
Scale =1.0: votlc1=-60*(-.0737)/exp(-.96708+.96708*rna)
Scale = 1.5: votlc15=-60*(-.07692)/exp(-.68278+1.0241*rna)
Scale = 2.0: votlc2=-60*(-.07874)/exp(-.54793+1.09587*rna)



[50] When you allow for correlation amongst attributes, the scale factor is not the standard deviation of the distribution, even in the normal distribution case. The standard deviation is the square root of the sum of squares of the elements in a row of the Cholesky matrix, and there is no way to make that square root equal to one of the parameters.

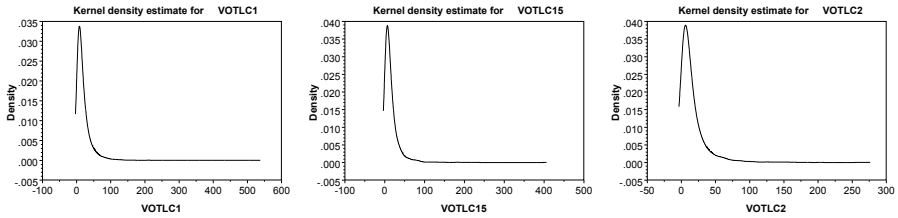


Figure 10 Constrained Distributions Relative to unconstrained distribution (VOTLC)

To investigate the implications of constraints on other distributions than the lognormal, we estimated a model using the triangular distribution imposing constraints on the spread. Setting the spread to 1.0 guarantees all the same sign. Any other value will lead to both signs. The reason is as follows. Define as before, $ß_i$ + scale*$ß_i$*t where t is the triangular distribution that ranges from -1 to +1. If the scale equal 1.0, the range is 0 to 2 $ß_1$. We found that the mean VTTS for spread equal to 1.0 is \$7.62 (with a range \$4.93 to \$14.1). Thus the entire distribution is within the positive VTTS range in contrast to the unbounded spread with a mean of \$2.51 and a range from -\$5848 to \$3112 (although 99% of values are in the range -\$200 to \$240). Figure 11 graphs these two distributions. We conclude on the basis of this evidence that a lower bounded triangular distribution has appeal in that it eliminates the long tail common to a lognormal while ensuring the behaviourally correct sign of WTP. This initial inquiry into constrained distributions suggests a major topic for ongoing research[51].

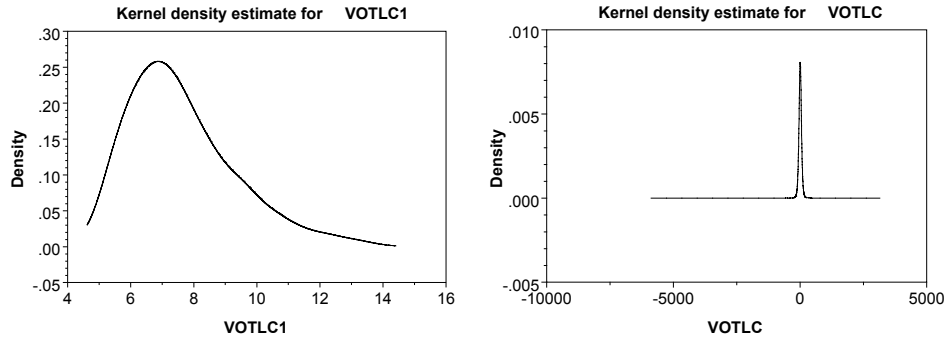


Figure 11. The VTTS distribution with and without the lower bound for the triangular distribution.

5. Conclusions

The continuing challenges we face with mixed logit models are derived in the main from the quality of the data. Mixed logit certainly demands better quality data than MNL since it offers an extended framework within which to capture a greater amount of true behavioural variability in choice making. It is, broadly speaking, aligning itself much more with reality where every individual has their own inter-related systematic and random components for each alternative in their perceptual choice set(s). Although there is a level of irreducible variability in everyone, its does have some basis in the fact that individuals do not do the same thing all the time for a variety of reasons that analysts cannot fully observe or explain (and probably neither can the individuals themselves).

[51] There are a number of views in the research community about ways of handling the information in distributions. These include using only the mean from a lognormal instead of the simulated distribution and selecting a more symmetrical distribution such as the triangular but constraining it to the non-negative range (as we have done in the text). The evidence herein suggests little gain from constraining a lognormal distribution; however promoting the use of only the mean (including any parameterisation of heterogeneity around the mean) from a lognormal is controversial since relevant information is being discarded. The possibility of eliminating the extreme values where they are small in number remains appealing if one wishes to use a lognormal.

As discrete choice models become less restrictive in their behavioural assumptions, the possibility of identifying sources of heterogeneity associated with the mean and variance of systematic and random components increases. Ultimately we want to improve on our modelling capability to improve the predictability of a model when individuals are faced with changes in the decision environment as represented by a set of attributes of alternatives, characteristics of decision makers and other contextual effects (which can include task complexity for data collection, especially stated choice experiments). The sources of explanatory power reside within the systematic and random components in potentially complex ways and can be captured by both the mean and the variance of parameters representing observed and unobserved effects. The mixed logit model certainly opens up new opportunities to research these behavioural phenomena.

What is important for modellers is the recognition that each individual's random component variance is perfectly confounded with their mean or systematic components (Louviere et al 2001). Thus, one needs extra information in order to achieve identification. It is an important and open question as to what that might be in a modelling setting where we abstract from reality to varying degrees and impose additional translation constraints in order to obtain preference and choice responses from individuals. These translational constraints include the actual design of the data collection instrument and how this translates into task complexity that intervenes in the decision making process. Recent work by DeShazo and Fermo (2001), Swait and Adamowicz (2001) amongst others[52] suggests that some individual differences can be used to put structure on the differences in variability.

There is always more research required, but at various junctures in the process it is prudent to take stock of progress and to highlight the major developments and warn about the continuing challenges. This paper has set itself this objective in the very specific setting of the application of the mixed logit model. If we have succeeded then a number of very practical issues discussed herein should assist analysts as they venture more into the practical detail of specifying, estimating, and interpreting mixed logit models and in applying their behavioural outputs.

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[52] Ongoing research by Hensher is investigating the role of design variability in the estimation of mixed logit models in order to control for the influence of design effects.

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