Correlation in Games

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“Intrinsic Correlation in Games,” with Amanda Friedenberg, *Journal of Economic Theory*, 2008

“The Power of Paradox: Some Recent Developments in Interactive Epistemology,” *International Journal of Game Theory*, 2007

“A Classification of Hidden-Variable Properties,” with Noson Yanofsky, *Journal of Physics A: Mathematical and Theoretical*, 2008

“The Relationship Between Quantum and Classical Correlation in Games,” at www.stern.nyu.edu/~abranden

Example of Correlation

	L	R
U	3	3
D	0	0

X

	L	R
U	2	0
D	0	2

Y

	L	R
U	0	0
D	3	3

Z

Strategy Y is optimal for the player choosing the matrix under, e.g.

$$\text{Prob}(U, L) = \text{Prob}(D, R) = \tfrac{1}{2}$$

There is no product measure under which Y is optimal

[Let p be the probability on U and q be the probability on L. We have

$$\max\{3p, 3(1-p)\} > 2pq + 2(1-p)(1-q) \ ]$$

Where Does Correlation Come From?

The two branches of game theory:*

In **non-cooperative** theory (matrices and trees), players choose strategies "independently"

In **cooperative** theory (characteristic functions), they may choose "jointly"

In non-cooperative theory, where does the correlation come from?

Answer: Hidden variables

Consider the players' characteristics not already described in the matrix—viz. what the players think about which strategies will be chosen, what they think other players think about this, etc. (In game theory jargon, these are the players' **hierarchies of beliefs**)

* Beware: Terminology here is misleading!

Epistemic Game Theory

Classically, a non-cooperative game was fully described as a game matrix or a game tree

The epistemic approach adds a description of what the players think about which strategies will be chosen, what they think other players think about this, etc.

Formally, this is done via **epistemic type structures**

[Harsanyi (1967-8) introduced types to talk about uncertainty over the payoffs

The epistemic program adapts the technique to talk about uncertainty over the strategies chosen—or about uncertainty over both the payoffs and the strategies]

Example of a Type Structure

	L	*R*
U	1,1,3	1,0,3
D	0,1,0	0,0,0

X

	L	*R*
U	1,1,2	0,0,0
D	0,0,0	1,1,2

Y

	L	*R*
U	1,1,0	1,0,0
D	0,1,3	0,0,3

Z

Ann's strategy set $S^a = \{U, D\}$

Bob's strategy set $S^b = \{L, R\}$

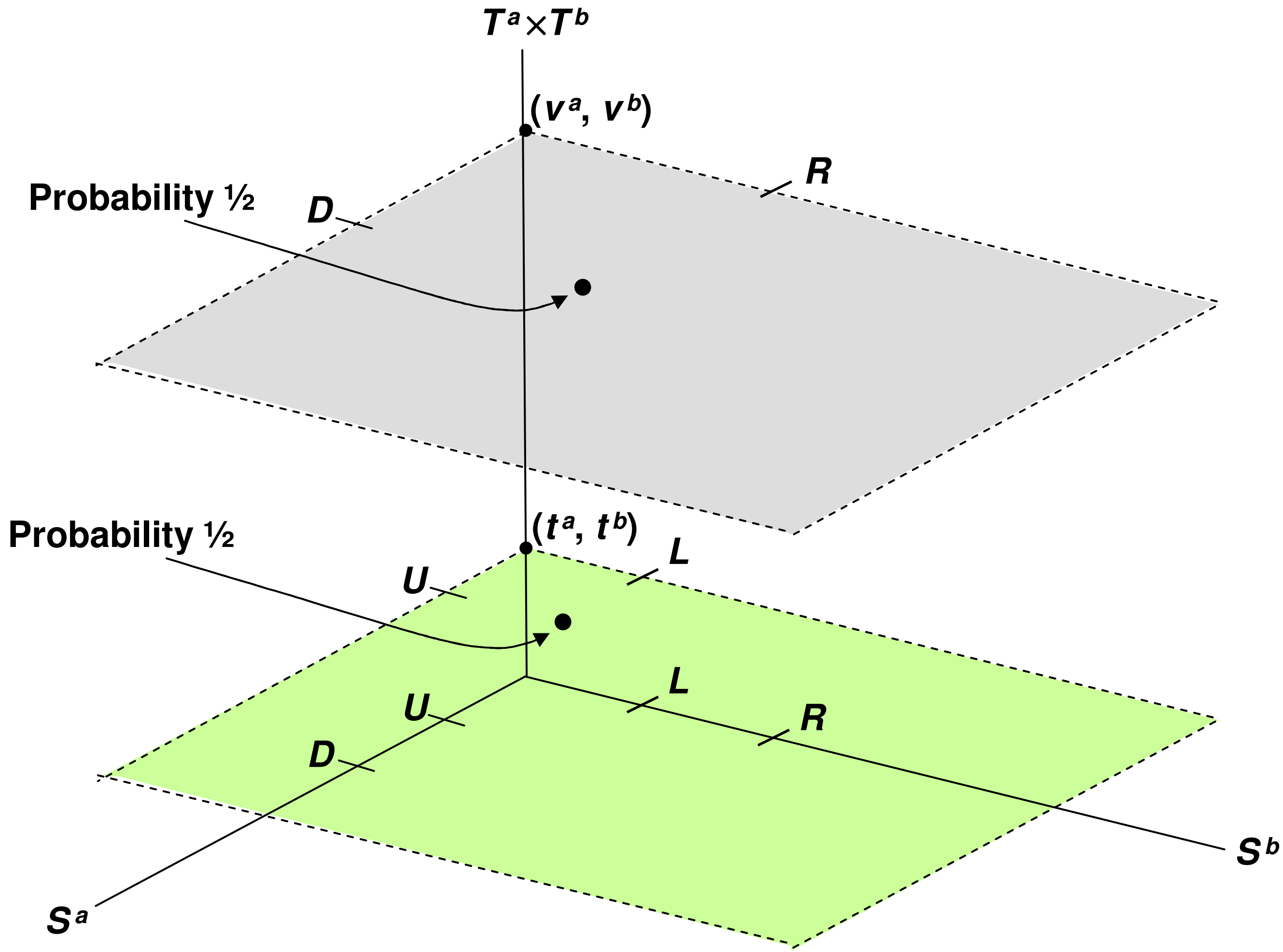
Charlie's strategy set $S^c = \{X, Y, Z\}$

Ann's type set $T^a = \{t^a, v^a\}$

Bob's type set $T^b = \{t^b, v^b\}$

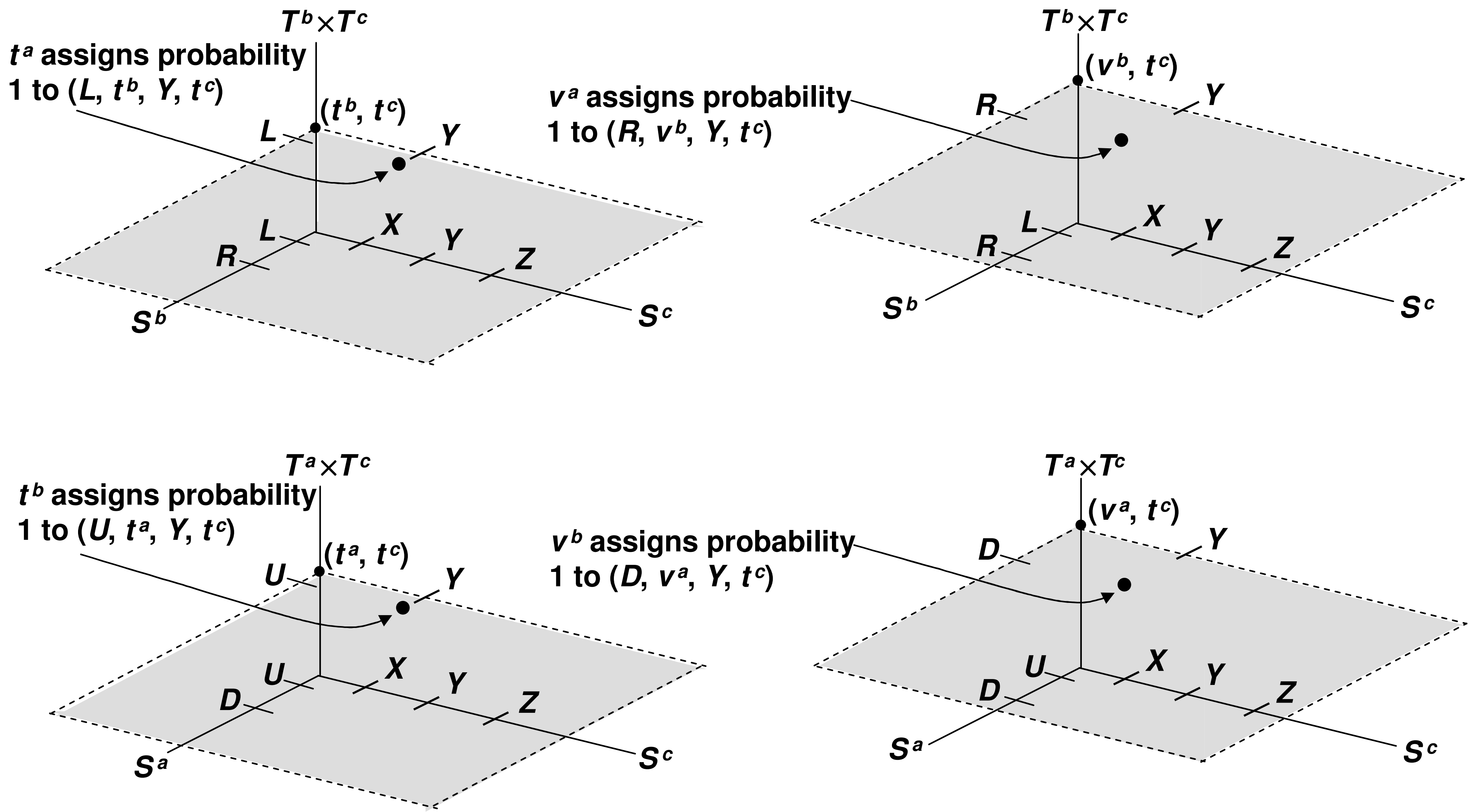
Charlie's type set $T^c = \{t^c\}$

Example continued



Type t^c assigns probability ½ to (U, t^a, L, t^b)
probability ½ to (D, v^a, R, v^b)

Example continued



Types induce hierarchies of beliefs

Conditions on Type Structures

Conditional Independence (**CI**):

Charlie's type t^c should satisfy

$p(s^a, s^b \mid t^a, t^b) = p(s^a \mid t^a, t^b) \times p(s^b \mid t^a, t^b)$ whenever $p(t^a, t^b) > 0$

(likewise for Ann and Bob)

Sufficiency (**SUFF**):

Charlie's type t^c should satisfy

$p(s^a \mid t^a, t^b) = p(s^a \mid t^a)$ whenever $p(t^a, t^b) > 0$

(likewise for *b*, and for Ann and Bob)

Under CI and SUFF:

If $p(t^a, t^b) = p(t^a) \times p(t^b)$, then $p(s^a, s^b) = p(s^a) \times p(s^b)$

In words, a correlated assessment about strategy choices implies a correlated assessment about types (no physical correlation)

Technical notes:
(i) If there are redundant types, then the conditioning must be on hierarchies
(ii) The definitions can be extended to infinite type spaces

Question

What strategies can be played under this notion of correlation?

The baseline:

Delete any dominated strategies from the matrix

Delete any dominated strategies from the resulting submatrix

...

(This characterizes the epistemic condition of **rationality and common belief of rationality**

The proof uses the Supporting Hyperplane Theorem

See Bernheim 1984, Pearce 1984, Brandenburger-Dekel 1987, Tan-Werlang 1988)

What is the effect of adding the CI and SUFF requirements?

Second Example

X

	L	*C*	*R*
U	0,0,2	0,0,2	0,1,2
M	0,0,0	0,0,0	0,1,0
D	1,0,2	1,0,2	1,1,2

Y

	L	*C*	*R*
U	1,1,1	0,1,0	0,1,0
M	1,0,0	0,0,1	0,1,0
D	1,0,0	1,0,0	1,1,0

Z

	L	*C*	*R*
U	0,0,0	0,0,0	0,1,0
M	0,0,2	0,0,2	0,1,2
D	1,0,2	1,0,2	1,1,2

U and *M* are each optimal if and only if Prob(*L*, *Y*) = 1

L and *C* are each optimal if and only if Prob(*U*, *Y*) = 1

Y is optimal if and only if Prob(*U*, *L*) = Prob(*M*, *C*) = ½

Every strategy is **iteratively undominated**

Second Example continued

	L	*C*	*R*
U	0,0,2	0,0,2	0,1,2
M	0,0,0	0,0,0	0,1,0
D	1,0,2	1,0,2	1,1,2

X

	L	*C*	*R*
U	1,1,1	0,1,0	0,1,0
M	1,0,0	0,0,1	0,1,0
D	1,0,0	1,0,0	1,1,0

Y

	L	*C*	*R*
U	0,0,0	0,0,0	0,1,0
M	0,0,2	0,0,2	0,1,2
D	1,0,2	1,0,2	1,1,2

Z

If (U, t^a) and (M, v^a) are rational, then the hierarchies induced by t^a and v^a must agree up to level 1

If (L, t^b) and (C, v^b) are rational, then the hierarchies induced by t^b and v^b must agree up to level 1

If (Y, t^c) and (Y, v^c) are rational, then the hierarchies induced by t^c and v^c must agree up to level 1

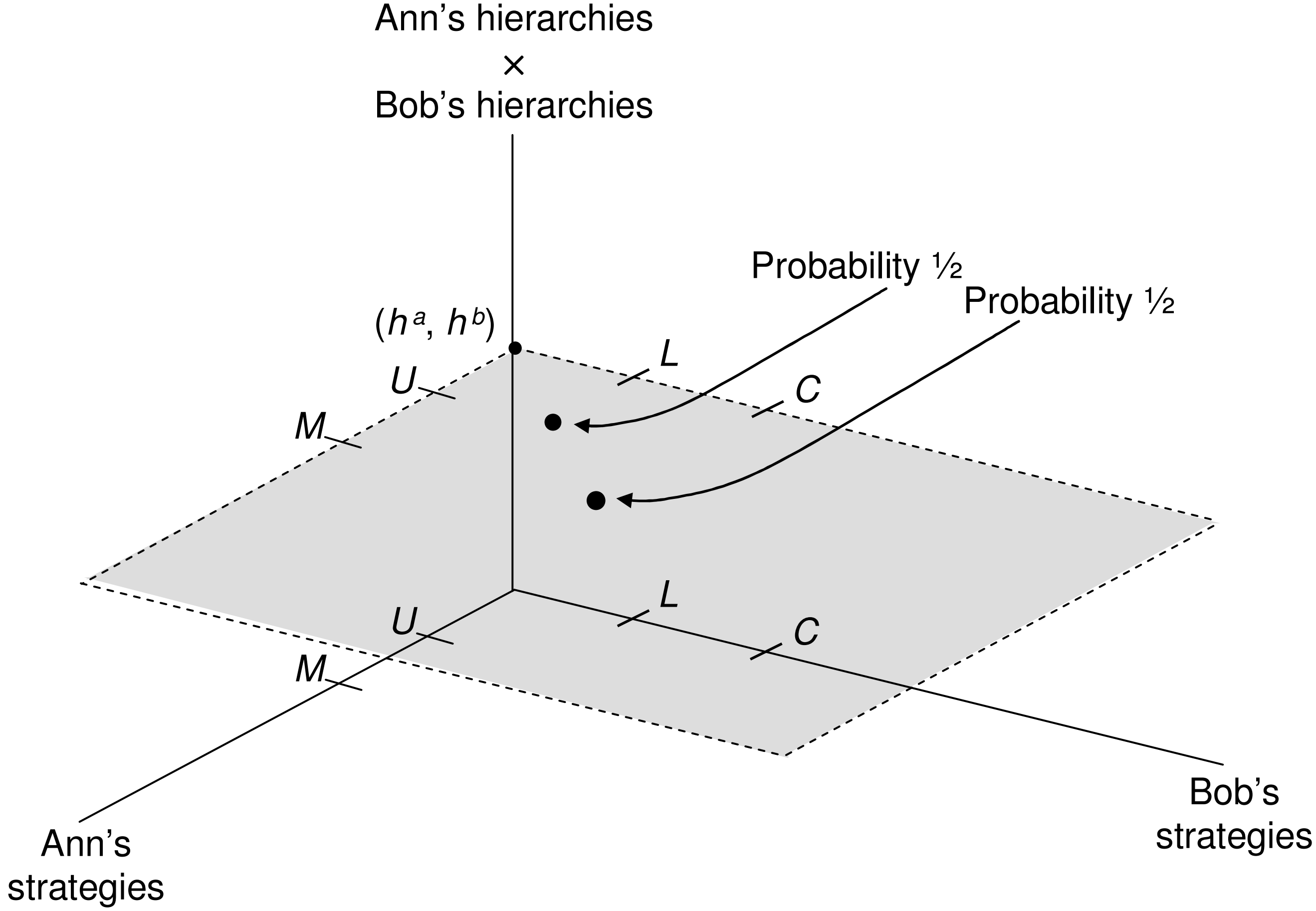
So, if (U, t^a) and (M, v^a) are rational and t^a and v^a believe Bob and Charlie are rational, then the hierarchies induced by t^a and v^a must agree up to level 2

And so on, inductively ...

Second Example continued

Consider a type t^c for Charlie satisfying RCBR (wrt Charlie)

By the previous slide (and a conjunction property), we get the picture:



But CI requires Charlie's conditional measure, conditional on any horizontal plane, to be a product measure—contradiction (even without SUFF)!

Solution #1

Do non-cooperative game theory by specifying hierarchies of beliefs and doing analysis relative to them (and to CI and SUFF)

Can we provide a characterization in terms only of the strategies that can be played?

Not obvious!

If not, then this route requires changing the very definition of a game to specify

strategy sets ("opportunity sets")

payoff functions ("tastes")

and

spaces of possible beliefs, beliefs about beliefs, … ("context")

This is **epistemic game theory**

Solution #2

Allow Charlie to think that Ann and Bob physically coordinate their choices—they jointly choose (U, L) or (M, C)

But we are analyzing the players as choosing separately

This is an awkward (conceptual) inconsistency

So, suppose all subsets of players can coordinate—this leads to **cooperative game theory** (von Neumann 1928, von Neumann and Morgenstern 1944)

(Formally, the characteristic function for a subset S of players is the maximin payoff to S in the associated zero-sum game between S and not-S)

This route appears to blur (erase?) the boundary between the non-cooperative and cooperative branches

Solution #3

Add extrinsic (payoff-irrelevant) **signals** to the game (Aumann 1974)

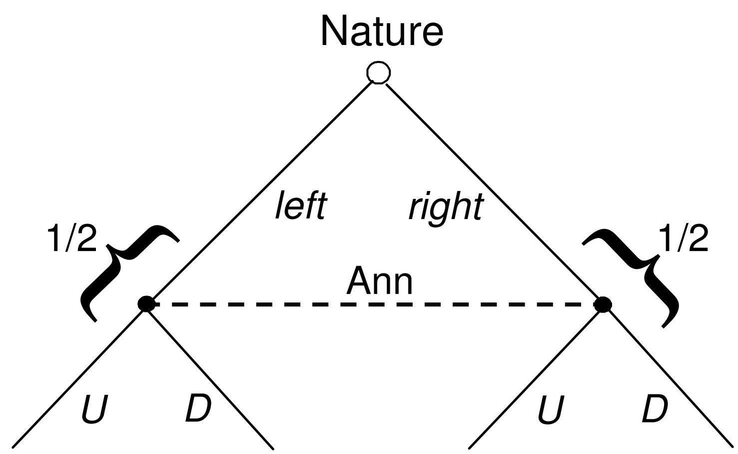
In the example, Ann and Bob observe a coin toss:

If Heads, they play *U* and *L*

If Tails, they play *M* and *C*

Can all correlations be explained via signals?

Third Example

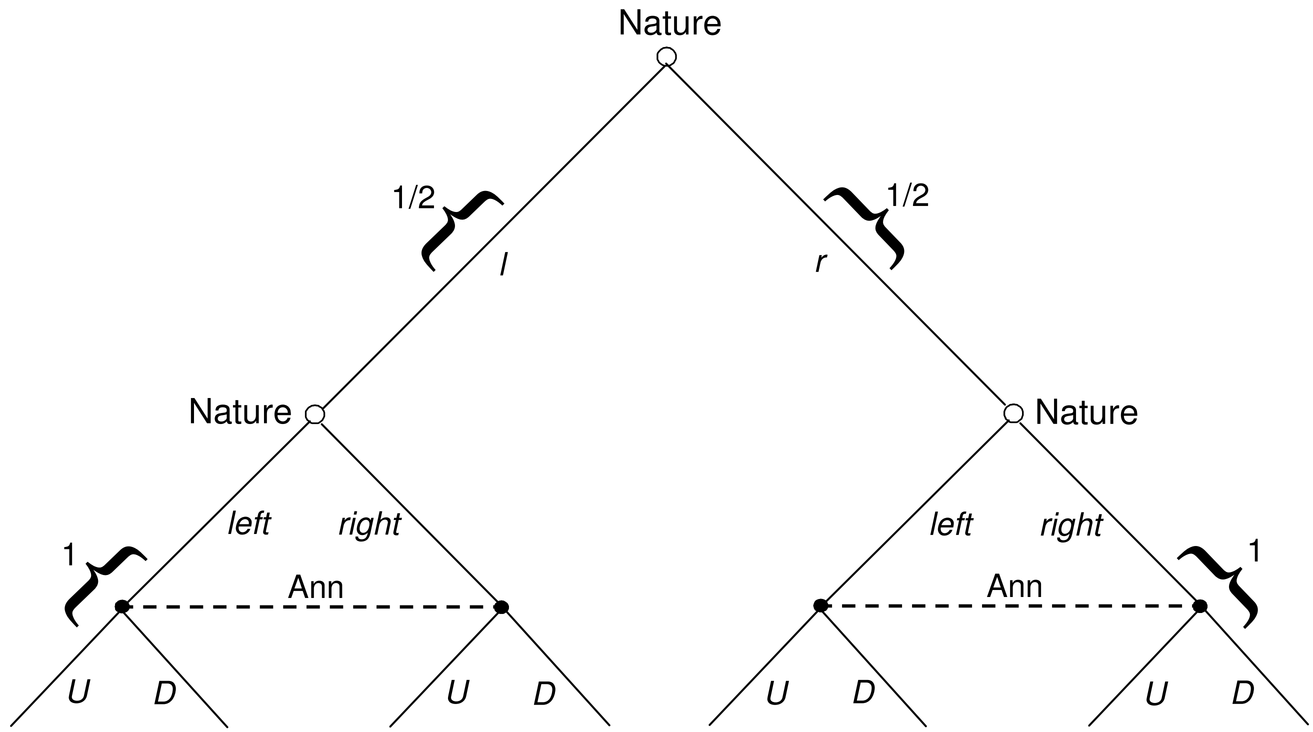


In the underlying game, Nature chooses between *left* and *right* (these moves could be payoff-relevant)

Charlie assigns probability ½ to (*left*, *U*) and probability ½ to (*right*, *D*)

Can we explain the correlation?

Third Example continued

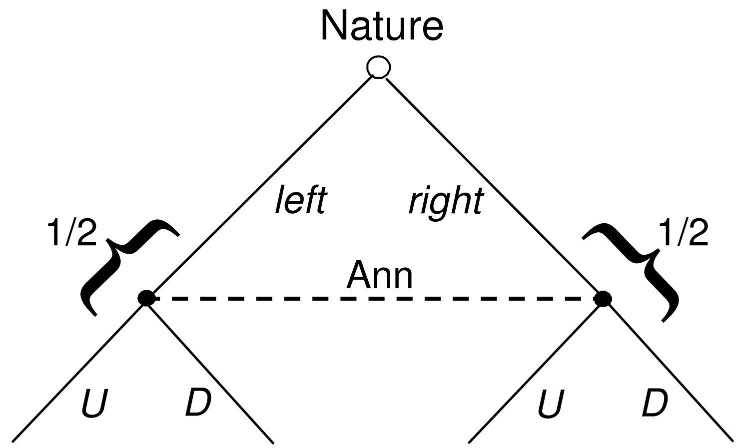


Yes, if we add an initial (payoff-irrelevant) choice by Nature between *l* and *r*, as shown

No, if we require the two choices by Nature—in the underlying game and in the augmented game—to be independent

(For example, suppose Nature's move in the underlying game corresponds to the outcome of a private coin toss by Bob)

Quantum Correlation



Depending on the outcome of his coin toss (*left* or *right*), Bob makes one or other of two measurements on a particle

Without seeing the coin toss or this measurement, Ann makes her own measurement on the particle—and, depending on the outcome of the measurement, chooses *U* or *D*

According to QM, the outcome of Ann's measurement can depend on what measurement Bob makes (Kochen-Specker 1967)

Arguably, Charlie's assessment is again consistent with non-cooperative play

Note: Quantum games were introduced by Eisert-Wilkens-Lewenstein (1999) and Meyer (1999)

Types of Correlation in Games

"Intrinsic"

Epistemic—via players' hierarchies of beliefs

Physical—via cooperative vs. non-cooperative behavior

"Extrinsic"

Classical—via classical signals external to the game

Quantum—via entanglement

The to-do list …