

Finite-Order Rationality*

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Very Preliminary

1 Introduction

Fix a (finite) strategic-form game and an associated epistemic type structure. Consider the states at which there is **rationality and common belief of rationality (RCBR)**. The projection of these states into the strategy sets constitutes a best-response **set (BRS)** of the game. But, suppose instead we consider, for some finite m , the states at which there is **rationality and m th-order belief of rationality (R m BR)**. Can we describe the projection of these states? In particular, if m is large enough, is the projection again equal to a BRS?

The question can be formulated in several different ways.

At one extreme, we could fix m before choosing the game (and type structure). In this case, by choosing a sufficiently large matrix, we can ensure that m rounds of eliminating strongly dominated strategies still leaves strategies that lie outside the **iteratively undominated (IU)** set. This strongly suggests—and it is true—that the projection of the R m BR set need not even lie in the IU set. Certainly, then, it cannot constitute a BRS, since every BRS lies in the IU set. This fact is surely well known.

At the other extreme, we could first fix both the game and the type structure. In this case, the projections of the R m BR sets, as a function of m , will eventually stop shrinking, since they are projections into a finite set (the product of the strategy sets). It is clear that the resulting constant projection must constitute a BRS. So, for sufficiently large m , the projection of the R m BR set is indeed a BRS. This, too, is straightforward.

The less obvious case is when we first fix the game, then choose m , and then are free to choose a type structure.

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There are some obvious conjectures about this case. To state them, fix a game $\langle S_a, S_b, \pi_a, \pi_b \rangle$. (Nothing hinges on two vs. more than two players.) Let $\langle T_a, T_b, \beta_a, \beta_b \rangle$ be an associated type structure, and, for any m , set

$$R_a^m = \{(s_a, t_a) : \begin{array}{l} \text{Ann is rational,} \\ \text{Ann is rational and believes Bob is rational,} \\ \dots \\ \text{Ann is rational and believes Bob is rational and ...} \end{array} \} \left. \vphantom{R_a^m} \right\} m \text{ times}$$

and define R_b^m similarly. Set $S = S_a \times S_b$ and $R^m = R_a^m \times R_b^m$.

Conjecture 1 *There is a finite M so that, for any type structure and each $m \geq M$, $\text{proj}_S R^m = \text{proj}_S R^M$.*

Conjecture 2 *There is a finite M so that, for any type structure and each $m \geq M$, $\text{proj}_S R^m$ is a BRS.*

Conjecture 3 *There is a finite M so that, for any type structure and each $m \geq M$, $\text{proj}_S R^m$ lies in the IU set.*

Notice the distinction between Conjectures 1 and 2: Conjecture 1 says that the projections of the Rm BR sets stop shrinking—which implies that they eventually constitute a BRS. Conjecture 2 says that, eventually, the projections constitute BRS’s—but, possibly, different BRS’s for different m ’s.

Conjecture 1 implies Conjecture 2 which implies Conjecture 3. Conjecture 3 is true, as we next show. In Section 2, we will show that Conjecture 2 is false, so that Conjecture 1 is also false.

Let S_a^m (resp. S_b^m) be the set of Ann’s (resp. Bob’s) strategies that survive m rounds of elimination of strongly dominated strategies. Conjecture 3 is true by virtue of the following:

Proposition 1 *For each m , $\text{proj}_S R^m \subseteq S^m$.*

Proof. By induction on m . For $m = 1$, fix $(s_a, t_a) \in R_a^1$. Then, s^a is optimal under $\text{marg}_{S_b} \beta_a(t_a)$, so that $s_a \in S_a^1$. Likewise, $\text{proj}_{S_b} R_b^1 \subseteq S_b^1$. Now assume the result holds for m . Fix $(s_a, t_a) \in R_a^{m+1}$. Then $\beta_a(t_a)(R_b^m) = 1$, from which $\text{marg}_{S_b} \beta_a(t_a)(\text{proj}_{S_b} R_b^m) = 1$. The result now follows from $\text{proj}_{S_b} R_b^m \subseteq S_b^m$ (the induction hypothesis) and the fact that s_a is optimal under $\text{marg}_{S_b} \beta_a(t_a)$ (indeed, with respect to all $r_a \in S_a$). ■

2 The Counter-Example

Consider the simple coordination game in Figure 1. The BRS’s are $\{(U, L)\}$, $\{(D, R)\}$, and $\{U, D\} \times \{L, R\}$ (the last is the IU set).

		Bob	
		L	R
Ann	U	1, 1	0, 0
	D	0, 0	1, 1

Figure 1

We will show that, given any m , we can build a type structure in which $\text{proj}_S R^m = \{U, D\} \times \{R\}$. That is, $\text{proj}_S R^m$ is not a BRS, so that Conjecture 2 is false. The type spaces are $T_a = \{U^1, \dots, U^m, D\}$ and $T_b = \{L^1, \dots, L^m, R\}$, and the maps β_a and β_b are given in Figure 2. (Ignore the arrows for now.) Thus, for each $n = 1, \dots, m$, $\beta_a(U^n)(L, L^n) = 1$, and $\beta_a(D)(R, R) = 1$. For each $n = 1, \dots, m - 1$, $\beta_b(L^n)(U, U^{n+1}) = 1$, and $\beta_b(L^m)(D, D) = \beta_b(R)(D, D) = 1$.

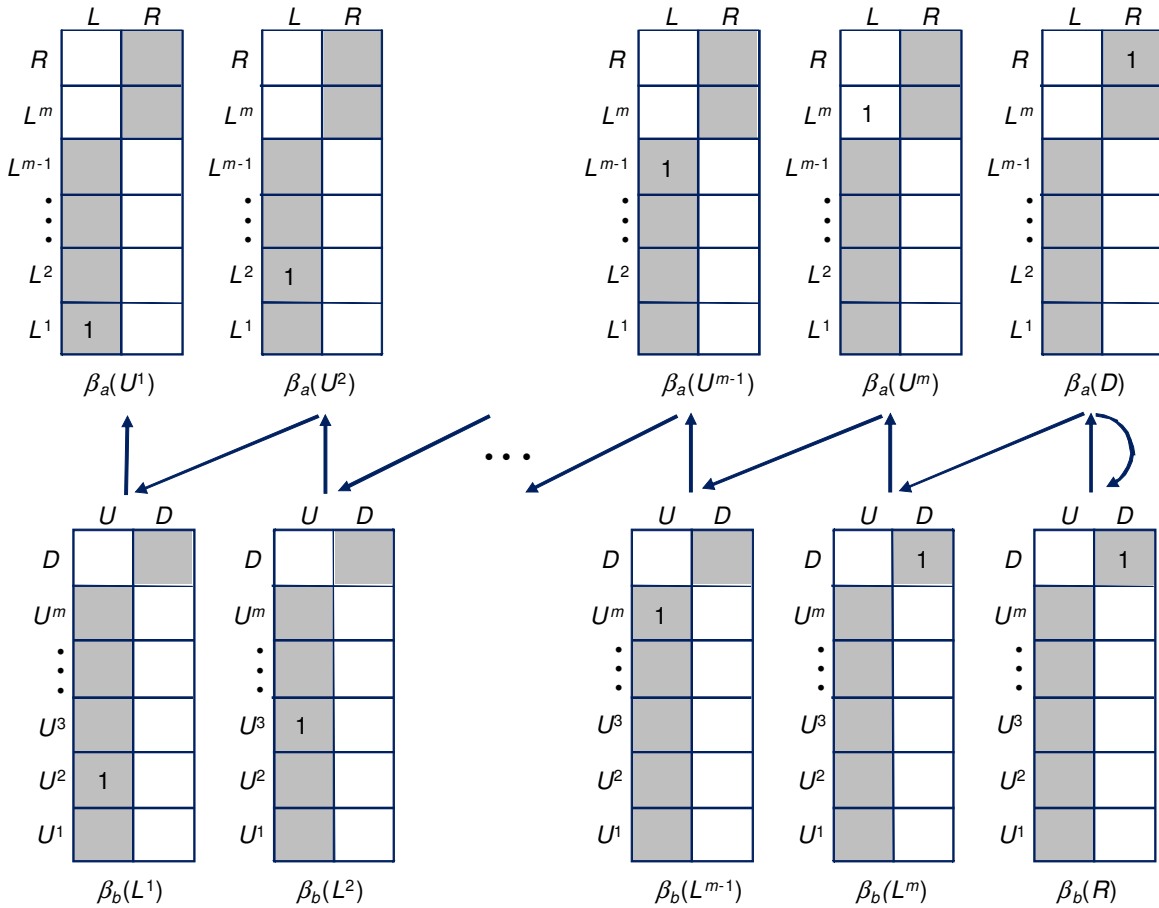


Figure 2

The rational strategy-type pairs for each player are shaded. Formally:

$$\begin{aligned} R_a^1 &= (\{U\} \times \{U^1, \dots, U^m\}) \cup \{(D, D)\}, \\ R_b^1 &= (\{L\} \times \{L^1, \dots, L^{m-1}\}) \cup (\{R\} \times \{L^m, R\}). \end{aligned}$$

So, all types except U^m believe [the other player is rational]. All types except U^m and L^{m-1} believe [the other player is rational and believes [the other player is rational]]. All types except U^m , L^{m-1} , and U^{m-1} believe [the other player is rational and believes [the other player is rational and believes [the other player is rational]]]. Etc. We can also follow the arrows to see how successive types are eliminated:

$$\begin{aligned} R^2 &\text{ rules out the type } U^m, \\ R^3 &\text{ rules out the types } U^m, L^{m-1}, \\ &\vdots \\ R^{2n} &\text{ rules out the types } U^m, L^{m-1}, \dots, U^{m-n+1}, \\ R^{2n+1} &\text{ rules out the types } U^m, L^{m-1}, \dots, L^{m-n}, \\ &\vdots \end{aligned}$$

Setting $n = m - 1$, we see that R^{2m-1} rules out U^m, L^{m-1}, \dots, L^1 . From this, $R_a^{2m-1} = \{(U, U^1), (D, D)\}$ and $R_b^{2m-1} = \{R\} \times \{L^m, R\}$, from which $\text{proj}_S R^{2m-1} = \{U, D\} \times \{R\}$. Since we can pick m arbitrarily, this establishes that Conjecture 2 is false.