Fixed Points in Epistemic Game Theory*

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Abstract

The epistemic conditions of "rationality and common belief of rationality" and "rationality and common assumption of rationality" are characterized by the solution concepts of a "best-response set" (Pearce [19, 1984]) and a "self-admissible set" (Brandenburger, Friedenberg, and Keisler [10, 2008]), respectively. We characterize each solution concept as the set of fixed points of a map on the lattice of rectangular subsets of the product of the strategy sets. Of note is that both maps we use are non-monotone.

1 Introduction

Topological fixed-point arguments were introduced into game theory by von Neumann [21, 1928] in proving his famous Minimax Theorem. Nash's existence proof for his equilibrium concept is also a topological fixed-point argument–using Brouwer's Theorem ([11, 1910]) in [16, 1950] and [18, 1951] and Kakutani's Theorem ([13, 1941]) in [17, 1950]. Subsequently, the Brouwer and Kakutani theorems became the standard tools in existence arguments in game theory.

More recently, an approach to game theory–called epistemic game theory (EGT)–has emerged which has ties to order-theoretic rather than topological fixed-point arguments. The purpose of this note is to explain this connection.

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An important stream of papers on order-theoretic fixed points in game theory is Apt [2, 2007a], [3, 2007b], [4, 2007c], and Apt and Zvesper [5, 2007]. These papers study iterated dominance concepts in general (infinite) games. The present note treats the best-response set and self-admissible set concepts in finite games. The definitions of these concepts and the relationships among them (for finite games) will be laid out later.

2 Epistemic Game Theory

We now give with a very brief sketch of how EGT proceeds. EGT starts from the position that the classical description of a game situation, as a game matrix or tree is inadequate, and should be extended to specify, in addition, what each player thinks about the game, what each player thinks each other player thinks about the game, and so on. By what a player "thinks about the game" we mean what s/he thinks about the strategies other players select, or what s/he thinks about the structure of the game (say, the payoff functions), or both. In this note, the focus is on uncertainty about the strategies played. The structure of the game is taken to be "transparent" among the players.

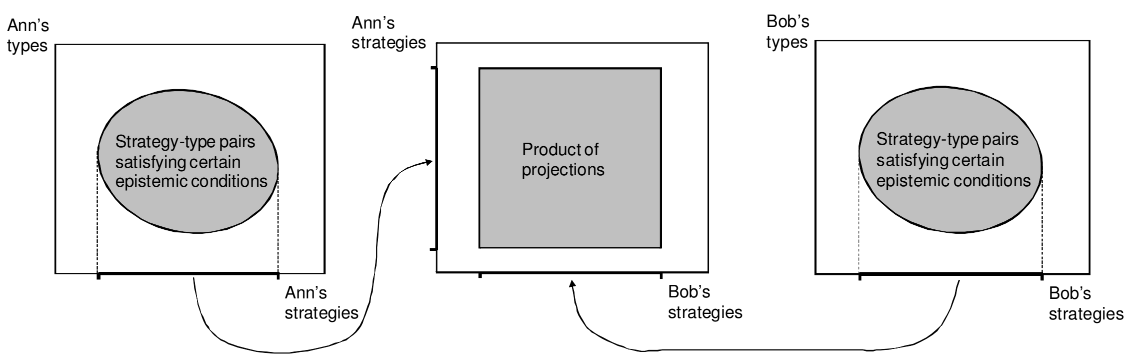

Figure 2.1

Formally, an epistemic description adds **types** for each player, where a type induces (via a natural inductive definition) what that player thinks, thinks other players think, ... (See Siniscalchi [20, 2008] for an exposition of this point and a survey of the relevant literature.) Figure 2.1 depicts the set-up in the two-player case.

The shaded set in the left-hand panel consists of those strategy-type pairs for Ann that satisfy whatever epistemic conditions we, the analysts, wish to study. For example, we might consider those strategy-type pairs for Ann that satisfy the condition that Ann is **rational**–i.e., that her strategy is optimal for her given what, according to her type, she thinks Bob will play. (Ann's remaining strategy-type pairs satisfy the condition that she is irrational. Below, we will be more precise about what "thinks" means.) A basic condition in EGT is that Ann is rational, thinks Bob is rational, and so on. Suppose the shaded set in the left-hand panel consists of the strategy-type pairs for Ann that satisfy this condition. Likewise, let the shaded set in the right-hand panel consist of the strategy-type pairs for Bob that are rational, think Ann is rational, and so on. The **characterization** question is whether we can identify the strategies that can be played under these conditions. That is, can we identify the projections of the shaded sets into the strategy sets–as depicted in the middle panel?

There are several such characterization results in EGT. They differ according to how exactly the terms "thinks" and "rationality" are formalized. The different formalizations, in turn, reflect different questions that we the analysts can ask about games.

We need some formal apparatus. Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$, where S^a and S^b are finite strategy sets for Ann and Bob, and $\pi^a : S^a \times S^b \to \mathbb{R}$ and $\pi^b : S^a \times S^b \to \mathbb{R}$ are their payoff functions.[1] Append Polish spaces T^a and T^b of types for Ann and Bob.

In the most basic formalization, a type t^a for Ann is associated with a probability measure on the Borel subsets of $S^b \times T^b$. In this case, rather than the general "thinks," we say that Ann **believes** $E^b \subseteq S^b \times T^b$ if E^b is Borel and the probability measure associated with type t^a assigns probability 1 to E^b. A strategy-type pair $(s^a, t^a) \in S^a \times T^a$ for Ann is **rational** if s^a maximizes her expected payoff, under the marginal on S^b of the probability measure associated with t^a. Of course, these and subsequent definitions have counterparts with Ann and Bob interchanged.

In another formalization (Brandenburger, Friedenberg, and Keisler [10, 2008]), a type t^a for Ann is associated with a lexicographic probability system on the Borel subsets of $S^b \times T^b$. A lexicographic probability system (Blume, Brandenburger, and Dekel [8, 1991]) is a finite sequence of probability measures satisfying certain conditions, and is to be thought of as a sequence of hypotheses held by Ann about Bob's strategy and type. In this case, we ask whether Ann **assumes** E^b. If T^b is finite, this is the condition that each point in E^b receives positive probability under an earlier probability measure in the sequence than does any point not in E^b. Alternatively put, the event E^b is "infinitely more likely" than the event not-E^b. This time, a strategy-type pair $(s^a, t^a) \in S^a \times T^a$ for Ann is (**lexicographically**) **rational** if s^a lexicographically maximizes her sequence of expected payoffs, calculated under the marginals on S^b of the sequence of probability measures associated with t^a. See [10, 2008] for a precise treatment and the motivation underlying these definitions.

For the purposes of this note, we will need one key property of both "believes" and "assumes." Returning to the general term "thinks," in order to subsume both cases, let

$$C^a(E^b) = \{t^a \in T^a : t^a \text{ thinks } E^b \text{ is true}\},$$

and define $C^b(E^a)$ similarly. Formally, C^a is a mapping from the family of Borel subsets of $S^b \times T^b$ into itself, and similarly for C^b.

Axiom 2.1 (Conjunction) *Fix a type $t^a \in T^a$ and Borel sets $E_1^b, E_2^b, \ldots$ in $S^b \times T^b$. Suppose, for each m, that $t^a \in C^a(E_m^b)$. Then $t^a \in C^a(\bigcap_m E_m^b)$.*

[1] We restrict attention to two-player games for notational simplicity, but our analysis readily extends to games with three or more players.

In words, if Ann thinks that each event E^b_m is true, then she thinks the joint event $\bigcap_m E^b_m$ is true. It is immediate from the rules of probability that "believes" satisfies this conjunction property. It is also satisfied for "assumption" ([10, 2008, Property 6.3]).

Given Borel sets $E^a \subseteq S^a \times T^a$ and $E^b \subseteq S^b \times T^b$, define $E^a_1 = E^a$, $E^b_1 = E^b$, and for $m \geq 1$,

$$E^a_{m+1} = E^a_m \cap [S^a \times C^a(E^b_m)],$$

and likewise with a and b interchanged. (For this to be well-defined, one checks that E^a_m, E^b_m are Borel.)

Definition 2.1 *The event that $E^a \times E^b$* ***is true and there is common thought of*** *$E^a \times E^b$ is*

$$\bigcap_{m=1}^{\infty} E^a_m \times \bigcap_{m=1}^{\infty} E^b_m.$$

In the situations we are interested in, $E^a \times E^b$ is the event that Ann and Bob are rational. Thus, the event $\bigcap_{m=1}^{\infty} E^a_m \times \bigcap_{m=1}^{\infty} E^b_m$ is either the event that there is **rationality and common belief of rationality** (**RCBR**) or the event that there is **rationality and common assumption of rationality** (**RCAR**).

3 Epistemic Fixed Points

The characterization question is whether we can identify the strategies playable under RCBR or RCAR. In this note, we spproach this question using fixed points. Given two Polish spaces P^a, P^b, let $\mathcal{B}(P^a, P^b)$ be the set of all rectangles $E^a \times E^b$ where E^a is a Borel subset of P^a and E^b is a Borel subset of P^b. Proposition 3.1 below will show that each of the events RCBR and RCAR is a fixed point of a mapping Γ from $\mathcal{B}(S^a \times T^a, S^b \times T^b)$ into itself. This suggests that we should be able to describe the strategies playable under RCBR or RCAR via fixed points of mappings from $\mathcal{B}(S^a, S^b)$ into itself. Sections 4-7 will show that this is indeed the case.

In sum, this section is about fixed points in $\mathcal{B}(S^a \times T^a, S^b \times T^b)$, while Sections 4-7 will be about fixed points in the smaller space $\mathcal{B}(S^a, S^b)$ where the type spaces T^a, T^b play no part.

Given $E^a \times E^b \in \mathcal{B}(S^a \times T^a, S^b \times T^b)$, define

$$\Gamma(E^a \times E^b) = (E^a \times E^b) \cap ([S^a \times C^a(E^b)] \times [S^b \times C^b(E^a)]).$$

Note that the mapping Γ depends on C^a and C^b, and maps $\mathcal{B}(S^a \times T^a, S^b \times T^b)$ into itself. In words, Γ maps an event $E^a \times E^b$ to the event that $E^a \times E^b$ is true, and Ann and Bob think their respective components of $E^a \times E^b$ are true. The next lemma is immediate:

Lemma 3.1 *The event* $E^a \times E^b$ *is a fixed point of* Γ*, i.e.* $\Gamma(E^a \times E^b) = E^a \times E^b$*, if and only if*

$$\begin{aligned} E^a &\subseteq S^a \times C^a(E^b), \\ E^b &\subseteq S^b \times C^b(E^a). \end{aligned}$$

When C^a stands for belief, Lemma 3.1 says the fixed points of Γ are the "belief-closed subsets" as defined by Mertens-Zamir [15, 1985, Definition 2.15]. When C^a stands for assumption, we can call the fixed points the "assumption-closed subsets." Lemma 3.1 will be used in the proofs of the following two propositions.

Proposition 3.1 *Fix* $E^a \times E^b \in \mathcal{B}(S^a \times T^a, S^b \times T^b)$*. The event*

$$\bigcap_{m=1}^{\infty} E^a_m \times \bigcap_{m=1}^{\infty} E^b_m$$

is a fixed point of Γ*.*

Proof. Using the definitions,

$$\bigcap_{m=1}^{\infty} E^a_m = E^a_1 \cap \bigcap_{m=1}^{\infty} [S^a \times C^a(E^b_m)] \subseteq \bigcap_{m=1}^{\infty} [S^a \times C^a(E^b_m)] = \\ S^a \times \bigcap_{m=1}^{\infty} C^a(E^b_m) \subseteq S^a \times C^a(\bigcap_{m=1}^{\infty} E^b_m),$$

where the last inclusion relies on conjunction (Axiom 2.1). ■

When C^a stands for belief, Proposition 3.1 says that the event RCBR is a fixed point of the corresponding map Γ. When C^a stands for assumption, Proposition 3.1 says that the event RCAR is a fixed point of Γ. We also have:

Proposition 3.2 *Suppose* $E^a \times E^b \in \mathcal{B}(S^a \times T^a, S^b \times T^b)$ *is a fixed point of* Γ*. Then* $E^a_m = E^a$ *and* $E^b_m = E^b$ *for all* m*.*

Proof. This is immediate for $m = 1$, so suppose it is true for m. We have

$$E^a_{m+1} = E^a_m \cap [S^a \times C^a(E^b_m)] = E^a \cap [S^a \times C^a(E^b)],$$

using the induction hypothesis. But since $E^a \times E^b$ is a fixed point,

$$E^a \cap [S^a \times C^a(E^b)] = E^a,$$

and so $E^a_{m+1} = E^a$, as required. ■

An important reference on fixed points on epistemic structures is Barwise [6, 1988].

4 Definitions

We now turn to the fixed-point characterizations of the strategies playable under RCBR or RCAR. We will need some additional definitions and some preliminary lemmas.

Given a finite set Ω, let $\mathcal{M}(\Omega)$ denote the set of all probability measures on Ω. Write $\mathrm{Supp}\,\sigma$ for the support of $\sigma \in \mathcal{M}(X)$. The definitions to come all have counterparts with a and b reversed. We extend π^a to $\mathcal{M}(S^a)\times\mathcal{M}(S^b)$ in the usual way, i.e. $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a,s^b)\in S^a\times S^b} \sigma^a(s^a)\sigma^b(s^b)\pi^a(s^a, s^b)$.

Definition 4.1 *Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$, and $\sigma^b \in \mathcal{M}(S^b)$. A strategy $s^a \in X$ is σ^b-**justifiable with respect to** $X \times Y$ if $\sigma^b(Y) = 1$ and $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$. Say s^a is **justifiable with respect to** $X \times Y$ if s^a is σ^b-justifiable with respect to $X \times Y$ for some σ^b.*

Definition 4.2 *Fix $Q^a \times Q^b \subseteq S^a \times S^b$. The set $Q^a \times Q^b$ is a **best-response set (BRS)** if for each $s^a \in Q^a$ there is a $\sigma^b \in \mathcal{M}(S^b)$ such that:*

(i) s^a is σ^b-justifiable with respect to $S^a \times Q^b$;

(ii) if r^a is also σ^b-justifiable with respect to $S^a \times Q^b$, then $r^a \in Q^a$;

and likewise for each $s^b \in Q^b$.

The original definition of a BRS is due to Pearce [19, 1984]. This definition is from Battigalli and Friedenberg [7, 2009]. It differs from Pearce's definition in two ways. First, players choose only pure (not mixed) strategies. Second, condition (ii) is new–we will see that it is important in the epistemic characterization.[2]

Definition 4.3 *Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is **strongly dominated with respect to** $X \times Y$ if there is a $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for every $s^b \in Y$. Otherwise, say s^a is **undominated with respect to** $X \times Y$.*

We have the usual equivalence:

Lemma 4.1 *Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is justifiable with respect to $X \times Y$ if and only if it is undominated with respect to $X \times Y$.*

Definition 4.4 *Set $S^i_0 = S^i$ for $i = a, b$, and define inductively*

$$S^i_{m+1} = \{s^i \in S^i_m : s^i \text{ is undominated with respect to } S^a_m \times S^b_m\}.$$

*A strategy $s^i \in S^i_m$ is called m-**undominated**. A strategy $s^i \in \bigcap_{m=0}^{\infty} S^i_m$ is called **iteratively undominated (IU)**.*

[2] David Pearce (private communication) told one of us that he was aware of this condition, but to keep things simple did not include it in his definition in [19, 1984].

By finiteness, there is a (first) number M such that $\bigcap_{m=0}^{\infty} S_m^i = S_M^i \neq \emptyset$ for $i = a, b$. It is well known that every BRS is contained in the IU set, which is itself a BRS.

Definition 4.5 *Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is* ***weakly dominated with respect to*** *$X \times Y$ if there is a $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y$. Otherwise, say s^a is* ***admissible with respect to*** *$X \times Y$.*

Definition 4.6 *Fix $Y \subseteq S^b$ with $Y \neq \emptyset$. Say r^a* ***supports*** *s^a* ***with respect to*** *Y if there is a $\sigma^a \in \mathcal{M}(S^a)$ with $r^a \in \operatorname{Supp} \sigma^a$ and $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ for all $s^b \in Y^b$. If r^a supports s^a with respect to S^b, say simply r^a* ***supports*** *s^a.*

Definition 4.7 (Brandenburger, Friedenberg, and Keisler [10, 2008]) *Fix $Q^a \times Q^b \subseteq S^a \times S^b$. The set $Q^a \times Q^b$ is a* ***self-admissible set (SAS)*** *if:*

(i) each $s^a \in Q^a$ is admissible with respect to $S^a \times S^b$;

(ii) each $s^a \in Q^a$ is admissible with respect to $S^a \times Q^b$;

(iii) if $r^a \in S^a$ supports some $s^a \in Q^a$, then $r^a \in Q^a$;

and likewise for each $s^b \in Q^b$.

We have the usual equivalence:

Lemma 4.2 *Fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. A strategy $s^a \in X$ is admissible with respect to $X \times Y$ if and only if there is a $\sigma^b \in \mathcal{M}(S^b)$, with $\operatorname{Supp} \sigma^b = Y$, such that s^a is σ^b-justifiable with respect to $X \times Y$.*

Using this and Lemma D.4 (on the geometry of polytopes) in Brandenburger, Friedenberg, and Keisler [10, 2008], we can rewrite Definition 4.7 in a way that brings out the comparison with BRS's.

Lemma 4.3 *A set $Q^a \times Q^b$ is an SAS if and only if:*

(i) for each $s^a \in Q^a$, there is a $\sigma^b \in \mathcal{M}(S^b)$, with $\operatorname{Supp} \sigma^b = S^b$, such that:

- *s^a is σ^b-justifiable with respect to $S^a \times S^b$,*
- *if $r^a \in S^a$ is also σ^b-justifiable with respect to $S^a \times S^b$, then $r^a \in Q^a$;*

(ii) for each $s^a \in Q^a$, there is a $\rho^b \in \mathcal{M}(S^b)$, with $\operatorname{Supp} \rho^b = Q^b$, such that s^a is ρ^b-justifiable with respect to $S^a \times Q^b$;

and likewise for each $s^b \in Q^b$.

Definition 4.8 *Set $\overline{S}_0^i = S^i$ for $i = a, b$, and define inductively*

$$\overline{S}_{m+1}^i = \{s^i \in \overline{S}_m^i : s^i \text{ is admissible with respect to } \overline{S}_m^a \times \overline{S}_m^b\}.$$

*A strategy $s^i \in \overline{S}_m^i$ is called m-**admissible**. A strategy $s^i \in \bigcap_{m=0}^{\infty} \overline{S}_m^i$ is called **iteratively admissible (IA)**.*

By finiteness, there is a (first) number N such that $\bigcap_{m=0}^{\infty} \overline{S}_m^i = \overline{S}_N^i \neq \emptyset$ for $i = a, b$. The IA set is an SAS (Brandenburger and Friedenberg [9, forthcoming, Proposition 5.1]). But, unlike the case with IU and BRS's, it need not be the case that every SAS is contained in the IA set. See Example 7.1 to come.

5 Fixed-Point Characterizations

BRS's (Definition 4.2) characterize the epistemic condition of RCBR, as follows. Fix a game $\langle S^a, S^b, \pi^a, \pi^b \rangle$ and an associated type structure, where each type is mapped to a (single) probability measure. Define "believes" and "rationality" as before. Then, the projection into $S^a \times S^b$ of the RCBR event (which lies in $S^a \times T^a \times S^b \times T^b$) constitutes a BRS of the game. Conversely, every BRS of $\langle S^a, S^b, \pi^a, \pi^b \rangle$ arises in this way, for a suitable choice of type structure. This follows from Theorem 5.1 in Battigalli and Friedenberg [7, 2009].

Likewise, fix a game $\langle S^a, S^b, \pi^a, \pi^b \rangle$ and an associated type structure, where each type is now mapped to a lexicographic probability system. Define "assumes" and "(lexicographic) rationality" as before. Then, the projection into $S^a \times S^b$ of the RCAR event constitutes an SAS of the game. Conversely, every SAS of $\langle S^a, S^b, \pi^a, \pi^b \rangle$ arises in this way, for a suitable choice of type structure. This is Theorem 8.1 in Brandenburger, Friedenberg, and Keisler [10, 2008].

So, to deliver on our promised fixed-point characterizations of the strategies that can be played, it remains to provide fixed-point characterizations of BRS's and SAS's.

6 Characterization of BRS's

We start with some lemmas. Throughout, fix $X \times Y \subseteq S^a \times S^b$ with $Y \neq \emptyset$. The proofs of the first two lemmas are standard.

Lemma 6.1 *A strategy $s^a \in X$ is σ^b-justifiable with respect to $X \times Y$ if and only if s^a is admissible with respect to $X \times \operatorname{Supp} \sigma^b$.*

Lemma 6.2 *Fix $s^a \in X$ and suppose there is a $\sigma^b \in \mathcal{M}(S^b)$ such that $\pi^a(s^a, \sigma^b) \geq \pi^a(q^a, \sigma^b)$ for all $q^a \in X$. Then if r^a supports s^a, $\pi^a(r^a, \sigma^b) \geq \pi^a(q^a, \sigma^b)$ for all $q^a \in X$.*

Lemma 6.3 *Suppose $s^a \in S^a$ is undominated (resp. admissible) with respect to $X \times Y$. If r^a supports s^a, then r^a is undominated (resp. admissible) with respect to $X \times Y$.*

Proof. By Lemma 4.1 (resp. Lemma 4.2) there is a $\sigma^b \in \mathcal{M}\left(S^b\right)$, with $\sigma^b(Y) = 1$ (resp. $\operatorname{Supp}\sigma^b = Y^b$), such that $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(q^a, \sigma^b\right)$ for all $q^a \in X$. By Lemma 6.2, r^a is then undominated (resp. admissible) with respect to $X \times Y$. ■

Given $\sigma^b \in \mathcal{M}\left(S^b\right)$, write $\mathcal{J}(\sigma^b)$ for the set of strategies $s^a \in S^a$ that are σ^b-justifiable.

Definition 6.1 *Say that σ^b **minimally justifies** s^a **with respect to** $X \times Y$ if s^a is σ^b-justifiable with respect to $X \times Y$ and, for each $\rho^b \in \mathcal{M}\left(S^b\right)$ such that s^a is ρ^b-justifiable with respect to $X \times Y$, $\mathcal{J}\left(\sigma^b\right) \subseteq \mathcal{J}\left(\rho^b\right)$.*

Lemma 6.4 *Fix $Y \neq \emptyset$. If s^a is justifiable with respect to $S^a \times Y$, there is a σ^b that minimally justifies s^a with respect to $S^a \times Y$.*

Proof. Suppose s^a is justifiable with respect to $S^a \times Y$. Then, by Lemma 6.1, there is $\emptyset \neq Z_k \subseteq Y$ such that s^a is admissible with respect to $S^a \times Z_k$. Let Z be the union of all such Z_k. Then, s^a is admissible with respect to $S^a \times Z$. To see this, suppose not, i.e., that there is $\sigma^a \in \mathcal{M}(S^a)$ with $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Z$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Z$. Then, we can find some $Z_k \subseteq Z$ such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Z_k$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Z_k$. This contradicts the fact that s^a is admissible with respect to each $S^a \times Z_k$.

We have established that s^a is admissible with respect to $S^a \times Z$. So, by Lemma D.4 in Brandenburger-Friedenberg-Keisler [10, 2008], there is a $\sigma^b \in \mathcal{M}\left(S^b\right)$, with $\operatorname{Supp}\sigma^b = Z$, such that $\mathcal{J}\left(\sigma^b\right)$ is the set of strategies that support s^a with respect to $S^a \times Z$. We will show that σ^b minimally justifies s^a with respect to $S^a \times Y$.

Fix $\rho^b \in \mathcal{M}\left(S^b\right)$ that justifies s^a with respect to $S^a \times Y$. We show that $\mathcal{J}\left(\sigma^b\right) \subseteq \mathcal{J}\left(\rho^b\right)$. Fix $r^a \in \mathcal{J}\left(\sigma^b\right)$. Then r^a supports s^a with respect to $S^a \times Z$. Note that $\operatorname{Supp}\rho^b \subseteq Z$, so that r^a also supports s^a with respect to $S^a \times \operatorname{Supp}\rho^b$. It follows from Lemma 6.2 that $r^a \in \mathcal{J}\left(\rho^b\right)$ as required.

■

Now consider the complete lattice $\Lambda = \mathcal{B}(S^a, S^b)$. (The join of two subsets is the component-by-component union. The meet is the intersection.) We will define a map $\Phi : \Lambda \to \Lambda$ so that the fixed points of Φ are the BRS's. Specifically, for $Q^a \times Q^b \in \mathcal{B}(S^a, S^b)$, put $(s^a, s^b) \in \Phi(Q^a \times Q^b)$ if either: (a) $s^a \in Q^a$ and s^a is justifiable with respect to $S^a \times Q^b$, or (b) $s^a \in \mathcal{J}\left(\sigma^b\right)$ for some σ^b that minimally justifies some $r^a \in Q^a$ with respect to $S^a \times Q^b$. The analogous conditions must hold for s^b.

Proposition 6.1 *If $Q^a \times Q^b$ is a BRS, then it is a fixed point of Φ, i.e., $\Phi(Q^a \times Q^b) = Q^a \times Q^b$. Conversely, if $Q^a \times Q^b$ is a fixed point of Φ, then it is a BRS.*

Proof. Fix a BRS $Q^a \times Q^b$. We will show that it is a fixed point of Φ. First, fix $\left(s^a, s^b\right) \in \Phi(Q^a \times Q^b)$. We will show that $\left(s^a, s^b\right) \in Q^a \times Q^b$. Indeed, suppose that $s^a \notin Q^a$. Then there is an $r^a \in Q^a$ such that r^a is justifiable with respect to $S^a \times Q^b$ and, for any σ^b such that r^a is σ^b-justifable with respect to $S^a \times Q^b$, $s^a \in \mathcal{J}\left(\sigma^b\right)$. It follows from condition (ii) of the definition of a BRS that $s^a \in Q^a$, a contradiction. Likewise, we reach a contradiction if we suppose that $s^b \notin Q^b$, so we conclude that $\Phi\left(Q^a \times Q^b\right) \subseteq Q^a \times Q^b$. Next, fix $\left(s^a, s^b\right) \in Q^a \times Q^b$. By condition (i) of the definition of a BRS and condition (a) of the definition of Φ, we get $\left(s^a, s^b\right) \in \Phi\left(Q^a \times Q^b\right)$, establishing that $Q^a \times Q^b \subseteq \Phi\left(Q^a \times Q^b\right)$.

For the converse, suppose $Q^a \times Q^b = \Phi\left(Q^a \times Q^b\right)$. Fix $s^a \in Q^a$ and note that, by condition (a) of the definition of Φ, there is a $\sigma^b \in \mathcal{M}\left(S^b\right)$ that justifies s^a with respect to $S^a \times Q^b$. By Lemma 6.4, we can choose σ^b to minimally justify s^a with respect to $S^a \times Q^b$. Then, by condition (b) of the definition of Φ, we get that s^a satisfies conditions (i)-(ii) of a BRS (using the measure σ^b). We can make the same argument for each each $s^b \in Q^b$. This establishes that $Q^a \times Q^b$ is a BRS. ■

Example 6.1 *The map Φ is not monotone (increasing). Consider the game in Figure 6.1. We have $\Phi(\{(U, L)\}) = \{U, D\} \times \{L, R\}$ but $\Phi(\{U\} \times \{L, R\}) = \{U\} \times \{L, R\}$.*

Ann \ Bob	*L*	*R*
U	1, 1	1, 1
D	1, 1	0, 0

Figure 6.1

7 Characterization of SAS's

We now define a map $\Psi : \Lambda \to \Lambda$ so that the fixed points of Ψ are the SAS's. Specifically, put $(s^a, s^b) \in \Psi(Q^a \times Q^b)$ if either: (a) $s^a \in Q^a$ and satisfies conditions (i)-(ii) of the definition of an SAS; or (b) s^a supports an $r^a \in Q^a$ that satisfies these conditions. The analogous conditions must hold for s^b.

Here is the analog to Proposition 6.1:

Proposition 7.1 *If $Q^a \times Q^b$ is an SAS, then it is a fixed point of Ψ. Conversely, if $Q^a \times Q^b$ is a fixed point of Ψ, then it is an SAS.*

Proof. Fix an SAS $Q^a \times Q^b$. If $s^a \in Q^a$, then s^a satisfies condition (a) for Ψ. Likewise for s^b. Thus $Q^a \times Q^b \subseteq \Psi\left(Q^a \times Q^b\right)$. Next, fix $\left(s^a, s^b\right) \in \Psi\left(Q^a \times Q^b\right)$. We need to show that $s^a \in Q^a$. If $s^a \notin Q^a$ then s^a supports r^a for some $r^a \in Q^a$. But then condition (iii) of an SAS implies $s^a \in Q^a$, a contradiction.

For the converse, fix $\left(s^a, s^b\right) \in Q^a \times Q^b = \Psi\left(Q^a \times Q^b\right)$. If s^a satisfies condition (a) for Ψ, then it satisfies conditions (i) and (ii) of an SAS. Next suppose s^a fails condition (a) for Ψ, i.e., is inadmissible with respect to $S^a \times S^b$, or $S^a \times Q^b$, or both. But then s^a must satisfy condition (b) for Ψ, i.e. s^a supports r^a for some $r^a \in Q^a$ satisfying conditions (i) and (ii) of an SAS. By Lemma 6.3,

s^a is then admissible with respect to both $S^a \times S^b$ and $S^a \times Q^b$, a contradiction. Finally, suppose q^a supports s^a. We just saw that s^a satisfies condition (a) for Ψ, so q^a satisfies condition (b) for Ψ. Thus $q^a \in \mathrm{proj}_{S^a} \Psi \left(Q^a \times Q^b\right) = Q^a$. This establishes condition (iii) of an SAS. ■

Example 7.1 *Like Φ, the map Ψ is non-monotone. Consider the game in Figure 7.1. We have $\Psi(\{U\} \times \{L, R\}) = \{U\} \times \{L, R\}$ but $\Psi(\{U, D\} \times \{L, R\}) = \{(U, R)\}$. Note also that the fixed points of Ψ are $\{U\} \times \{L, R\}$, $\{(U, R)\}$, and $\{(M, L)\}$. The SAS $\{(M, L)\}$ is the IA set. We see that, different from BRS vs. IU, the SAS's need not be contained in the IA set.*

		Bob	
		L	*R*
	U	2, 2	2, 2
Ann	*M*	3, 1	0, 0
	D	0, 0	1, 3

Figure 7.1

8 Comment

A potentially valuable comparison is with the occurrence of non-monotone maps in theoretical computer science. (Abramsky and Jung [1, 1994] and Davey and Priestley [12, 2002] are standard references on order-theoretic methods in this area.) A notable instance of the non-monotone case is Martin's [14, 2000] theory of measurements on domains. We leave it to future work to investigate whether what is known in computer science about the non-monotone case can be usefully applied in game theory.

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