Fixed Points for Strong and Weak Dominance*

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In this note, we provide fixed-point characterizations for two solution concepts in finite games: **best-response sets** (**BRS's**) and **self-admissible sets** (**SAS's**). The BRS concept is due to Pearce [7, 1984]. The SAS concept is a weak-dominance analog to a BRS, and is due to Brandenburger-Friedenberg-Keisler [4, 2006].

BRS's are important because they characterize the epistemic condition of **rationality and common belief of rationality** (**RCBR**) in a game. Similarly, SAS's characterize the condition of **rationality and common assumption of rationality** (**RCAR**). (See [4, 2006] for the meanings of "rationality" in a weak-dominance setting and of "assumption.")

We imagine that the material here on BRS's is well known to researchers in the area. In particular, Apt [1, 2006] is a lattice-theoretic treatment of strong dominance in general (infinite) games. The focus of most work is on a particular BRS–the **iteratively undominated** (**IU**) set, i.e., the set of strategies that survive iterated deletion of strongly dominated strategies. We cover all BRS's in this note, to make clear the comparison with SAS's.

Here is a summary on BRS's: BRS's are the fixed points of a certain map defined on the complete lattice of rectangular subsets of the product of the players' strategy sets. There is a greatest fixed point, which is the IU set. The map Φ is monotone increasing: If $x \leq y$, then $\Phi(x) \leq \Phi(y)$. So, the existence of a greatest fixed point also follows from the Knaster-Tarski Theorem [5, 1928], [8, 1955]. (This map already appears in Apt [1, 2006].)

SAS's, too, are the fixed points of a map Ψ. Unlike Φ, the map Ψ need not be monotone increasing and so need not have a greatest fixed point. But Ψ does satisfy: $\Psi(\top) \geq \Psi(\Psi(\top)) \geq \ldots$, where $\top$ is the top element of the lattice. So, induction gives a fixed point. The fixed point obtained this way is the **iteratively admissible** (**IA**) set, i.e., the set of strategies that survive

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iterated deletion of weakly dominated strategies. But other fixed points (SAS's) can even be disjoint from the IA set.

Given that BRS's and SAS's have fixed-point characterizations, we would expect that the corresponding epistemic conditions of RCBR and RCAR also have fixed-point characterizations. We show this is so.

Section 1 defines BRS's, the IU set, SAS's, and the IA set. Sections 2-3 gives the fixed-point characterizations of BRS's and SAS's. Section 4 gives the epistemic fixed-point results.

1 Preliminaries

Fix a two-player finite strategic-form game $\langle S^a, S^b, \pi^a, \pi^b \rangle$, where S^a, S^b are the (finite) strategy sets and π^a, π^b are the payoff functions for Ann and Bob, respectively.[1] Given a finite set Ω, let $\mathcal{M}(\Omega)$ denote the set of all probability measures on Ω. The definitions to come all have counterparts with a and b reversed. We extend π^a to $\mathcal{M}(S^a) \times \mathcal{M}(S^b)$ in the usual way, i.e. $\pi^a(\sigma^a, \sigma^b) = \sum_{(s^a,s^b) \in S^a \times S^b} \sigma^a(s^a)\sigma^b(s^b)\pi^a(s^a, s^b)$.

Definition 1.1 *Fix $X \times Y \subseteq S^a \times S^b$. A strategy $s^a \in X$ is **strongly dominated with respect to** $X \times Y$ if: (i) $Y \neq \emptyset$; and (ii) there exists $\sigma^a \in \mathcal{M}(S^a)$, with $\sigma^a(X) = 1$, such that $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for every $s^b \in Y$. Otherwise, say s^a is **undominated with respect to** $X \times Y$. If s^a is undominated with respect to $S^a \times S^b$, simply say that s^a is **undominated**.*

We have the usual equivalence:

Lemma 1.1 *Fix $X \times Y \subseteq S^a \times S^b$, where $Y \neq \emptyset$. A strategy $s^a \in X$ is undominated with respect to $X \times Y$ if and only if there exists $\sigma^b \in \mathcal{M}(S^b)$, with $\sigma^b(Y) = 1$, such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in X$.*

We now define BRS's and the IU set.

Definition 1.2 *Fix $Q^a \times Q^b \subseteq S^a \times S^b$. The set $Q^a \times Q^b$ is a **best-response set (BRS)** if:*

(i) each $s^a \in Q^a$ is undominated with respect to $S^a \times Q^b$;

and likewise for each $s^b \in Q^b$.

This definition is as in Pearce [7, 1984], except that we restrict attention to pure strategies and (in the most immediate extension to three or more players) there would be no independence requirement.

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[1] We restrict attention to two-player games for notational simplicity, but our analysis readily extends to games with three or more players.

Definition 1.3 *Set* $S_0^i = S^i$ *for* $i = a, b$*, and define inductively*

$$S_{m+1}^i = \{s^i \in S_m^i : s^i \text{ is undominated with respect to } S_m^a \times S_m^b\}.$$

A strategy $s^i \in S_m^i$ *is called* m***-undominated****. A strategy* $s^i \in \bigcap_{m=0}^{\infty} S_m^i$ *is called* ***iteratively undominated (IU)****.*

Note there is an M such that $\bigcap_{m=0}^{\infty} S_m^i = S_M^i \neq \emptyset$ for $i = a, b$.

Next are SAS's and the IA set.

Definition 1.4 *Fix* $X \times Y \subseteq S^a \times S^b$*. A strategy* $s^a \in X$ *is* ***weakly dominated with respect to*** $X \times Y$ *if: (i)* $Y \neq \emptyset$*; and (ii) there exists* $\sigma^a \in \mathcal{M}(S^a)$*, with* $\sigma^a(X) = 1$*, such that* $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ *for every* $s^b \in Y$*, and* $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ *for some* $s^b \in Y$*. Otherwise, say* s^a *is* ***admissible with respect to*** $X \times Y$*. If* s^a *is admissible with respect to* $S^a \times S^b$*, simply say that* s^a *is* ***admissible****.*

Lemma 1.2 *Fix* $X \times Y \subseteq S^a \times S^b$*, where* $Y \neq \emptyset$*. A strategy* $s^a \in X$ *is admissible with respect to* $X \times Y$ *if and only if there exists* $\sigma^b \in \mathcal{M}(S^b)$*, with* $\operatorname{Supp} \sigma^b = Y$*, such that* $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ *for every* $r^a \in X$*.*

Definition 1.5 *Say* r^a ***supports*** s^a *if there exists some* $\sigma^a \in \mathcal{M}(S^a)$ *with* $r^a \in \operatorname{Supp} \sigma^a$ *and* $\pi^a(\sigma^a, s^b) = \pi^a(s^a, s^b)$ *for all* $s^b \in S^b$*. Write* $\mathrm{su}(s^a)$ *for the set of* $r^a \in S^a$ *that support* s^a*.*

In words, strategy r^a supports strategy s^a if it is part of a convex combination of Ann's strategies that is equivalent for her to s^a.

Definition 1.6 *Fix* $Q^a \times Q^b \subseteq S^a \times S^b$*. The set* $Q^a \times Q^b$ *is a* ***self-admissible set (SAS)*** *if:*

(i) each $s^a \in Q^a$ *is admissible with respect to* $S^a \times S^b$*;*

(ii) each $s^a \in Q^a$ *is admissible with respect to* $S^a \times Q^b$*;*

(iii) for any $s^a \in Q^a$*, if* r^a *supports* s^a *then* $r^a \in Q^a$*;*

and likewise for each $s^b \in Q^b$*.*

Of course, conditions (i) and (ii) in Definition 1.6 reduce to one condition in the case of strong dominance (Definition 1.2). Also, we could add condition (iii) to Definition 1.2. It is without loss of generality in the following sense: Any BRS $Q^a \times Q^b$ not satisfying (iii) is contained in a larger set $P^a \times P^b$ that also satisfies (iii).[2] By contrast, condition (iii) plays an essential role in SAS's; see [4, 2006, Section 2.3].

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[2] See Lemma 2.2 below. David Pearce (private communication) has told us that he was aware of condition (iii), but to keep things simple didn't include the condition in his [7, 1984].

Definition 1.7 *Set* $\overline{S}^i_0 = S^i$ *for* $i = a, b$, *and define inductively*

$$\overline{S}^i_{m+1} = \{s^i \in \overline{S}^i_m : s^i \text{ is admissible with respect to } \overline{S}^a_m \times \overline{S}^b_m\}.$$

A strategy $s^i \in \overline{S}^i_m$ *is called* m-***admissible****. A strategy* $s^i \in \bigcap_{m=0}^{\infty} \overline{S}^i_m$ *is called* ***iteratively admissible (IA)****.*

Note there is an N such that $\bigcap_{m=0}^{\infty} \overline{S}^i_m = \overline{S}^i_N \neq \emptyset$ for $i = a, b$.

We will make use of the following lemmas:

Lemma 1.3 ([4, 2006, Lemma D.2]) *Fix* $X \subseteq S^a$ *and* $s^a \in X$, *and suppose there is a* $\sigma^b \in \mathcal{M}\left(S^b\right)$ *such that* $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(q^a, \sigma^b\right)$ *for all* $q^a \in X$. *Then if* $r^a \in \mathrm{su}(s^a)$, $\pi^a\left(r^a, \sigma^b\right) \geq \pi^a\left(q^a, \sigma^b\right)$ *for all* $q^a \in X$.

Lemma 1.4 *Suppose* $s^a \in S^a$ *is undominated (resp. admissible) with respect to* $X \times Y$. *If* $r^a \in \mathrm{su}(s^a)$, *then* r^a *is undominated (resp. admissible) with respect to* $X \times Y$.

Proof. If $Y = \emptyset$ the result is immediate, so assume $Y \neq \emptyset$. By Lemma 1.1 (resp. Lemma 1.2) there is a $\sigma^b \in \mathcal{M}\left(S^b\right)$, with $\sigma^b(Y) = 1$ (resp. $\mathrm{Supp}\, \sigma^b = Y^b$), such that $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(q^a, \sigma^b\right)$ for all $q^a \in X$. By Lemma 1.3, r^a is then undominated (resp. admissible) with respect to $X \times Y$. ■

2 Characterization of BRS's

Consider the complete lattice Λ of rectangular subsets $Q^a \times Q^b$ of $S^a \times S^b$. (The join of two subsets is the component-by-component union. The meet is the intersection.) Define a map $\Phi : \Lambda \to \Lambda$ as follows.[3] Put $(s^a, s^b) \in \Phi(Q^a \times Q^b)$ if and only if $s^a \in Q^a$ and satisfies condition (i) in Definition 1.2, and likewise for s^b.

Proposition 2.1 *If* $Q^a \times Q^b$ *is a BRS, then it is a fixed point of* Φ, *i.e.,* $\Phi(Q^a \times Q^b) = Q^a \times Q^b$. *Conversely, if* $Q^a \times Q^b$ *is a fixed point of* Φ, *then it is a BRS.*

Proof. Fix a BRS $Q^a \times Q^b$. Certainly, $\Phi\left(Q^a \times Q^b\right) \subseteq Q^a \times Q^b$ so we will show $Q^a \times Q^b \subseteq \Phi\left(Q^a \times Q^b\right)$. Fix $\left(s^a, s^b\right) \in Q^a \times Q^b$. Then s^a satisfies condition (i) of a BRS, and likewise for s^b, and so $\left(s^a, s^b\right) \in \Phi\left(Q^a \times Q^b\right)$.

For the converse, suppose $Q^a \times Q^b = \Phi\left(Q^a \times Q^b\right)$. Then, each $s^a \in Q^a$ satisfies condition (i), and likewise for each $s^b \in Q^b$. It follows that $Q^a \times Q^b$ is a BRS. ■

Lemma 2.1 *If* $s^a \in S^a_m$, *then there exists* $\sigma^b \in \mathcal{M}\left(S^b\right)$, *with* $\sigma^b(S^b_{m-1}) = 1$, *such that* $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(r^a, \sigma^b\right)$ *for all* $r^a \in S^a$.

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[3] The same map is mentioned in Apt [1, 2006, Section 9], where it is called $\overline{MSS}$.

Proof. Suppose that for every $\sigma^b \in \mathcal{M}\left(S^b\right)$ with $\sigma^b(S^b_{m-1}) = 1$, there is an $r^a \in S^a$ (possibly different for different σ^b) such that $\pi^a\left(r^a, \sigma^b\right) > \pi^a\left(s^a, \sigma^b\right)$. In fact, for any σ^b, we can choose $r^a \in S^a$ so that $\pi^a\left(r^a, \sigma^b\right) \geq \pi^a\left(q^a, \sigma^b\right)$ for all $q^a \in S^a$. Since $\sigma^b\left(S^b_{m-1}\right) = 1$ we also have $\sigma^b\left(S^b_l\right) = 1$ for all $l \leq m-1$. It follows that $r^a \in S^a_{m-1}$. Since this holds for every $\sigma^b \in \mathcal{M}\left(S^b\right)$ with $\sigma^b(S^b_{m-1}) = 1$, Lemma 1.1 implies $s^a \notin S^a_m$. ■

Write $\Phi^1\left(S^a \times S^b\right) = \Phi\left(S^a \times S^b\right)$ and, for $m \geq 1$, $\Phi^{m+1}\left(S^a \times S^b\right) = \left(\Phi \circ \Phi^m\right)\left(S^a \times S^b\right)$.

Proposition 2.2

(i) The map Φ is monotone increasing, i.e., if $Q^a \times Q^b \subseteq P^a \times P^b$, then $\Phi(Q^a \times Q^b) \subseteq \Phi(P^a \times P^b)$.

(ii) There is a greatest fixed point of Φ, which is the IU set.

Proof. For part (i), fix $Q^a \times Q^b \subseteq P^a \times P^b$ and some $\left(s^a, s^b\right) \in \Phi\left(Q^a \times Q^b\right)$. Then s^a is undominated with respect to $S^a \times Q^b$. But $Q^b \subseteq P^b$, so certainly s^a is undominated with respect to $S^a \times P^b$. Likewise for s^b. With this, $\left(s^a, s^b\right) \in \Phi\left(P^a \times P^b\right)$.

For part (ii), we will first show that, for all $m \geq 1$, $\Phi^m\left(S^a \times S^b\right) = S^a_m \times S^b_m$. The proof is by induction on m.

$m = 1$: It is immediate that $\left(s^a, s^b\right) \in \Phi^1\left(S^a \times S^b\right)$ if and only if $\left(s^a, s^b\right) \in S^a_1 \times S^b_1$.

$m > 1$: Assume $\Phi^m\left(S^a \times S^b\right) = S^a_m \times S^b_m$, and fix $\left(s^a, s^b\right) \in \Phi\left(\Phi^m\left(S^a \times S^b\right)\right)$. By the induction hypothesis, $s^a \in S^a_m$ and is undominated with respect to $S^a \times S^b_m$. Thus $s^a \in S^a_{m+1}$. Likewise for s^b. This establishes that $\left(s^a, s^b\right) \in S^a_{m+1} \times S^b_{m+1}$.

Conversely, fix $\left(s^a, s^b\right) \in S^a_{m+1} \times S^b_{m+1}$. Then, by the induction hypothesis, $\left(s^a, s^b\right) \in S^a_m \times S^b_m = \Phi^m\left(S^a \times S^b\right)$. Moreover, by Lemmas 1.1 and Lemma 2.1, s^a is also undominated with respect to $S^a \times S^b_m$. Likewise for s^b. So, $\left(s^a, s^b\right) \in \Phi\left(\Phi^m\left(S^a \times S^b\right)\right)$, as required.

It remains to show that $S^a_M \times S^b_M = \Phi^M\left(S^a \times S^b\right)$ is the greatest fixed point of Φ. Fix an arbitrary fixed point $Q^a \times Q^b$. Then $Q^a \times Q^b = \Phi^M\left(Q^a \times Q^b\right)$. By part (i) and induction, $\Phi^M\left(Q^a \times Q^b\right) \subseteq \Phi^M\left(S^a \times S^b\right)$. ■

Here, we proved directly the existence of a greatest fixed point. As noted in the Introduction, since Φ is monotone, the result also follows from the Knaster-Tarski Theorem.

The set of fixed points of a monotone map Φ on a complete lattice is itself a complete lattice (Tarski [8, 1955]). It is not, in general, a complete sublattice. Here is an example in the present setting:

	L	*C*	*R*
U	3, 0	3, 1	0, 0
M	3, 0	3, 0	0, 0
D	0, 1	0, 0	3, 0

Figure 2.1

Example 2.1 *In the game of Figure 2.1,* $\{U, M\} \times \{L, R\}$ *and* $\{U, M\} \times \{C, R\}$ *are BRS's, but the meet* $\{U, M\} \times \{R\}$ *is not a BRS. This shows that the set of fixed points is not a sublattice of the underlying lattice; a fortiori, it is not a complete sublattice. (But* $\{U, M\} \times \emptyset$ *is a fixed point of* Φ *and, within the set of fixed points of* Φ*, is the meet of* $\{U, M\} \times \{L, R\}$ *and* $\{U, M\} \times \{C, R\}$*.)*

Finally, we prove the observation in Footnote 2:

Lemma 2.2 *Fix a BRS* $Q^a \times Q^b$*. Let*

$$P^a \times P^b = \{r^a \in \mathrm{su}\,(s^a) : s^a \in Q^a\} \times \left\{r^b \in \mathrm{su}\left(s^b\right) : s^b \in Q^b\right\}.$$

Then $P^a \times P^b$ *is a BRS containing* $Q^a \times Q^b$ *that satisfies condition (iii) of Definition 1.6.*

Proof. Since $s^a \in \mathrm{su}\,(s^a)$ for any $s^a \in S^a$, $Q^a \subseteq P^a$. And likewise for b. From this, it also follows that $P^a \times P^b$ satisfies condition (i) of a BRS: Fix $r^a \in P^a$. Then $r^a \in \mathrm{su}\,(s^a)$ where $s^a \in Q^a$, i.e., s^a is undominated with respect to $S^a \times Q^b$. Since $Q^b \subseteq P^b$, s^a is also undominated with respect to $S^a \times P^b$. By Lemma 1.4, $r^a \in \mathrm{su}\,(s^a)$ is then undominated with respect to $S^a \times P^b$.

Next, we show that condition (iii) is satisfied, by showing that if $r^a \in \mathrm{su}\,(s^a)$ and $q^a \in \mathrm{su}\,(r^a)$, then $q^a \in \mathrm{su}\,(s^a)$.

So, suppose: (1) there is some $\sigma^a \in \mathcal{M}\,(S^a)$ with $r^a \in \mathrm{Supp}\,\sigma^a$ and $\pi^a\left(\sigma^a, s^b\right) = \pi^a\left(s^a, s^b\right)$ for all $s^b \in S^b$; and (2) there is some $\rho^a \in \mathcal{M}\,(S^a)$ with $q^a \in \mathrm{Supp}\,\rho^a$ and $\pi^a\left(\rho^a, s^b\right) = \pi^a\left(r^a, s^b\right)$ for all $s^b \in S^b$.

Let $\sigma^a \backslash r^a$ denote the measure (not a probability measure) which agrees with σ^a except for assigning measure 0 to r^a. Let τ^a be the measure given by $\tau^a = \sigma^a \backslash r^a + \sigma^a(r^a)\rho^a$. Clearly, for each $v^a \in S^a$, $0 \leq \tau^a\,(v^a) \leq 1$. Also,

$$\sum\nolimits_{v^a \in S^a} \tau^a\,(v^a) = \sum\nolimits_{v^a \in S^a \backslash \{r^a\}} [\sigma^a\,(v^a) + \sigma^a(r^a)\rho^a\,(v^a)] + \sigma^a\,(r^a)\,\rho^a\,(r^a) = 1,$$

so that τ^a is, in fact, a probability measure. Notice

$$\begin{aligned}
\pi^a\left(\tau^a, s^b\right) &= \sum\nolimits_{v^a \in S^a \backslash \{r^a\}} [\sigma^a\,(v^a) + \sigma^a(r^a)\rho^a\,(v^a)]\pi^a\left(v^a, s^b\right) + \sigma^a\,(r^a)\,\rho^a\,(r^a)\,\pi^a\left(r^a, s^b\right) \\
&= \sum\nolimits_{v^a \in S^a \backslash \{r^a\}} \sigma^a\,(v^a)\,\pi^a\left(v^a, s^b\right) + \sigma^a(r^a) \sum\nolimits_{v^a \in S^a} \rho^a\,(v^a)\,\pi^a\left(v^a, s^b\right) \\
&= \sum\nolimits_{v^a \in S^a \backslash \{r^a\}} \sigma^a\,(v^a)\,\pi^a\left(v^a, s^b\right) + \sigma^a(r^a)\pi^a\left(r^a, s^b\right) \\
&= \sum\nolimits_{v^a \in S^a} \sigma^a\,(v^a)\,\pi^a\left(v^a, s^b\right) \\
&= \pi^a\left(s^a, s^b\right).
\end{aligned}$$

Finally, observe that $q^a \in \mathrm{Supp}\,\tau^a$, since $q^a \in \mathrm{Supp}\,\rho^a$ and $\mathrm{Supp}\,\rho^a \subseteq \mathrm{Supp}\,\tau^a$. This establishes that $q^a \in \mathrm{su}\,(s^a)$, as desired. ■

3 Characterization of SAS's

Now weak dominance. We define $\Psi : \Lambda \to \Lambda$ as follows. Put $(s^a, s^b) \in \Psi(Q^a \times Q^b)$ if and only if: (a) $s^a \in Q^a$ and satisfies conditions (i) and (ii) in Definition 1.6; or (b) s^a supports an $r^a \in Q^a$ that satisfies these conditions; and likewise for s^b.

Here is the weak-dominance analog to Proposition 2.1:

Proposition 3.1 *If $Q^a \times Q^b$ is an SAS, then it is a fixed point of Ψ. Conversely, if $Q^a \times Q^b$ is a fixed point of Ψ, then it is an SAS.*

Proof. Fix an SAS $Q^a \times Q^b$. If $s^a \in Q^a$, then s^a satisfies condition (a) for Ψ. Likewise for s^b. Thus $Q^a \times Q^b \subseteq \Psi\left(Q^a \times Q^b\right)$. Next, fix $\left(s^a, s^b\right) \in \Psi\left(Q^a \times Q^b\right)$. We need to show that $s^a \in Q^a$. If $s^a \notin Q^a$ then $s^a \in \mathrm{su}(r^a)$ for some $r^a \in Q^a$. But then condition (iii) of an SAS implies $s^a \in Q^a$, a contradiction.

For the converse, fix $\left(s^a, s^b\right) \in Q^a \times Q^b = \Psi\left(Q^a \times Q^b\right)$. If s^a satisfies condition (a) for Ψ, then it satisfies conditions (i) and (ii) of an SAS. Next suppose s^a fails condition (a) for Ψ, i.e., is inadmissible with respect to $S^a \times S^b$, or $S^a \times Q^b$, or both. But then s^a must satisfy condition (b) for Ψ, i.e. $s^a \in \mathrm{su}(r^a)$ for some $r^a \in Q^a$ satisfying conditions (i) and (ii) of an SAS. By Lemma 1.4, s^a is then admissible with respect to both $S^a \times S^b$ and $S^a \times Q^b$, a contradiction. Finally, suppose $q^a \in \mathrm{su}(s^a)$. We just saw that s^a satisfies condition (a) for Ψ, so q^a satisfies condition (b) for Ψ. Thus $q^a \in \mathrm{proj}_{S^a}\, \Psi\left(Q^a \times Q^b\right) = Q^a$. This establishes condition (iii) of an SAS. ■

Lemma 3.1 *For all m, if $s^a \in \overline{S}^a_m$ and $r^a \in \mathrm{su}(s^a)$, then $r^a \in \overline{S}^a_m$.*

Proof. By induction on m. The result is immediate for $m = 0$. Assume it is true for m, and fix $s^a \in \overline{S}^a_{m+1}$ and $r^a \in \mathrm{su}(s^a)$. By the induction hypothesis, $r^a \in \overline{S}^a_m$. Since s^a is admissible with respect to $\overline{S}^a_m \times \overline{S}^b_m$, Lemma 1.4 says that r^a is also admissible with respect to $\overline{S}^a_m \times \overline{S}^b_m$, establishing $r^a \in \overline{S}^a_{m+1}$. ■

Lemma 3.2 ([4, 2006, Lemma E.1]) *If $s^a \in S^a_m$, then there exists $\sigma^b \in \mathcal{M}\left(S^b\right)$, with $\mathrm{Supp}\,\sigma^b = S^b_{m-1}$, such that $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(r^a, \sigma^b\right)$ for all $r^a \in S^a$.*

Write $\Psi^1\left(S^a \times S^b\right) = \Psi\left(S^a \times S^b\right)$ and, for $m \geq 1$, $\Psi^{m+1}\left(S^a \times S^b\right) = \left(\Psi \circ \Psi^m\right)\left(S^a \times S^b\right)$.

Proposition 3.2 *The IA set $\overline{S}^a_N \times \overline{S}^b_N$ is a fixed point of Ψ.*

Proof. We show that, for all $m \geq 1$, $\Psi^m\left(S^a \times S^b\right) = \overline{S}^a_m \times \overline{S}^b_m$. The proof is by induction on m.

$m = 1$: Fix $\left(s^a, s^b\right) \in \Psi^1\left(S^a \times S^b\right)$. If s^a satisfies condition (a) for Ψ, it is immediate that s^a is admissible. If s^a satisfies condition (b) for Ψ, then $s^a \in \mathrm{su}(r^a)$ for some admissible $r^a \in S^a$. By Lemma 1.4, s^a is also admissible. Likewise for s^b. This establishes that $\left(s^a, s^b\right) \in \overline{S}^a_1 \times \overline{S}^b_1$. Conversely, fix $\left(s^a, s^b\right) \in \overline{S}^a_1 \times \overline{S}^b_1$. It is then immediate that s^a and s^b satisfy condition (a) for $\Psi\left(S^a \times S^b\right)$, and so $\left(s^a, s^b\right) \in \Psi^1\left(S^a \times S^b\right)$.

$m > 1$: Assume $\Psi^m\left(S^a \times S^b\right) = \overline{S}^a_m \times \overline{S}^b_m$, and fix $\left(s^a, s^b\right) \in \Psi\left(\Psi^m\left(S^a \times S^b\right)\right)$. If s^a satisfies condition (a) for Ψ, evaluated at $\Psi^m\left(S^a \times S^b\right)$, then, using the induction hypothesis, $s^a \in \overline{S}^a_m$ and is admissible with respect to $S^a \times \overline{S}^b_m$. Thus $s^a \in \overline{S}^a_{m+1}$. If s^a satisfies condition (b) for Ψ, evaluated at $\Psi^m\left(S^a \times S^b\right)$, then, again using the induction hypothesis, $s^a \in \mathrm{su}(r^a)$ for some $r^a \in \overline{S}^a_m$ that is admissible with respect to $S^a \times \overline{S}^b_m$. Thus $s^a \in \overline{S}^a_m$, by Lemma 3.1, and s^a is admissible with respect to $S^a \times \overline{S}^b_m$, by Lemma 1.4. So, certainly $s^a \in \overline{S}^a_{m+1}$. Likewise for s^b. This establishes that $\left(s^a, s^b\right) \in \overline{S}^a_{m+1} \times \overline{S}^b_{m+1}$.

Conversely, fix $\left(s^a, s^b\right) \in S^a_{m+1} \times S^b_{m+1}$. Then, by the induction hypothesis, $\left(s^a, s^b\right) \in S^a_m \times S^b_m = \Phi^m\left(S^a \times S^b\right)$. Moreover, by Lemmas 1.1 and Lemma 2.1, s^a is also undominated with respect to $S^a \times S^b_m$. Likewise for s^b. So, $\left(s^a, s^b\right) \in \Phi\left(\Phi^m\left(S^a \times S^b\right)\right)$, as required.

Conversely, fix $\left(s^a, s^b\right) \in \overline{S}^a_{m+1} \times \overline{S}^b_{m+1}$. Then, by the induction hypothesis, $\left(s^a, s^b\right) \in \overline{S}^a_m \times \overline{S}^b_m = \Psi^m\left(S^a \times S^b\right)$. Certainly, s^a and s^b satisfy condition (i) of an SAS. Moreover, by Lemma 1.2 and 3.2, s^a is admissible with respect to $S^a \times \overline{S}^b_m$. Likewise for s^b. So, s^a and s^b satisfy condition (ii) of an SAS evaluated for $\Psi^m\left(S^a \times S^b\right)$. So, $\left(s^a, s^b\right) \in \Psi\left(\Psi^m\left(S^a \times S^b\right)\right)$, as required. ■

Propositions 3.1 and 3.2 give:

Corollary 3.1 *The IA set is an SAS.*

As noted in the Introduction, the map Ψ satisfies: $\Psi(S^a \times S^b) \supseteq \Psi(\Psi((S^a \times S^b)) \cdots$. From this we immediately get a fixed point by induction. But unlike Φ in the previous section, the map Ψ is not necessarily monotone increasing, and there need not be a greatest fixed point.

	L	*R*
U	2, 2	2, 2
M	3, 1	0, 0
D	0, 0	1, 3

Figure 3.1

Example 3.1 *In the game of Figure 3.1,* $\Psi(\{U\} \times \{L, R\}) = \{U\} \times \{L, R\}$*, while* $\Psi(\{U, D\} \times \{L, R\}) = \{(U, R)\}$*, a failure of monotonicity. The nonempty fixed points are* $\{U\} \times \{L, R\}$*,* $\{(U, R)\}$*, and* $\{(M, L)\}$*, with no greatest one. The last is the IA set. (By contrast, the BRS's are the preceding sets, and* $\{U, M\} \times \{L, R\}$*, which is the IU set.)*

4 Epistemic Fixed Points

We now give a definition from epistemic game theory. Fix a game $\langle S^a, S^b, \pi^a, \pi^b\rangle$ and append Polish spaces T^a, T^b of **epistemic types** for each player. Given a Borel set $E^b \subseteq S^b \times T^b$, suppose we are given what it means to say that a type t^a for Ann **thinks** E^b happens. Let

$$C^a(E^b) = \{t^a \in T^a : t^a \text{ thinks } E^b \text{ happens}\},$$

and define $C^b(E^a)$ similarly.

In the standard case, types are associated with (Borel) probability measures, and we ask whether type t^a **believes** E^b. In Brandenburger-Friedenberg-Keisler [4, 2006], types are associated with lexicographic probability systems (Blume-Brandenburger-Dekel [3, 2001]), and we ask whether type t^a **assumes** E^b. We omit the details here, since we will only need a property of belief and assumption. The terminology "thinks" and the notation C^a are meant to subsume both cases.

Axiom 4.1 (Conjunction) *Fix a type $t^a \in T^a$ and Borel sets $E^b_1, E^b_2, \ldots$ in $S^b \times T^b$. Suppose, for each m, that $t^a \in C^a(E^b_m)$. Then $t^a \in C^a(\bigcap_m E^b_m)$.*

In words, if Ann thinks that each event E^b_m happens, then she thinks the joint event $\bigcap_m E^b_m$ happens. It is well known that this conjunction property is satisfied for belief. It is also satisfied for assumption; see [4, 2006, Property 4.3].

Given Borel sets $E^a \subseteq S^a \times T^a$ and $E^b \subseteq S^b \times T^b$, define $E^a_1 = E^a$, $E^b_1 = E^b$, and for $m \geq 1$,

$$E^a_{m+1} = E^a_m \cap [S^a \times C^a(E^b_m)],$$

and likewise with a and b interchanged.

Definition 4.1 *The event that $E^a \times E^b$ **happens and there is common thought of** $E^a \times E^b$ is*

$$\bigcap\nolimits_{m=1}^{\infty} E^a_m \times \bigcap\nolimits_{m=1}^{\infty} E^b_m.$$

In the application we are interested in, $E^a \times E^b$ is the event that Ann and Bob are rational, and so $\bigcap_{m=1}^{\infty} E^a_m \times \bigcap_{m=1}^{\infty} E^b_m$ is the event that there is rationality and common belief (resp. assumption) of rationality. That is, this is the RCBR (resp. RCAR) event mentioned in the Introduction. But we won't need to use this information here.

Given $E^a \times E^b \subseteq S^a \times T^a \times S^b \times T^b$, define

$$\Gamma(E^a \times E^b) = (E^a \times E^b) \cap ([S^a \times C^a(E^b)] \times [S^b \times C^b(E^a)]).$$

In words, Γ takes an event $E^a \times E^b$ to the event that $E^a \times E^b$ happens, and Ann and Bob think their respective components of $E^a \times E^b$ happen. The next lemma is immediate:

Lemma 4.1 *The event $E^a \times E^b$ is a fixed point of Γ if and only if*

$$\begin{aligned} E^a &\subseteq S^a \times C^a(E^b), \\ E^b &\subseteq S^b \times C^b(E^a). \end{aligned}$$

In the case of belief, Lemma 4.1 says the fixed points Γ are the belief-closed subsets (Mertens-Zamir [6, 1985, Definition 2.15]). If C^a stands for assumption, we can call the fixed points the "assumption-closed subsets."

Proposition 4.1 *Fix* $E^a \times E^b \subseteq S^a \times T^a \times S^b \times T^b$. *The event*

$$\bigcap\nolimits_{m=1}^{\infty} E_m^a \times \bigcap\nolimits_{m=1}^{\infty} E_m^b$$

is a fixed point of Γ.

Proof. We have

$$\bigcap\nolimits_{m=1}^{\infty} E_m^a = E_1^a \cap \bigcap\nolimits_{m=1}^{\infty} [S^a \times C^a(E_m^b)] \subseteq \bigcap\nolimits_{m=1}^{\infty} [S^a \times C^a(E_m^b)] = S^a \times \bigcap\nolimits_{m=1}^{\infty} C^a(E_m^b) \subseteq S^a \times C^a(\bigcap\nolimits_{m=1}^{\infty} E_m^b),$$

where the last inclusion uses conjunction (Axiom 4.1). ■

Proposition 4.2 *Suppose* $E^a \times E^b$ *is a fixed point of* Γ. *Then* $E_m^a = E^a$ *and* $E_m^b = E^b$ *for all* m.

Proof. This is immediate for $m = 1$, so suppose it is true for m. We have

$$E_{m+1}^a = E_m^a \cap [S^a \times C^a(E_m^b)] = E^a \cap [S^a \times C^a(E^b)],$$

using the induction hypothesis. But since $E^a \times E^b$ is a fixed point,

$$E^a \cap [S^a \times C^a(E^b)] = E^a,$$

and so $E_{m+1}^a = E^a$, as required. ■

Barwise [2, 1988] offers a general treatment of fixed points in an epistemic setting.

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