

Modelling Knowledge

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In the note “Common Knowledge,” we represented the information a person possesses by a partition \mathcal{P} of the finite set Ω of possible states of the world. The interpretation is that, if the true state is ω , the person is informed of the member of \mathcal{P} that contains ω , denoted by $\mathcal{P}(\omega)$. It is then natural to define the event that the person knows an event $E \subseteq \Omega$, to be written by KE , by

$$KE = \{\omega \in \Omega : \mathcal{P}(\omega) \subseteq E\}.$$

It is not hard to see that K , considered as a map from $2^\Omega \rightarrow 2^\Omega$ (where 2^Ω is the set of all subsets of Ω), has the properties listed below. (Give an interpretation of each of these properties.)

P1: For any $E \subseteq \Omega$, $KE \subseteq E$.

P2: For any $E, F \subseteq \Omega$, if $E \subseteq F$, then $KE \subseteq KF$.

P3: For any $E \subseteq \Omega$, $(KE)^c \subseteq K(KE)^c$ where the superscript c denotes complement.

Here are the proofs:

Proof of P1: If $\omega \in KE$, then $\mathcal{P}(\omega) \subseteq E$. But $\omega \in \mathcal{P}(\omega)$, so $\omega \in E$.

Proof of P2: If $\omega \in KE$, then $\mathcal{P}(\omega) \subseteq E$. But then certainly $\mathcal{P}(\omega) \subseteq F$, i.e. $\omega \in KF$.

Proof of P3: If $\omega \in (KE)^c$, then $\mathcal{P}(\omega) \not\subseteq E$. Suppose there exists some $\omega' \in \mathcal{P}(\omega) \cap KE$. Then $\omega' \in \mathcal{P}(\omega)$ implies $\mathcal{P}(\omega') = \mathcal{P}(\omega) \not\subseteq E$, contradicting $\omega' \in KE$. Thus $\mathcal{P}(\omega) \cap KE = \emptyset$. This says that $\mathcal{P}(\omega) \subseteq (KE)^c$, or $\omega \in K(KE)^c$.

Let's list some further properties of the map K :

P4: $K\Omega = \Omega$. (Proof: Setting $E = (K\Omega)^c$ and $F = \Omega$ in P2 gives $K(K\Omega)^c \subseteq (K\Omega)^c$. Thus $(K\Omega)^c \subseteq K\Omega$, using P3, from which $K\Omega = \Omega$ as required.)

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P5: $KE \subseteq KKE$. (Proof: We have $(KE)^c = K(KE)^c$ by P1 and P3, and so $KE = (K(KE)^c)^c$. Thus $KKE = K(K(KE)^c)^c \supseteq (K(KE)^c)^c$, using P3. But $(K(KE)^c)^c = KE$, again using P1 and P3.)

P6: $K(E \cap F) = KE \cap KF$. (Proof: Immediate from P2.)

A nice result is that the properties P1-P3 actually characterize partition information in the following sense. (In the proof below, we use P4-P6 as well, which we can do since we derived these using only P1-P3.)

Theorem 1 *If a function $K : 2^\Omega \rightarrow 2^\Omega$ satisfies P1-P3 then there is a partition \mathcal{P} of Ω such that $KE = \{\omega \in \Omega : \mathcal{P}(\omega) \subseteq E\}$.*

Proof. For any $\omega \in \Omega$, define

$$\mathcal{P}(\omega) = \bigcap_{\{E: \omega \in KE\}} E.$$

The following must be shown:

- (i) $\omega \in KE$ if and only if $\mathcal{P}(\omega) \subseteq E$;
- (ii) $\omega \in \mathcal{P}(\omega)$;
- (iii) if $\omega' \in \mathcal{P}(\omega)$, then $\mathcal{P}(\omega') = \mathcal{P}(\omega)$.

First, use P1 to write for any ω :

$$\bigcap_{\{E: \omega \in KE\}} KE \subseteq \bigcap_{\{E: \omega \in KE\}} E. \quad (\text{a})$$

Now, from P4 we certainly have $\omega \in K\Omega$. Thus $\{E : \omega \in KE\} \neq \emptyset$, and so we get

$$\omega \in \bigcap_{\{E: \omega \in KE\}} KE \subseteq \bigcap_{\{E: \omega \in KE\}} E = \mathcal{P}(\omega),$$

establishing (ii).

Next, use P1 and P5 to get $KE = KKE$, so that

$$\bigcap_{\{E: \omega \in KE\}} KE = \bigcap_{\{E: \omega \in KKE\}} KE \supseteq \bigcap_{\{E: \omega \in KE\}} E. \quad (\text{b})$$

By (a) and (b),

$$\mathcal{P}(\omega) = \bigcap_{\{E: \omega \in KE\}} E = \bigcap_{\{E: \omega \in KE\}} KE, \quad (\text{c})$$

so that

$$(\mathcal{P}(\omega))^c = \bigcup_{\{E: \omega \in KE\}} (KE)^c.$$

Using P2 and P3,

$$\begin{aligned} K(\mathcal{P}(\omega))^c &= K \bigcup_{\{E:\omega \in KE\}} (KE)^c \supseteq \bigcup_{\{E:\omega \in KE\}} K(KE)^c = \\ & \bigcup_{\{E:\omega \in KE\}} (KE)^c = \left(\bigcap_{\{E:\omega \in KE\}} KE \right)^c = (\mathcal{P}(\omega))^c. \end{aligned}$$

We have shown that $(K(\mathcal{P}(\omega))^c)^c \subseteq \mathcal{P}(\omega)$, from which $(K(\mathcal{P}(\omega))^c)^c = K(K(\mathcal{P}(\omega))^c)^c \subseteq K\mathcal{P}(\omega)$, using P2 and P3. Thus $(K\mathcal{P}(\omega))^c \subseteq K(\mathcal{P}(\omega))^c \subseteq (\mathcal{P}(\omega))^c$, by P1, from which $\mathcal{P}(\omega) \subseteq K\mathcal{P}(\omega)$.

To establish (i), notice that by definition of $\mathcal{P}(\omega)$, if $\omega \in KE$ then $\mathcal{P}(\omega) \subseteq E$. Conversely, suppose $\mathcal{P}(\omega) \subseteq E$. Using $\mathcal{P}(\omega) \subseteq K\mathcal{P}(\omega)$, (ii), and P2, we get $\omega \in \mathcal{P}(\omega) \subseteq K\mathcal{P}(\omega) \subseteq KE$.

Finally, (iii) must be shown. If $\omega' \in \mathcal{P}(\omega)$ then by (c),

$$\omega' \in \bigcap_{\{E:\omega \in KE\}} KE.$$

Thus $\omega \in KE$ implies $\omega' \in KE$. From this we get

$$\mathcal{P}(\omega') = \bigcap_{\{E:\omega' \in KE\}} E \subseteq \bigcap_{\{E:\omega \in KE\}} E = \mathcal{P}(\omega). \quad (d)$$

If $\omega \in \mathcal{P}(\omega')$, an analogous argument yields $\mathcal{P}(\omega) \subseteq \mathcal{P}(\omega')$, thereby establishing (iii). So suppose $\omega \in \mathcal{P}(\omega) \cap \mathcal{P}(\omega')^c$.

We know that $K\mathcal{P}(\omega') \subseteq \mathcal{P}(\omega')$, and so $K\mathcal{P}(\omega') = \mathcal{P}(\omega')$, using P1. Thus, using P4,

$$\mathcal{P}(\omega')^c = (K\mathcal{P}(\omega'))^c = K[(K\mathcal{P}(\omega'))^c] = K[(\mathcal{P}(\omega'))^c].$$

We also have $K\mathcal{P}(\omega) = \mathcal{P}(\omega)$, so that, using P6,

$$\mathcal{P}(\omega) \cap \mathcal{P}(\omega')^c = K\mathcal{P}(\omega) \cap K[(\mathcal{P}(\omega'))^c] = K[\mathcal{P}(\omega) \cap (\mathcal{P}(\omega'))^c].$$

Using the definition of $\mathcal{P}(\omega)$ we then get $\mathcal{P}(\omega) \subseteq \mathcal{P}(\omega) \cap \mathcal{P}(\omega')^c$, i.e. $\mathcal{P}(\omega) \subseteq \mathcal{P}(\omega')^c$. But this contradicts (d). It follows that $\omega \in \mathcal{P}(\omega')$, and (iii) is established. ■

Reference:

Aumann, R., "Interactive Epistemology I: Knowledge," *Int. J. Game Theory*, 28, 1999, 263-300.