

Nash Equilibrium: Existence

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Version 01/04/07

An n -player game Γ in strategic form is a $2n$ -tuple $\Gamma = \langle S^1, \dots, S^n, \pi^1, \dots, \pi^n \rangle$ where S^i is player i 's set of pure strategies and $\pi^i : \prod_{j=1}^n S^j \rightarrow \mathfrak{R}$ is i 's payoff function. If each S^i is a finite set, then Γ is called a finite game. A mixed strategy for player i is a probability measure on i 's set of pure strategies. So, player i 's set of mixed strategies is

$$\Delta(S^i) = \left\{ \sigma^i : S^i \rightarrow \mathfrak{R}_+ \mid \sum_{s^i \in S^i} \sigma(s^i) = 1 \right\},$$

where $\sigma^i(s^i)$ is the probability with which i chooses the pure strategy $s^i \in S^i$. If the players choose the n -tuple of mixed strategies $\sigma = (\sigma^1, \dots, \sigma^n)$, then the expected payoff to player i is

$$v^i(\sigma) = \sum_{(s^1, \dots, s^n) \in S^1 \times \dots \times S^n} \prod_{j=1}^n \sigma^j(s^j) \pi^i(s^1, \dots, s^n).$$

The following notation will be helpful:

$$\begin{aligned} \sigma^{-i} &= (\sigma^1, \dots, \sigma^{i-1}, \sigma^{i+1}, \dots, \sigma^n), \\ (\sigma^{-i}, \tilde{\sigma}^i) &= (\sigma^1, \dots, \sigma^{i-1}, \tilde{\sigma}^i, \sigma^{i+1}, \dots, \sigma^n). \end{aligned}$$

Definition 1 An n -tuple of mixed strategies $\sigma = (\sigma^1, \dots, \sigma^n)$ is a **Nash equilibrium** if for every i it is true that $v^i(\sigma) \geq v^i(\sigma^{-i}, \tilde{\sigma}^i)$ for every $\tilde{\sigma}^i \in \Delta(S^i)$.

Theorem 1 (Nash) Every finite game possesses a Nash equilibrium.

*With the assistance of Amanda Friedenberg. nashproof-01-04-07

Nash's theorem will be proved as an easy corollary of a more general existence theorem.

Theorem 2 Consider an n -player game $\Gamma = \langle D^1, \dots, D^n, v^1, \dots, v^n \rangle$ where D^i is the set of pure strategies available to player i and $v^i : \prod_{j=1}^n D^j \rightarrow \mathfrak{R}$ is i 's payoff function. If each D^i is a compact and convex subset of Euclidean space, and each v^i is quasiconcave in d^i and continuous, then Γ has a Nash equilibrium in pure strategies.

(To say that v^i is quasiconcave in d^i means that

$$v^i(d^{-i}, \alpha d^i + (1 - \alpha) \tilde{d}^i) \geq \min \left\{ v^i(d^{-i}, d^i), v^i(d^{-i}, \tilde{d}^i) \right\}$$

for every $\alpha \in [0, 1]$.) Theorem 2 is in turn an immediate consequence of Kakutani's Fixed Point Theorem. Before stating Kakutani's theorem, a definition will be useful.

Definition 2 A correspondence ϕ from a subset T of Euclidean space to a compact subset V of Euclidean space is **upper hemicontinuous at a point** $x \in T$ if $x_r \rightarrow x$, $y_r \rightarrow y$, where $y_r \in \phi(x_r)$ for every r , implies $y \in \phi(x)$. The correspondence ϕ is **upper hemicontinuous** if it is upper hemicontinuous at every $x \in T$.

Theorem 3 (Kakutani) If T is a nonempty compact and convex subset of Euclidean space, and ϕ is an upper hemicontinuous, nonempty, and convex-valued correspondence from T to T , then ϕ has a fixed point, that is, there is an $x \in T$ such that $x \in \phi(x)$.

Proof of Theorem 2. For each i define a *best reply* correspondence ϕ^i from $\prod_{j=1}^n D^j$ to D^i as follows. For any $d \in \prod_{j=1}^n D^j$, let

$$\phi^i(d) = \left\{ \hat{d}^i \in D^i \mid v^i(d^{-i}, \hat{d}^i) \geq v^i(d^{-i}, \tilde{d}^i) \text{ for every } \tilde{d}^i \in D^i \right\}.$$

The set $\phi^i(d)$ is the set of strategies that maximizes i 's payoff given the strategies of the other players prescribed by d ; it is nonempty since D^i is compact and v^i is continuous. The correspondence $\phi^i(d)$ is convex-valued since v^i is quasiconcave in d^i . To see that ϕ^i is upper hemicontinuous, consider a sequence d_r in $\prod_{j=1}^n D^j$ converging to d , and a sequence \hat{d}_r^i in D^i converging to \hat{d}^i , where $\hat{d}_r^i \in \phi^i(d_r)$

for every r . For any $\tilde{d}_i \in D^i$, we have $v^i(d_r^{-i}, \tilde{d}_r^i) \geq v^i(d_r^{-i}, \tilde{d}^i)$. Therefore $v^i(d^{-i}, \tilde{d}^i) \geq v^i(d^{-i}, \tilde{d}^i)$, since v^i is continuous, i.e. $\tilde{d}^i \in \phi(d)$. This shows that ϕ^i is upper hemicontinuous. Now define a correspondence ϕ from $\prod_{i=1}^n D^i$ to

$$\prod_{i=1}^n D^i \text{ by}$$

$$\phi(d) = \phi^1(d) \times \cdots \times \phi^n(d).$$

The set $\prod_{i=1}^n D^i$ is a compact and convex subset of Euclidean space since each D^i is. The correspondence ϕ is upper hemicontinuous, nonempty, and convex-valued since each ϕ^i is. And it is easy to see that a fixed point of ϕ is just a Nash equilibrium of Γ . ■

Finally, we have to say why Theorem 1 follows from Theorem 2. To see this, just let $D^i = \Delta(S^i)$. Each D^i is then a compact and convex subset of a Euclidean space, and each v^i is quasiconcave in d^i (in fact, linear in each variable) and continuous. So, by Theorem 2, there is a Nash equilibrium of the game in which each player i chooses a pure strategy from $\Delta(S^i)$. But this is, of course, just a Nash equilibrium in mixed strategies of the original game.