# The Relationship Between Rationality on the Matrix and the Tree*

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The relationship between the matrix and the tree has been the subject of intensive investigation ever since the beginning of game theory. The issue goes back even to Borel and von Neumann.

Later, Thompson [11, 1952], followed by Elmes-Reny [4, 1994], uncovered the structural relationship between the matrix and the tree. They showed that, up to the duplication of pure strategies, two games have the same strategic form if and only if they differ by a certain sequence of elementary transformations.

There is also the question of the relationship between Nash equilibrium defined on the matrix (perfect, proper, stable equilibria, etc.) and equilibrium defined on the tree (extensive-form perfect, sequential equilibria, etc.). For example, Kohlberg-Mertens [6, 1986] and van Damme [12, 1984] showed that a strategic-form proper equilibrium induces a sequential-equilibrium outcome in any tree with that strategic form.

But a very basic question has remained: What is the relationship between dominance in the matrix and dominance in the tree? And, following from this, what is the relationship between iterated dominance in the matrix and iterated dominance in the tree? This note addresses this question. See Gilli [5, 2002] and Shimoji [10, 2004] for related analyses.

## 1 Set-Up

For a finite set $X$, let $\mathcal{M}(X)$ be the set of probability measures on $X$ and $\mathcal{M}^{+}(X)$ the set of full-support measures on $X$. Given a set $Y \subseteq X$, we will often identify $\mathcal{M}(Y)$ (resp. $\mathcal{M}^{+}(Y)$) with

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the set of probability measures on $X$ with support contained in (resp. equal to) $Y$. We adopt the convention that if $X \times Y = \emptyset$ then $X = \emptyset$ and $Y = \emptyset$.

We begin with a finite **extensive-form game** $\Gamma$ between Ann and Bob.[1] Let $S^a$ (resp. $S^b$) be the set of strategies for Ann (resp. Bob). Note that $S^a$ and $S^b$ are finite. Let $H^a$ (resp. $H^b$) be the family of information sets at which Ann (resp. Bob) moves, and let $H = H^a \cup H^b$. Write $S^a(h)$ (resp. $S^b(h)$) for the set of Ann's strategies that allow information set $h$. Let $Z$ be the set of terminal nodes, and $\zeta : S^a \times S^b \to Z$ map each strategy profile to the terminal node it reaches. Extensive-form payoff functions are maps $\Pi^a : Z \to \mathbb{R}$ and $\Pi^b : Z \to \mathbb{R}$.

We restrict attention to extensive-form games with perfect recall (Kuhn [7, 1950], [8, 1953]). These games satisfy an important property. In perfect-recall games, we have: For all information sets $h$ and $i$, either $S^a(h) \subseteq S^a(i)$, $S^a(i) \subseteq S^a(h)$, or $S^a(h) \cap S^a(i) = \emptyset$.

An extensive-form game $\Gamma$ induces a **strategic-form game** $G = \langle S^a, S^b, \pi^a, \pi^b \rangle$, where $\pi^a = \Pi^a \circ \zeta$ and $\pi^b = \Pi^b \circ \zeta$. We extend $\pi^a$ to $\mathcal{M}(S^a) \times \mathcal{M}(S^b)$ in the usual way, i.e. $\pi^a(\sigma^a, \sigma^b) = \sum_{s^a \in S^a} \sum_{s^b \in S^b} \pi^a(s^a, s^b)\, \sigma^b(s^b)\, \sigma^a(s^a)$.

The following definitions all have counterparts with $a$ and $b$ reversed.

**Definition 1.1** *Fix $Y^a \times Y^b \subseteq S^a \times S^b$. A strategy $s^a \in Y^a$ is **(strongly) dominated with respect to** $Y^a \times Y^b$ if there exists $\sigma^a \in \mathcal{M}(Y^a)$ such that $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for every $s^b \in Y^b$. Otherwise, say $s^a$ is **undominated with respect to** $Y^a \times Y^b$. If $s^a$ is undominated with respect to $S^a \times S^b$, simply say that $s^a$ is **undominated**.*

**Definition 1.2** *Fix $Y^a \times Y^b \subseteq S^a \times S^b$. A strategy $s^a \in Y^a$ is **weakly dominated with respect to** $Y^a \times Y^b$ if there exists $\sigma^a \in \mathcal{M}(Y^a)$ such that $\pi^a(\sigma^a, s^b) \geq \pi^a(s^a, s^b)$ for every $s^b \in Y^b$, and $\pi^a(\sigma^a, s^b) > \pi^a(s^a, s^b)$ for some $s^b \in Y^b$. Otherwise, say $s^a$ is **admissible with respect to** $Y^a \times Y^b$. If $s^a$ is admissible with respect to $S^a \times S^b$, simply say that $s^a$ is **admissible**.*

We have the usual equivalences:

**Lemma 1.1** *A strategy $s^a \in Y^a$ is undominated with respect to $Y^a \times Y^b$ if and only if there exists $\sigma^b \in \mathcal{M}(Y^b)$ such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in Y^a$.*

**Lemma 1.2** *A strategy $s^a \in Y^a$ is admissible with respect to $Y^a \times Y^b$ if and only if there exists $\sigma^b \in \mathcal{M}^+(Y^b)$ such that $\pi^a(s^a, \sigma^b) \geq \pi^a(r^a, \sigma^b)$ for every $r^a \in Y^a$.*

We now define four procedures. The first two procedures are on the matrix. Let $G_0^a \times G_0^b = G_0^{+,a} \times G_0^{+,b} = S^a \times S^b$. Define $G_m^a \times G_m^b$ by induction, where $G_{m+1}^a$ is the set of strategies for Ann that are undominated with respect to $G_m^a \times G_m^b$ (and likewise with $a$ and $b$ reversed). Define $G_m^{+,a} \times G_m^{+,b}$ by induction, where $G_{m+1}^{+,a}$ is the set of strategies for Ann that are admissible with respect to $G_m^{+,a} \times G_m^{+,b}$ (and likewise with $a$ and $b$ reversed).

[1] The restriction to two-player games is for notational simplicity only.

**Definition 1.3** *Say $s^a$ is $m$-**rationalizable** if $s^a \in G_m^a$. Say $s^a$ is $m$-**admissible** if $s^a \in G_m^{+,a}$.*

The next two procedures are on the tree. Let $\Gamma_0^a \times \Gamma_0^b = \Gamma_0^{+,a} \times \Gamma_0^{+,b} = S^a \times S^b$. Define $\Gamma_m^a \times \Gamma_m^b$ by induction, as follows. A strategy $s^a$ lies in $\Gamma_{m+1}^a$ if and only if, for each information set $h$ with $s^a \in S^a(h)$, $s^a$ is undominated with respect to $[\Gamma_m^a \cap S^a(h)] \times [\Gamma_m^b \cap S^b(h)]$ (and likewise with $a$ and $b$ reversed). Define $\Gamma_m^{+,a} \times \Gamma_m^{+,b}$ by induction, as follows. A strategy $s^a$ is contained $\Gamma_{m+1}^{+,a}$ if and only if, for each information set $h$ with $s^a \in S^a(h)$, $s^a$ is admissible with respect to $[\Gamma_m^{+,a} \cap S^a(h)] \times [\Gamma_m^{+,b} \cap S^b(h)]$ (and likewise with $a$ and $b$ reversed).

**Definition 1.4** *Say $s^a$ is $m$-**extensive-form rationalizable** if $s^a \in \Gamma_m^a$. Say $s^a$ is $m$-**extensive-form admissible** if $s^a \in \Gamma_m^{+,a}$.*

The concept of extensive-form rationalizability is due to Pearce [9, 1984]. (See, also, Battigalli [1, 1997].)

## 2 Summary of Results

It is well known that:

$$G_m^{+,a} \times G_m^{+,b} \subseteq G_m^a \times G_m^b \text{ for all } m.$$

In Section 3 we show that:

$$\Gamma_m^{+,a} \times \Gamma_m^{+,b} = G_m^{+,a} \times G_m^{+,b} \text{ for all } m.$$

In Section 4 we show that under a condition on the tree we call No Relevant Convexities:

$$\Gamma_m^a \times \Gamma_m^b = \Gamma_m^{+,a} \times \Gamma_m^{+,b} \text{ for all } m.$$

## 3 Admissibility in the Matrix and the Tree

**Proposition 3.1** *Fix an extensive-form game $\Gamma$ with associated strategic form $G$. Then $\Gamma_m^{+,a} \times \Gamma_m^{+,b} = G_m^{+,a} \times G_m^{+,b}$ for all $m$.*

The proposition will follow immediately from:

**Lemma 3.1** *A strategy $s^a$ is admissible with respect to $Y^a \times Y^b$ if and only if, for each information set $h$ allowed by $s^a$, $s^a$ is (extensive-form) admissible with respect to $[Y^a \cap S^a(h)] \times \left[Y^b \cap S^b(h)\right]$.*

**Proof.** First, fix an $s^a$ that is admissible with respect to $Y^a \times Y^b$. Then there is a measure $\sigma^b \in \mathcal{M}^+\left(Y^b\right)$ with $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(r^a, \sigma^b\right)$ for all $r^a \in Y^a$. Fix an information set $h$ with $s^a \in S^a(h)$. We will show that if $Y^b \cap S^b(h) \neq \emptyset$, then $\pi^a\left(s^a, \sigma^b\left(\cdot|S^b(h)\right)\right) \geq \pi^a\left(r^a, \sigma^b\left(\cdot|S^b(h)\right)\right)$

for all $r^a \in Y^a \cap S^a(h)$. (Note, in this case, $\sigma^b\left(\cdot|S^b(h)\right)$ is well defined, since $Y^b \cap S^b(h) \neq \emptyset$ implies $\sigma^b\left(Y^b \cap S^b(h)\right) > 0$.)

Suppose not. Then there is an $r^a \in S^a(h)$ with $\pi^a\left(r^a, \sigma^b\left(\cdot|S^b(h)\right)\right) > \pi^a\left(s^a, \sigma^b\left(\cdot|S^b(h)\right)\right)$. Since $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(r^a, \sigma^b\right)$, we must have $\sigma^b\left(S^b(h)\right) < 1$. Let $q^a$ be the strategy that agrees with $r^a$ at $h$ onwards but otherwise agrees with $s^a$. We have

$$\begin{aligned}
\pi^a\left(q^a, \sigma^b\right) &= \sigma^b\left(S^b(h)\right)\pi^a\left(r^a, \sigma^b\left(\cdot|S^b(h)\right)\right) + \left(1-\sigma^b\left(S^b(h)\right)\right)\pi^a\left(s^a, \sigma^b\left(\cdot|S^b(h)\right)\right) \\
&> \sigma^b\left(S^b(h)\right)\pi^a\left(s^a, \sigma^b\left(\cdot|S^b(h)\right)\right) + \left(1-\sigma^b\left(S^b(h)\right)\right)\pi^a\left(s^a, \sigma^b\left(\cdot|S^b(h)\right)\right) \\
&= \pi^a\left(s^a, \sigma^b\right),
\end{aligned}$$

a contradiction.

Next suppose that, for each information set $h$ with $s^a \in S^a(h)$, $s^a$ is (extensive-form) admissible with respect to $\left[Y^a \cap S^a(h)\right] \times \left[Y^b \cap S^b(h)\right] \neq \emptyset$. If Ann moves at the root of the tree, it is immediate that $s^a$ is admissible with respect to $Y^a \times Y^b$. If not, we can find information sets $1, .., K \in H$ so that: (i) $S^a = S^a(k)$ for each $k = 1, .., K$; and (ii) the sets $S^b(1), \ldots, S^b(K)$ form a partition of $S^b$.

Fix $J \leq K$ so that $Y^b \cap S^b(k) \neq \emptyset$ for all $1 \leq k \leq J$ and $Y^b \cap S^b(k) = \emptyset$ for all $J < k \leq K$. For each $k \leq J$, there is a measure $\sigma_k^b \in \mathcal{M}^+(Y^b \cap S^b(k))$ with $\pi^a\left(s^a, \sigma_k^b\right) \geq \pi^a\left(r^a, \sigma_k^b\right)$ for all $r^a \in Y^a$. Build $\sigma^b$ so that, for each $s^b \in Y^b$, $\sigma^b\left(s^b\right) = \frac{1}{J}\sigma_k^b\left(s^b\right)$, where $s^b \in S^b(k)$. It is immediate that this defines a probability measure in $\mathcal{M}^+(Y^b)$, and, moreover,

$$\pi^a\left(q^a, \sigma^b\right) = \frac{1}{J}\sum_{k=1}^{J}\pi^a\left(q^a, \sigma_k^b\right),$$

for each $q^a \in Y^a$. Therefore $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(r^a, \sigma^b\right)$ for all $r^a \in Y^a$. This establishes that $s^a$ is admissible with respect to $Y^a \times Y^b$. ■

**Proof of Proposition 3.1.** The result is immediate for $m = 0$. Assuming the result for $m \geq 1$ and applying Lemma 3.1 gives the result for $m + 1$. ■

# 4 No Relevant Convexities

**Definition 4.1** *Say $r^a$ **supports** $s^a$ **with respect to** $Y^b \subseteq S^b$ if there exists $\sigma^a \in \mathcal{M}(S^a)$ such that $r^a \in \operatorname{Supp}\sigma^a$ and $\pi^a\left(\sigma^a, s^b\right) = \pi^a\left(s^a, s^b\right)$ for all $s^b \in Y^b$.*

**Definition 4.2** *An extensive-form game $\Gamma$ satisfies **No Relevant Convexities (NRC)** if whenever $r^a$ supports $s^a$ with respect to some $Y^b \subseteq S^b$, then $\zeta\left(s^a, s^b\right) = \zeta\left(r^a, s^b\right)$ for each $s^b \in Y^b$.*

NRC is a strengthening of the No Relevant Ties condition, due to Battigalli [1, 1997].

**Proposition 4.1** *Fix an extensive-form game $\Gamma$ satisfying NRC. Then $\Gamma_m^a \times \Gamma_m^b = \Gamma_m^{+,a} \times \Gamma_m^{+,b}$ for all $m$.*

The proof will make use of the following implication of NRC.

**Lemma 4.1** *Fix an extensive-form game* $\Gamma$ *satisfying NRC and let* $Y^a \times Y^b \subseteq S^a \times S^b$*. Then the following are equivalent:*

*(a) The strategy* $s^a$ *is undominated with respect to* $Y^a \times Y^b$*.*

*(b) There exists* $\sigma^b \in \mathcal{M}\left(Y^b\right)$ *with*

*(i)* $\pi^a\left(s^a, \sigma^b\right) \geq \pi^a\left(r^a, \sigma^b\right)$ *for all* $r^a \in Y^a$*;*

*(ii) if* $r^a \in Y^a$ *satisfies* $\pi^a\left(r^a, \sigma^b\right) = \pi^a\left(s^a, \sigma^b\right)$*, then* $\zeta\left(r^a, s^b\right) = \zeta\left(r^a, s^b\right)$ *for every* $s^b \in \operatorname{Supp} \sigma^b$*.*

**Proof.** Certainly (b) implies (a) (for any game). We will show (a) implies (b). Part (i) is immediate by Lemma 1.1. We will show that we can choose $\sigma^b$ so that part (ii) holds.

We claim that we can find $Z^b \subseteq Y^b$ and $\rho^b \in \mathcal{M}^+\left(Z^b\right)$ satisfying the following: Strategy $r^a$ satisfies $\pi^a\left(r^a, \rho^b\right) \geq \pi^a\left(q^a, \rho^b\right)$ for all $q^a \in Y^a$ if and only if $r^a$ supports $s^a$ with respect to $Z^b$. (Note, $s^a$ supports $s^a$ with respect to $Z^b$, for any choice of $Z^b$.) To see this, note that, since $s^a$ is undominated with respect to $Y^a \times Y^b$, $s^a$ must be admissible with respect to $Y^a \times Z^b$ for some $Z^b \subseteq Y^b$. Lemma D.4 in Brandenburger-Friedenberg-Keisler [2, 2007] then guarantees we can find $\rho^b$ satisfying the above conditions.

Certainly $\pi^a\left(s^a, \rho^b\right) \geq \pi^a\left(r^a, \rho^b\right)$ for all $r^a \in Y^a$. Suppose there is some $r^a \in Y^a$ with $\pi^a\left(s^a, \rho^b\right) = \pi^a\left(r^a, \rho^b\right)$. Then, $r^a$ supports $s^a$ with respect to $Z^b = \operatorname{Supp} \rho^b$. So, by NRC, $\zeta\left(s^a, s^b\right) = \zeta\left(r^a, s^b\right)$ for each $s^b \in \operatorname{Supp} \rho^b$. ■

**Proof of Proposition 4.1.** The proof is by induction on $m$. The result is immediate for $m = 0$. Assume the result holds for $m \geq 1$. By the induction hypothesis, $\Gamma^{+,a}_{m+1} \times \Gamma^{+,b}_{m+1} \subseteq \Gamma^a_{m+1} \times \Gamma^b_{m+1}$. We will show that $\Gamma^a_{m+1} \times \Gamma^b_{m+1} \subseteq \Gamma^{+,a}_{m+1} \times \Gamma^{+,b}_{m+1}$.

Fix $s^a \in \Gamma^a_{m+1}$. Let $\overline{H}$ be the family of information sets $h$ allowed by $\{s^a\} \times \Gamma^b_m$. It suffices to show that, for each $h \in \overline{H}$, $s^a$ is admissible with respect to $\Gamma^a_m \times \Gamma^b_m$. If so, the result follows from the induction hypothesis.

Suppose not. That is, there is an information set $h \in \overline{H}$ such that $s^a$ is inadmissible with respect to $[\Gamma^a_m \cap S^a(h)] \times [\Gamma^b_m \cap S^b(h)]$. In particular, pick $h$ so that, for each $i \in \overline{H}$ with $S(i) \subsetneq S(h)$, $s^a$ is admissible with respect to $[\Gamma^a_m \cap S^a(i)] \times [\Gamma^b_m \cap S^b(i)]$.

Begin with the fact that $s^a \in \Gamma^a_{m+1}$. By Lemma 4.1, there exists $\sigma^b_h \in \mathcal{M}\left(\Gamma^b_m \cap S^b(h)\right)$ such that: (1) $\pi^a\left(s^a, \sigma^b_h\right) \geq \pi^a\left(r^a, \sigma^b_h\right)$ for all $r^a \in \Gamma^a_m \cap S^a(h)$; and (2) if there is $r^a \in \Gamma^a_m \cap S^a(h)$ with $\pi^a\left(s^a, \sigma^b_h\right) = \pi^a\left(r^a, \sigma^b_h\right)$, then $\zeta\left(s^a, s^b\right) = \zeta\left(r^a, s^b\right)$ for all $s^b \in \operatorname{Supp} \sigma^b_h$.

We divide the remainder of the proof into two cases. The first is where Ann moves at $h$ and the second is where she does not.

**Case I: Ann moves at** $h$**.** Here, we can find information sets $1, \ldots, K$ so that $S^a(1), \ldots, S^a(K)$ form a partition of $S^a(h)$ and $S^b(k) = S^b(h)$ for all $k = 1, \ldots, K$. Suppose there is a strategy

$r^a$ such that $\pi^a\left(s^a,\sigma^b_h\right)=\pi^a\left(r^a,\sigma^b_h\right)$. Then $r^a$ must specify the same move as $s^a$ at $h$. (This is property (2) above.) Recall that $s^a$ is admissible with respect to $[\Gamma^a_m\cap S^a(k)]\times[\Gamma^b_m\cap S^b(k)]$. Since $S^b(k)=S^b(h)$, there is a $\rho^b_h\in\mathcal{M}^+\left(\Gamma^b_m\cap S^b(h)\right)$ with $\pi^a\left(s^a,\rho^b_h\right)\geq\pi^a\left(q^a,\rho^b_h\right)$ for all $q^a\in\Gamma^a_m\cap S^a(k)$. Moreover, if $\pi^a\left(s^a,\sigma^b_h\right)=\pi^a\left(r^a,\sigma^b_h\right)$, then $\pi^a\left(s^a,\rho^b_h\right)\geq\pi^a\left(r^a,\rho^b_h\right)$. So, we can build a $\delta^b_h\in\mathcal{M}^+\left(\Gamma^b_m\cap S^b(h)\right)$ by setting $\delta^b_h\left(s^b\right)=(1-\varepsilon)\,\sigma^b_h\left(s^b\right)+\varepsilon\rho^b_h\left(s^b\right)$, where $0<\varepsilon<1$. Then, there exists $\varepsilon>0$ so that $\pi^a\left(s^a,\delta^b_h\right)\geq\pi^a\left(q^a,\delta^b_h\right)$ for all $q^a\in\Gamma^a_m\cap S^a(h)$. (Here, we use the fact that if $\pi^a\left(s^a,\sigma^b_h\right)=\pi^a\left(r^a,\sigma^b_h\right)$, then $\pi^a\left(s^a,\rho^b_h\right)\geq\pi^a\left(r^a,\rho^b_h\right)$.) This contradicts the fact that $s^a$ is inadmissible with respect to $[\Gamma^a_m\cap S^a(h)]\times[\Gamma^b_m\cap S^b(h)]$.

**Case II: Ann doesn't move at $h$.** Here, we can find information sets $1,\ldots,K$ so that $S^b(1),\ldots,S^b(K)$ form a partition of $S^b(h)$ and $S^a(k)=S^a(h)$ for all $k=1,\ldots,K$. Order the information sets so that $\Gamma^b_m\cap S^b(k)\neq\emptyset$ if $1\leq k\leq J\leq K$ and $\Gamma^b_m\cap S^b(k)=\emptyset$ if $J<k\leq K$. For each $k=1,\ldots,J$, there exists $\sigma^b_k\in\mathcal{M}^+\left(\Gamma^b_m\times S^b(k)\right)$ such that $\pi^a\left(s^a,\sigma^b_k\right)\geq\pi^a\left(q^a,\sigma^b_k\right)$ for all $q^a\in\Gamma^a_m\cap S^a(h)$. Build $\sigma^b\in\mathcal{M}^+\left(\Gamma^b_m\cap S^b(h)\right)$ so that $\sigma^b\left(s^b\right)=\frac{1}{J}\sigma^b_k\left(s^b\right)$ where $s^b\in S^b(k)$. Then

$$\pi^a\left(q^a,\sigma^b\right)=\frac{1}{J}\textstyle\sum_{k=1}^{J}\pi^a\left(q^a,\sigma^b_k\right)$$

for any $q^a\in\Gamma^a_m\cap S^a(h)$. It follows (using $\sigma^b$) that $s^a$ is admissible with respect to $[\Gamma^a_m\cap S^a(h)]\times[\Gamma^b_m\cap S^b(h)]$, again a contradiction. ■

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