Cooperative Game Theory: The Core

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1 The Core

Given a cooperative game (N, v), recall the following definitions from the note " Cooperative Game Theory: Characteristic Functions, Allocations, Marginal Contribution":

(i) an *allocation* is a collection $(x_1, x_2, \ldots, x_n)$ of numbers;

(ii) an allocation $(x_1, x_2, \ldots, x_n)$ is *individually rational* if $x_i \geq v(\{i\})$ for all i;

(iii) an allocation $(x_1, x_2, \ldots, x_n)$ is *efficient* if $\sum_{i=1}^{n} x_i = v(N)$;

(iv) an (individually rational and efficient) allocation $(x_1, x_2, \ldots, x_n)$ satisfies the *Marginal-Contribution Principle* if $x_i \leq \mathrm{MC}_i$ for all i.

Some additional notation will be useful. For any subset S of the set of players N, let $x(S) = \sum_{i \in S} x_i$. In words, the term $x(S)$ denotes the sum of the values received by each of the players i in the subset S.

Definition 1 *An allocation $(x_1, x_2, \ldots, x_n)$ is said to lie in the* ***core*** *of the game if it is efficient and is such that for every subset S of N we have $x(S) \geq v(S)$.*

Two observations are in order. First, an allocation that lies in the core is individually rational. To see this, let $S = \{i\}$ for some $i = 1, 2, \ldots, n$. Note that $x(\{i\}) = x_i$. (Both are the value received by player i.). Thus, the core condition that $x(\{i\}) \geq v(\{i\})$ is precisely the individual rationality condition.

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Second, note that if an allocation lies in the core then certainly it satisfies the Marginal-Contribution Principle. To see this, consider a particular player i and let $S = N\backslash\{i\}$. The core condition says that $x(N\backslash\{i\}) \geq v(N\backslash\{i\})$. The efficiency condition says that $x(N) = v(N)$. But $x_i = x(N) - x(N\backslash\{i\})$ by definition. Putting all this together gives $x_i \leq v(N) - v(N\backslash\{i\})$, which is exactly the condition of the Marginal-Contribution Principle.

In fact, the core can be thought of as a generalization of the Marginal-Contribution Principle. To demonstrate this, we first need to define the marginal contribution of a group of players. (So far, we have considered only the marginal contribution of an individual player.)

Definition 2 *The **marginal contribution** of a subset S of players is $v(N) - v(N\backslash S)$, to be denoted by MC_S.*

(Under this new notation, the marginal contribution of the subset of players consisting of player i alone should be denoted by $\mathrm{MC}_{\{i\}}$. But no confusion will result if we continue to write MC_i for $\mathrm{MC}_{\{i\}}$.)

Theorem 1 *An efficient allocation $(x_1, x_2, \ldots, x_n)$ lies in the core if and only if for every subset S of N we have $x(S) \leq \mathrm{MC}_S$.*

Proof. Suppose that the allocation $(x_1, x_2, \ldots, x_n)$ is efficient and lies in the core. Then $x(N) = v(N)$ by efficiency. Now consider the subset $N\backslash S$, and use the core condition $x(N\backslash S) \geq v(N\backslash S)$. Since $x(N) = x(N\backslash S) + x(S)$, we can rearrange terms to get $x(S) \leq v(N) - v(N\backslash S) = \mathrm{MC}_S$, as required.

Conversely, suppose that the allocation $(x_1, x_2, \ldots, x_n)$ is efficient and satisfies $x(S) \leq \mathrm{MC}_S$ for every subset S of N. Then $x(N) = v(N)$ by efficiency. Now consider the subset $N\backslash S$, and use the condition $x(N\backslash S) \leq \mathrm{MC}_{N\backslash S} = v(N) - v(S)$. Since $x(N) = x(N\backslash S) + x(S)$, we can rearrange terms to get $x(S) \geq v(S)$, as required. ■

Theorem 1 makes clear that the motivations for the Marginal-Contribution Principle and the core are similar. Indeed, the core is another expression of the idea that 'no good deal goes undone.' If any group of players, say S, anticipated capturing less value in total than the group could create on its own, i.e. if $x(S) < v(S)$, then the players in this group would do better to create and divide the value $v(S)$ by themselves. This is the 'good deal' that can't go 'undone' according to the core, and is why the core imposes the condition that $x(S) \geq v(S)$.

2 Examples

Example 1 *Consider a cooperative game with two sellers and two buyers. Each seller is offering to sell one unit of a product. The first seller can make its product available at a cost of* \$2. *The second seller can make its product available at a cost of* \$4. *The first buyer has a willingness-to-pay for either product of* \$8, *and is interested in acquiring only one unit. The second buyer has a willingness-to-pay for either product of* \$6, *and also is interested in acquiring only one unit.*

(i) What divisions of value are possible in the core?

(ii) What divisions of value satisfy the Marginal-Contribution Principle?

(iii) Consider a division of value that satisfies the Marginal-Contribution Principle, but that is not in the core. Provide an argument as to why it is a reasonable outcome, then provide an argument as to why it is not reasonable.

Example 2 *Consider a cooperative game with two sellers and three buyers. Each seller has two units to sell at a cost of* \$0 *per unit. Each buyer is interested in buying one unit at a willingness-to-pay of* \$1 *for either seller's product.*[1]

(i) What divisions of value are possible in the core?

(ii) What divisions of value satisfy the Marginal-Contribution Principle?

(iii) What divisions of value do you consider plausible in this game?

Example 3 *Consider a cooperative game with two suppliers (labelled s_1 and s_2), two firms (labelled f_1 and f_2), and two buyers (labelled b_1 and b_2). For value to be created, a supplier, firm, and buyer must come together and transact, as follows. Each of the combinations*

$$\{s_1, f_1, b_1\},$$
$$\{s_2, f_2, b_1\},$$
$$\{s_2, f_1, b_2\},$$
$$\{s_1, f_2, b_2\},$$

creates \$1 *of value.*

What divisions of value satisfy the Marginal-Contribution Principle?

[1] Adapted from Postlewaite, A., and R. Rosenthal, "Disadvantageous Syndicates," *Journal of Economic Theory*, 9, 1974, 324-326.