Modelling Knowledge

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Version 01/04/07

In the note "Common Knowledge," we represented the information a person possesses by a partition $\mathcal{P}$ of the finite set Ω of possible states of the world. The interpretation is that, if the true state is ω, the person is informed of the member of $\mathcal{P}$ that contains ω, denoted by $\mathcal{P}(\omega)$. It is then natural to define the event that the person knows an event $E \subseteq \Omega$, to be written by $\mathrm{K}E$, by

$$\mathrm{K}E = \{\omega \in \Omega : \mathcal{P}(\omega) \subseteq E\}.$$

It is not hard to see that K, considered as a map from $2^{\Omega} \to 2^{\Omega}$ (where 2^{Ω} is the set of all subsets of Ω), has the properties listed below. (Give an interpretation of each of these properties.)

P1: For any $E \subseteq \Omega$, $\mathrm{K}E \subseteq E$.

P2: For any $E, F \subseteq \Omega$, if $E \subseteq F$, then $\mathrm{K}E \subseteq \mathrm{K}F$.

P3: For any $E \subseteq \Omega$, $(\mathrm{K}E)^{\mathrm{c}} \subseteq \mathrm{K}(\mathrm{K}E)^{\mathrm{c}}$ where the superscript c denotes complement.

Here are the proofs:

Proof of P1: If $\omega \in \mathrm{K}E$, then $\mathcal{P}(\omega) \subseteq E$. But $\omega \in \mathcal{P}(\omega)$, so $\omega \in E$.

Proof of P2: If $\omega \in \mathrm{K}E$, then $\mathcal{P}(\omega) \subseteq E$. But then certainly $\mathcal{P}(\omega) \subseteq F$, i.e. $\omega \in \mathrm{K}F$.

Proof of P3: If $\omega \in (\mathrm{K}E)^{\mathrm{c}}$, then $\mathcal{P}(\omega) \nsubseteq E$. Suppose there exists some $\omega' \in \mathcal{P}(\omega) \cap \mathrm{K}E$. Then $\omega' \in \mathcal{P}(\omega)$ implies $\mathcal{P}(\omega') = \mathcal{P}(\omega) \nsubseteq E$, contradicting $\omega' \in \mathrm{K}E$. Thus $\mathcal{P}(\omega) \cap \mathrm{K}E = \emptyset$. This says that $\mathcal{P}(\omega) \subseteq (\mathrm{K}E)^{\mathrm{c}}$, or $\omega \in \mathrm{K}(\mathrm{K}E)^{\mathrm{c}}$.

Let's list some further properties of the map K:

P4: $\mathrm{K}\Omega = \Omega$. (Proof: Setting $E = (\mathrm{K}\Omega)^{\mathrm{c}}$ and $F = \Omega$ in P2 gives $\mathrm{K}(\mathrm{K}\Omega)^{\mathrm{c}} \subseteq \mathrm{K}\Omega$. Thus $(\mathrm{K}\Omega)^{\mathrm{c}} \subseteq \mathrm{K}\Omega$, using P3, from which $\mathrm{K}\Omega = \Omega$ as required.)

*With the assistance of Amanda Friedenberg. knowledge-01-04-07

P5: $\mathrm{K}E \subseteq \mathrm{KK}E$. (Proof: We have $(\mathrm{K}E)^{\mathrm{c}} = \mathrm{K}(\mathrm{K}E)^{\mathrm{c}}$ by P1 and P3, and so $\mathrm{K}E = (\mathrm{K}\,(\mathrm{K}E)^{\mathrm{c}})^{\mathrm{c}}$. Thus $\mathrm{KK}E = \mathrm{K}(\mathrm{K}\,(\mathrm{K}E)^{\mathrm{c}})^{\mathrm{c}} \supseteq (\mathrm{K}\,(\mathrm{K}E)^{\mathrm{c}})^{\mathrm{c}}$, using P3. But $(\mathrm{K}\,(\mathrm{K}E)^{\mathrm{c}})^{\mathrm{c}} = \mathrm{K}E$, again using P1 and P3.)

P6: $\mathrm{K}(E \cap F) = \mathrm{K}E \cap \mathrm{K}F$. (Proof: Immediate from P2.)

A nice result is that the properties P1-P3 actually characterize partition information in the following sense. (In the proof below, we use P4-P6 as well, which we can do since we derived these using only P1-P3.)

Theorem 1 *If a function $K : 2^{\Omega} \to 2^{\Omega}$ satisfies P1-P3 then there is a partition $\mathcal{P}$ of Ω such that $KE = \{\omega \in \Omega : \mathcal{P}(\omega) \subseteq E\}$.*

Proof. For any $\omega \in \Omega$, define

$$\mathcal{P}(\omega) = \bigcap_{\{E:\omega\in\, \mathrm{K}E\}} E.$$

The following must be shown:

(i) $\omega \in \mathrm{K}E$ if and only if $\mathcal{P}(\omega) \subseteq E$;

(ii) $\omega \in \mathcal{P}(\omega)$;

(iii) if $\omega' \in \mathcal{P}(\omega)$, then $\mathcal{P}(\omega') = \mathcal{P}(\omega)$.

First, use P1 to write for any ω:

$$\bigcap_{\{E:\omega\in\, \mathrm{K}E\}} \mathrm{K}E \subseteq \bigcap_{\{E:\omega\in\, \mathrm{K}E\}} E. \tag{a}$$

Now, from P4 we certainly have $\omega \in \mathrm{K}\Omega$. Thus $\{E : \omega \in \mathrm{K}E\} \neq \emptyset$, and so we get

$$\omega \in \bigcap_{\{E:\omega\in\, \mathrm{K}E\}} \mathrm{K}E \subseteq \bigcap_{\{E:\omega\in\, \mathrm{K}E\}} E = \mathcal{P}(\omega),$$

establishing (ii).

Next, use P1 and P5 to get $\mathrm{K}E = \mathrm{KK}E$, so that

$$\bigcap_{\{E:\omega\in\, \mathrm{K}E\}} \mathrm{K}E = \bigcap_{\{E:\omega\in\, \mathrm{KK}E\}} \mathrm{K}E \supseteq \bigcap_{\{E:\omega\in\, \mathrm{K}E\}} E. \tag{b}$$

By (a) and (b),

$$\mathcal{P}(\omega) = \bigcap_{\{E:\omega\in\, \mathrm{K}E\}} E = \bigcap_{\{E:\omega\in\, \mathrm{K}E\}} \mathrm{K}E, \tag{c}$$

so that

$$(\mathcal{P}(\omega))^{\mathrm{c}} = \bigcup_{\{E:\omega\in\, \mathrm{K}E\}} (\mathrm{K}E)^{\mathrm{c}}.$$

Using P2 and P3,

$$\mathrm{K}\left(\mathcal{P}\left(\omega\right)\right)^{\mathrm{c}}=\mathrm{K}\bigcup_{\{E:\omega\in\,\mathrm{K}E\}}\left(\mathrm{K}E\right)^{\mathrm{c}}\supseteq\bigcup_{\{E:\omega\in\,\mathrm{K}E\}}\mathrm{K}\left(\mathrm{K}E\right)^{\mathrm{c}}=$$
$$\bigcup_{\{E:\omega\in\,\mathrm{K}E\}}\left(\mathrm{K}E\right)^{\mathrm{c}}=(\bigcap_{\{E:\omega\in\,\mathrm{K}E\}}\mathrm{K}E)^{\mathrm{c}}=\left(\mathcal{P}\left(\omega\right)\right)^{\mathrm{c}}.$$

We have shown that $(\mathrm{K}\left(\mathcal{P}\left(\omega\right)\right)^{\mathrm{c}})^{\mathrm{c}}\subseteq\mathcal{P}\left(\omega\right)$, from which $(\mathrm{K}\left(\mathcal{P}\left(\omega\right)\right)^{\mathrm{c}})^{\mathrm{c}}=\mathrm{K}(\mathrm{K}\left(\mathcal{P}\left(\omega\right)\right)^{\mathrm{c}})^{\mathrm{c}}\subseteq\mathrm{K}\mathcal{P}\left(\omega\right)$, using P2 and P3. Thus $\left(\mathrm{K}\mathcal{P}\left(\omega\right)\right)^{\mathrm{c}}\subseteq\mathrm{K}(\mathcal{P}\left(\omega\right))^{\mathrm{c}}\subseteq\left(\mathcal{P}\left(\omega\right)\right)^{\mathrm{c}}$, by P1, from which $\mathcal{P}\left(\omega\right)\subseteq\mathrm{K}\mathcal{P}\left(\omega\right)$.

To establish (i), notice that by definition of $\mathcal{P}\left(\omega\right)$, if $\omega\in\mathrm{K}E$ then $\mathcal{P}\left(\omega\right)\subseteq E$. Conversely, suppose $\mathcal{P}\left(\omega\right)\subseteq E$. Using $\mathcal{P}\left(\omega\right)\subseteq\mathrm{K}\mathcal{P}\left(\omega\right)$, (ii), and P2, we get $\omega\in\mathcal{P}\left(\omega\right)\subseteq\mathrm{K}\mathcal{P}\left(\omega\right)\subseteq\mathrm{K}E$.

Finally, (iii) must be shown. If $\omega'\in\mathcal{P}\left(\omega\right)$ then by (c),

$$\omega'\in\bigcap_{\{E:\omega\in\,\mathrm{K}E\}}\mathrm{K}E.$$

Thus $\omega\in\mathrm{K}E$ implies $\omega'\in\mathrm{K}E$. From this we get

$$\mathcal{P}\left(\omega'\right)=\bigcap_{\{E:\omega'\in\,\mathrm{K}E\}}E\subseteq\bigcap_{\{E:\omega\in\,\mathrm{K}E\}}E=\mathcal{P}\left(\omega\right). \tag{d}$$

If $\omega\in\mathcal{P}\left(\omega'\right)$, an analogous argument yields $\mathcal{P}\left(\omega\right)\subseteq\mathcal{P}\left(\omega'\right)$, thereby establishing (iii). So suppose $\omega\in\mathcal{P}\left(\omega\right)\cap\mathcal{P}\left(\omega'\right)^{\mathrm{c}}$.

We know that $\mathrm{K}\mathcal{P}\left(\omega'\right)\subseteq\mathcal{P}\left(\omega'\right)$, and so $\mathrm{K}\mathcal{P}\left(\omega'\right)=\mathcal{P}\left(\omega'\right)$, using P1. Thus, using P4,

$$\mathcal{P}\left(\omega'\right)^{\mathrm{c}}=\left(\mathrm{K}\mathcal{P}\left(\omega'\right)\right)^{\mathrm{c}}=\mathrm{K}\left[\left(\mathrm{K}\mathcal{P}\left(\omega'\right)\right)^{\mathrm{c}}\right]=\mathrm{K}\left[\left(\mathcal{P}\left(\omega'\right)\right)^{\mathrm{c}}\right].$$

We also have $\mathrm{K}\mathcal{P}\left(\omega\right)=\mathcal{P}\left(\omega\right)$, so that, using P6,

$$\mathcal{P}\left(\omega\right)\cap\mathcal{P}\left(\omega'\right)^{\mathrm{c}}=\mathrm{K}\mathcal{P}\left(\omega\right)\cap\ \mathrm{K}\left[\left(\mathcal{P}\left(\omega'\right)\right)^{\mathrm{c}}\right]=\ \mathrm{K}\left[\mathcal{P}\left(\omega\right)\cap\left(\mathcal{P}\left(\omega'\right)\right)^{\mathrm{c}}\right].$$

Using the definition of $\mathcal{P}\left(\omega\right)$ we then get $\mathcal{P}\left(\omega\right)\subseteq\mathcal{P}\left(\omega\right)\cap\mathcal{P}\left(\omega'\right)^{\mathrm{c}}$, i.e. $\mathcal{P}\left(\omega\right)\subseteq\mathcal{P}\left(\omega'\right)^{\mathrm{c}}$. But this contradicts (d). It follows that $\omega\in\mathcal{P}\left(\omega'\right)$, and (iii) is established. ■

Reference:

Aumann, R., "Interactive Epistemology I: Knowledge," *Int. J. Game Theory*, 28, 1999, 263-300.