

# Nash Equilibrium: Existence

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An  $n$ -player game  $\Gamma$  in strategic form is a  $2n$ -tuple  $\Gamma = \langle S^1, \dots, S^n, \pi^1, \dots, \pi^n \rangle$  where  $S^i$  is player  $i$ 's set of pure strategies and  $\pi^i : \prod_{j=1}^n S^j \rightarrow \mathfrak{R}$  is  $i$ 's payoff function. If each  $S^i$  is a finite set, then  $\Gamma$  is called a finite game. A mixed strategy for player  $i$  is a probability measure on  $i$ 's set of pure strategies. So, player  $i$ 's set of mixed strategies is

$$\Delta(S^i) = \left\{ \sigma^i : S^i \rightarrow \mathfrak{R}_+ \mid \sum_{s^i \in S^i} \sigma(s^i) = 1 \right\},$$

where  $\sigma^i(s^i)$  is the probability with which  $i$  chooses the pure strategy  $s^i \in S^i$ . If the players choose the  $n$ -tuple of mixed strategies  $\sigma = (\sigma^1, \dots, \sigma^n)$ , then the expected payoff to player  $i$  is

$$v^i(\sigma) = \sum_{(s^1, \dots, s^n) \in S^1 \times \dots \times S^n} \prod_{j=1}^n \sigma^j(s^j) \pi^i(s^1, \dots, s^n).$$

The following notation will be helpful:

$$\begin{aligned} \sigma^{-i} &= (\sigma^1, \dots, \sigma^{i-1}, \sigma^{i+1}, \dots, \sigma^n), \\ (\sigma^{-i}, \tilde{\sigma}^i) &= (\sigma^1, \dots, \sigma^{i-1}, \tilde{\sigma}^i, \sigma^{i+1}, \dots, \sigma^n). \end{aligned}$$

**Definition 1** An  $n$ -tuple of mixed strategies  $\sigma = (\sigma^1, \dots, \sigma^n)$  is a **Nash equilibrium** if for every  $i$  it is true that  $v^i(\sigma) \geq v^i(\sigma^{-i}, \tilde{\sigma}^i)$  for every  $\tilde{\sigma}^i \in \Delta(S^i)$ .

**Theorem 1 (Nash)** Every finite game possesses a Nash equilibrium.

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Nash's theorem will be proved as an easy corollary of a more general existence theorem.

**Theorem 2** Consider an  $n$ -player game  $\Gamma = \langle D^1, \dots, D^n, v^1, \dots, v^n \rangle$  where  $D^i$  is the set of pure strategies available to player  $i$  and  $v^i : \prod_{j=1}^n D^j \rightarrow \mathfrak{R}$  is  $i$ 's payoff function. If each  $D^i$  is a compact and convex subset of Euclidean space, and each  $v^i$  is quasiconcave in  $d^i$  and continuous, then  $\Gamma$  has a Nash equilibrium in pure strategies.

(To say that  $v^i$  is quasiconcave in  $d^i$  means that

$$v^i(d^{-i}, \alpha d^i + (1 - \alpha) \tilde{d}^i) \geq \min \left\{ v^i(d^{-i}, d^i), v^i(d^{-i}, \tilde{d}^i) \right\}$$

for every  $\alpha \in [0, 1]$ .) Theorem 2 is in turn an immediate consequence of Kakutani's Fixed Point Theorem. Before stating Kakutani's theorem, a definition will be useful.

**Definition 2** A correspondence  $\phi$  from a subset  $T$  of Euclidean space to a compact subset  $V$  of Euclidean space is **upper hemicontinuous at a point**  $x \in T$  if  $x_r \rightarrow x$ ,  $y_r \rightarrow y$ , where  $y_r \in \phi(x_r)$  for every  $r$ , implies  $y \in \phi(x)$ . The correspondence  $\phi$  is **upper hemicontinuous** if it is upper hemicontinuous at every  $x \in T$ .

**Theorem 3 (Kakutani)** If  $T$  is a nonempty compact and convex subset of Euclidean space, and  $\phi$  is an upper hemicontinuous, nonempty, and convex-valued correspondence from  $T$  to  $T$ , then  $\phi$  has a fixed point, that is, there is an  $x \in T$  such that  $x \in \phi(x)$ .

**Proof of Theorem 2.** For each  $i$  define a *best reply* correspondence  $\phi^i$  from  $\prod_{j=1}^n D^j$  to  $D^i$  as follows. For any  $d \in \prod_{j=1}^n D^j$ , let

$$\phi^i(d) = \left\{ \hat{d}^i \in D^i \mid v^i(d^{-i}, \hat{d}^i) \geq v^i(d^{-i}, \tilde{d}^i) \text{ for every } \tilde{d}^i \in D^i \right\}.$$

The set  $\phi^i(d)$  is the set of strategies that maximizes  $i$ 's payoff given the strategies of the other players prescribed by  $d$ ; it is nonempty since  $D^i$  is compact and  $v^i$  is continuous. The correspondence  $\phi^i(d)$  is convex-valued since  $v^i$  is quasiconcave in  $d^i$ . To see that  $\phi^i$  is upper hemicontinuous, consider a sequence  $d_r$  in  $\prod_{j=1}^n D^j$  converging to  $d$ , and a sequence  $\hat{d}_r^i$  in  $D^i$  converging to  $\hat{d}^i$ , where  $\hat{d}_r^i \in \phi^i(d_r)$

for every  $r$ . For any  $\tilde{d}_i \in D^i$ , we have  $v^i(d_r^{-i}, \hat{d}_r^i) \geq v^i(d_r^{-i}, \tilde{d}_r^i)$ . Therefore  $v^i(d^{-i}, \hat{d}^i) \geq v^i(d^{-i}, \tilde{d}^i)$ , since  $v^i$  is continuous, i.e.  $\hat{d}^i \in \phi(d)$ . This shows that  $\phi^i$  is upper hemicontinuous. Now define a correspondence  $\phi$  from  $\prod_{i=1}^n D^i$  to

$\prod_{i=1}^n D^i$  by

$$\phi(d) = \phi^1(d) \times \cdots \times \phi^n(d).$$

The set  $\prod_{i=1}^n D^i$  is a compact and convex subset of Euclidean space since each  $D^i$  is. The correspondence  $\phi$  is upper hemicontinuous, nonempty, and convex-valued since each  $\phi^i$  is. And it is easy to see that a fixed point of  $\phi$  is just a Nash equilibrium of  $\Gamma$ . ■

Finally, we have to say why Theorem 1 follows from Theorem 2. To see this, just let  $D^i = \Delta(S^i)$ . Each  $D^i$  is then a compact and convex subset of a Euclidean space, and each  $v^i$  is quasiconcave in  $d^i$  (in fact, linear in each variable) and continuous. So, by Theorem 2, there is a Nash equilibrium of the game in which each player  $i$  chooses a pure strategy from  $\Delta(S^i)$ . But this is, of course, just a Nash equilibrium in mixed strategies of the original game.