In general, the value of any asset is the present value of the expected cash flows on that asset. In this appendix, we will consider an exception to that rule when we will look at assets with two specific characteristics:

- They derive their value from the values of other assets.
- The cash flows on the assets are contingent on the occurrence of specific events.

These assets are called options, and the present value of the expected cash flows on them will understate their true value. We will describe the cash flow characteristics of options, consider the factors that determine their value, and examine how best to value them.

**Cash Flows on Options**

There are two types of options. A *call option* gives the buyer of the option the right to buy the underlying asset at a fixed price, whereas a *put option* gives the buyer the right to sell the underlying asset at a fixed price. In both cases, the fixed price at which the underlying asset can be bought or sold is called the *strike* or *exercise price*.

To look at the payoffs on an option, consider first the case of a call option. When you acquire the right to buy an asset at a fixed price, you want the price of the asset to increase above that fixed price. If it does, you make a profit, because you can buy at the fixed price and then sell at the much higher price; this profit has to be netted against the cost initially paid for the option. However, if the price of the asset decreases below the strike price, it does not make sense to exercise your right to buy it at a higher price. In this scenario, you lose what you originally paid for the option. Figure A4.1 summarizes the cash payoff at expiration to the buyer of a call option.

With a put option, you get the right to sell at a fixed price, and you want the price of the asset to decrease below the exercise price. If it does, you buy the asset at the current price and then sell it back at the exercise price, claiming the difference as a gross profit. When the initial cost of buying the option is netted against the gross profit, you arrive at an estimate of the net profit. If the value of the asset rises above the exercise price, you will not exercise the right to sell at a lower price. Instead, the option will be allowed to expire without being exercised, resulting in a net loss of the original price paid for the put option. Figure A4.2 summarizes the net payoff on buying a put option.

With both call and put options, the potential for profit to the buyer is significant, but the potential for loss is limited to the price paid for the option.
Determinants of Option Value

What is it that determines the value of an option? At one level, options have expected cash flows just like all other assets, and that may seem like good candidates for discounted cash flow valuation. The two key characteristics of options—that they derive their value from some other traded asset, and the fact that their cash flows are contingent on the occurrence of a specific event—does suggest an easier alternative. We can create a portfolio that has the same cash flows as the option being valued by combining a position in the underlying asset with borrowing or lending. This portfolio is called a replicating portfolio and should cost the same amount as the option. The principle that two assets (the option and the replicating portfolio) with identical cash flows cannot sell at different prices is called the arbitrage principle.

The Binomial Model

The simplest model for illustrating the replicating portfolio and arbitrage principles on which option pricing is based is the binomial model. The binomial option pricing model is based on a simple formulation for the asset price process in which the asset, in any time period, can move to one of two possible prices. The general formulation of a stock price process that follows the binomial is shown in Figure A4.3.

In this figure, $S$ is the current stock price; the price moves up to $S_u$ with probability $p$ and down to $S_d$ with probability $1 - p$ in any time period. For instance, if the stock price today is $100$, $u$ is $1.1$ and $d$ is $0.9$, the stock price in the next period can either be $110$ (if $u$ is the outcome) and $90$ (if $d$ is the outcome).

The objective in creating a replicating portfolio is to use a combination of risk-free borrowing/lending and the underlying asset to create the same cash flows as the option being valued. In the case of the general formulation, where stock prices can either move up to $S_u$ or down to $S_d$ in any time period, the replicating portfolio for a call with a given strike price will involve borrowing $SB$ and acquiring $\Delta$ of the underlying asset. Of course, this formulation is of no use if we cannot determine how much we need to borrow and what $\Delta$ is. There is a way, however, of identifying both variables. To do this, note that the value of this position has to be same as the value of the call, no matter what the stock price does. Let us assume that the value of the call is $C_u$ if the stock price goes to $S_u$, and $C_d$ if the stock price goes down to $S_d$. If we had borrowed $SB$ and bought $\Delta$ shares of stock with the money, the value of this position under the two scenarios would have been as follows.

<table>
<thead>
<tr>
<th>Value of Position</th>
<th>Value of Call</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta S_u - SB(1 + r)$</td>
<td>$C_u$</td>
</tr>
<tr>
<td>$\Delta S_d - SB(1 + r)$</td>
<td>$C_d$</td>
</tr>
</tbody>
</table>

Note that in either case, we have to pay back the borrowing with interest. Because the position has to have the same cash flows as the call, we get

$\Delta S_u - SB(1 + r) = C_u$

$\Delta S_d - SB(1 + r) = C_d$

Solving for $\Delta$, we get

$\Delta = \text{Number of units of the underlying asset bought} = (C_u - C_d)/(S_u - S_d)$

where $C_u = \text{Value of the call if the stock price is } S_u$ and $C_d = \text{Value of the call if the stock price is } S_d$.

When there are multiple periods involved, we have to begin with the last period, where we know what the cash flows on the call will be, solve for the replicating portfolio, and then estimate how much it would cost us to create this portfolio. We use this value as
the estimated value of the call and estimate the replicating portfolio in the previous period. We continue to do this until we get to the present. The replicating portfolio we obtain for the present can’t be priced to yield a current value for the call.

Value of the Call = Current Value of Underlying Asset \times Option Delta − Borrowing Needed to Replicate the Option

Illustration A4.1: An Example of Binomial Valuation

Assume that the objective is to value a call with a strike price of $50, which is expected to expire in two time periods, on an underlying asset whose price currently is $50 and is expected to follow a binomial process. Figure A4.4 illustrates the path of underlying asset prices and the value of the call (with a strike price of 50) at the expiration.

Note that because the call has a strike price of $50, the gross cash flows at expiration are as follows:

- If the stock price moves to $100: Cash Flow on Call = $100 − $50 = $50
- If the stock price moves to $50: Cash Flow on Call = $50 − $50 = $0
- If the stock price moves to $25: Cash Flow on Call = $0 (option is not exercised)

Now assume that the interest rate is 11%. In addition, define

\[ \Delta = \text{Number of shares in the replicating portfolio} \]

\[ B = \text{Dollars of borrowing in replicating portfolio} \]

The objective in this analysis is to combine \( \Delta \) shares of stock and \( \$B \) of borrowing to replicate the cash flows from the call with a strike price of $50.

The first step in doing this is to start with the last period and work backward. Consider, for instance, one possible outcome at \( t = 1 \). The stock price has jumped to $70 and is poised to change again, either to $100 or $50. We know the cash flows on the call under either scenario, and we also have a replicating portfolio composed of \( \Delta \) shares of the underlying stock and \( \$B \) of borrowing. Writing out the cash flows on the replicating portfolio under both scenarios (stock price of $100 and $50), we get the replicating portfolios in Figure A4.5:

In other words, if the stock price is $70 at \( t = 1 \), borrowing $45 and buying one share of the stock will give the same cash flows as buying the call. The value of the call at \( t = 1 \), if the stock price is $70, should therefore be the cash flow associated with creating this replicating position and it can be estimated as follows:

\[ 70\Delta − B = 70 − 45 = 25 \]

The cost of creating this position is only $25, because $45 of the $70 is borrowed. This should also be the price of the call at \( t = 1 \) if the stock price is $70.

Consider now the other possible outcome at \( t = 1 \) where the stock price is $35 and is poised to jump to either $50 or $25. Here again, the cash flows on the call can be
estimated, as can the cash flows on the replicating portfolio composed of \( \Delta \) shares of stock and $B$ of borrowing. Figure A4.6 illustrates the replicating portfolio.

**Figure A4.6 Replicating Portfolio when Price Is $35**

<table>
<thead>
<tr>
<th>( t = 0 )</th>
<th>Call Value</th>
<th>Replicating portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>0</td>
<td>( (50 \times 0) - (1.11 \times B) = 0 )</td>
</tr>
<tr>
<td>25</td>
<td>0</td>
<td>( (25 \times 0) - (1.11 \times B) = 0 )</td>
</tr>
</tbody>
</table>

Solving for \( D \) and \( B \)

\[ D = 0, \quad B = 0 \]

Because the call is worth nothing under either scenario, the replicating portfolio also is empty. The cash flow associated with creating this position is obviously zero, which becomes the value of the call at \( t = 1 \) if the stock price is $35.

We now have the value of the call under both outcomes at \( t = 1 \); it is worth $25 if the stock price goes to $70 and $0 if it goes to $35. We now move back to today (\( t = 0 \)), and look at the cash flows on the replicating portfolio. Figure A4.7 summarizes the replicating portfolios as viewed from today.

**Figure A4.7 Replicating Portfolios for Call Value**

<table>
<thead>
<tr>
<th>( t = 1 )</th>
<th>Replicating portfolio</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>( (70 \times D) - (B \times 1.11) = 25 ) (from step 1)</td>
</tr>
<tr>
<td>35</td>
<td>( (25 \times D) - (1.11 \times B) = 0 ) (from step 1)</td>
</tr>
</tbody>
</table>

Solving for \( D \) and \( B \)

\[ D = 57.1 \%, \quad B = 22.5 \]

Buy 5/7 shares, Borrow 22.5;

Using the same process as in the previous step, we find that borrowing $22.50 and buying 5/7 of a share will provide the same cash flows as a call with a strike price of $50. The cost to the investor of borrowing $22.5 and buying 5/7 of a share at the current stock price of $50 yields:

\[
\text{Cost of replicating position} = 5/7 \times 50 - 22.5 = 13.20
\]

This should also be the value of the call.

**The Black-Scholes Model**

While the binomial model provides an intuitive feel for the determinants of option value, it requires a large number of inputs, in terms of expected future prices at each node. As we make time periods shorter in the binomial model, we can make one of two assumptions about asset prices. We can assume that price changes become smaller as periods get shorter; this leads to price changes becoming infinitesimally small as time periods approach zero, leading to a **continuous price process**. Alternatively, we can assume that price changes stay large even as the period gets shorter; this leads to a **jump price process**, where prices can jump in any period.\(^1\) When the price process is continuous, the binomial model for pricing options converges on the Black-Scholes model. The model, named after its co-creators, Fischer Black and Myron Scholes, allows us to estimate the value of any option using a small number of inputs and has been shown to be remarkably robust in valuing many listed options.\(^2\)

**The Model**

While the derivation of the Black-Scholes model is far too complicated to present here, it is also based upon the idea of creating a portfolio of the underlying asset and the riskless asset with the same cashflows and hence the same cost as the option being valued. The value of a call option in the Black-Scholes model can be written as a function of the five variables:

\[ S, K, t, r, \Delta \]

Where:

- \( S \) = Current value of the underlying asset
- \( K \) = Strike price of the option
- \( t \) = Life to expiration of the option
- \( r \) = Riskless interest rate corresponding to the life of the option

\(^1\) While we do not consider jump process option pricing models in this appendix, they do exist but are not widely used because of the difficulties we face in estimating jump process parameters.

$\sigma^2 = \text{Variance in the ln(value) of the underlying asset}$

The value of a call is then:

$\text{Value of call} = S N(d_1) - K e^{-rt} N(d_2)$

where

$d_1 = \frac{\ln \left( \frac{S}{K} \right) + (r + \frac{\sigma^2}{2}) \cdot t}{\sigma \sqrt{t}}$

$d_2 = d_1 - \sigma \sqrt{t}$

Note that $e^{-rt}$ is the present value factor and reflects the fact that the exercise price on the call option does not have to be paid until expiration. $N(d_1)$ and $N(d_2)$ are probabilities, estimated by using a cumulative standardized normal distribution and the values of $d_1$ and $d_2$ obtained for an option. The cumulative distribution is shown in Figure A4.8:

$\text{Figure A4.8: Cumulative Normal Distribution}$

In approximate terms, these probabilities yield the likelihood that an option will generate positive cash flows for its owner at exercise, i.e., when $S > K$ in the case of a call option and when $K > S$ in the case of a put option. The portfolio that replicates the call option is created by buying $N(d_1)$ units of the underlying asset, and borrowing $Ke^{-rt}N(d_2)$. The portfolio will have the same cash flows as the call option and thus the same value as the option. $N(d_1)$, which is the number of units of the underlying asset that are needed to create the replicating portfolio, is called the option delta.

$\text{Model Limitations and Fixes}$

The Black-Scholes model was designed to value options that can be exercised only at maturity and on underlying assets that do not pay dividends. In addition, options are valued based upon the assumption that option exercise does not affect the value of the underlying asset. In practice, assets do pay dividends, options sometimes get exercised early and exercising an option can affect the value of the underlying asset. Adjustments exist. While they are not perfect, adjustments provide partial corrections to the Black-Scholes model.

$1. \text{Dividends}$

The payment of a dividend reduces the stock price; note that on the ex-dividend day, the stock price generally declines. Consequently, call options will become less valuable and put options more valuable as expected dividend payments increase. There are two ways of dealing with dividends in the Black Scholes:

- **Short-term Options**: One approach to dealing with dividends is to estimate the present value of expected dividends that will be paid by the underlying asset during the option life and subtract it from the current value of the asset to use as $S$ in the model.

  Modifed Stock Price = Current Stock Price $- \text{Present value of expected dividends during the life of the option}$

- **Long Term Options**: Since it becomes impractical to estimate the present value of dividends as the option life becomes longer, we would suggest an alternate approach. If the dividend yield ($y = \text{dividends/current value of the asset}$) on the underlying asset is expected to remain unchanged during the life of the option, the Black-Scholes model can be modified to take dividends into account.

  $C = S e^{yt} N(d_1) - K e^{-rt} N(d_2)$

  where

  $d_2 = d_1 - \sigma \sqrt{t}$

  From an intuitive standpoint, the adjustments have two effects. First, the value of the asset is discounted back to the present at the dividend yield to take into account the present value of expected dividends.
account the expected drop in asset value resulting from dividend payments. Second, the interest rate is offset by the dividend yield to reflect the lower carrying cost from holding the asset (in the replicating portfolio). The net effect will be a reduction in the value of calls estimated using this model.

2. Early Exercise

The Black-Scholes model was designed to value options that can be exercised only at expiration. Options with this characteristic are called European options. In contrast, most options that we encounter in practice can be exercised any time until expiration. These options are called American options. The possibility of early exercise makes American options more valuable than otherwise similar European options; it also makes them more difficult to value. In general, though, with traded options, it is almost always better to sell the option to someone else rather than exercise early, since options have a time premium, i.e., they sell for more than their exercise value. There are two exceptions. One occurs when the underlying asset pays large dividends, thus reducing the asset value as a consequence of the dividend payment. The other exception arises when an investor holds both the underlying asset and deep in-the-money puts, i.e., puts with strike prices well above the current price of the underlying asset, on that asset and at a time when interest rates are high. In this case, the time premium on the put may be less than the potential gain from exercising the put early and earning interest on the exercise price.

There are two basic ways of dealing with the possibility of early exercise. One is to continue to use the unadjusted Black-Scholes model and regard the resulting value as a floor or conservative estimate of the true value. The other is to try to adjust the value of the option for the possibility of early exercise. There are two approaches for doing so. One uses the Black-Scholes to value the option to each potential exercise date. With options on stocks, this basically requires that we value options to each ex-dividend day and choose the maximum of the estimated call values. The second approach is to use a modified version of the binomial model to consider the possibility of early exercise. In this version, the up and down movements for asset prices in each period can be estimated from the variance and the length of each period.

3. The Impact of Exercise On The Value Of The Underlying Asset

The Black-Scholes model is based upon the assumption that exercising an option does not affect the value of the underlying asset. This may be true for listed options on stocks, but it is not true for some types of options. For instance, the exercise of warrants increases the number of shares outstanding and brings fresh cash into the firm, both of which will affect the stock price. The expected negative impact (dilution) of exercise will decrease the value of warrants compared to otherwise similar call options. The adjustment for dilution in the Black-Scholes to the stock price is fairly simple. The stock price is adjusted for the expected dilution from the exercise of the options. In the case of warrants, for instance:

\[
\text{Dilution-adjusted S} = \frac{S_n + W_n}{n_s + n_w}
\]

where
S = Current value of the stock
n_W = Number of warrants outstanding
W = Value of warrants outstanding
n_S = Number of shares outstanding

When the warrants are exercised, the number of shares outstanding will increase, reducing the stock price. The numerator reflects the market value of equity, including both stocks and warrants outstanding. The reduction in S will reduce the value of the call option.

3 To illustrate, if \( \sigma \) is the variance in \( \ln(\text{stock prices}) \), the up and down movements in the binomial can be estimated as follows:

\[
u = e^{\left[-\frac{\sigma^2}{2}\right] \frac{m}{2}}\]

\[
d = e^{\left[-\frac{\sigma^2}{2}\right] \frac{m}{2}}\]

where u and d are the up and down movements per unit time for the binomial, T is the life of the option and m is the number of periods within that lifetime.  

4 Warrants are call options issued by firms, either as part of management compensation contracts or to raise equity. We will discuss them in chapter 16.
There is an element of circularity in this analysis, since the value of the warrant is needed to estimate the dilution-adjusted $S$ and the dilution-adjusted $S$ is needed to estimate the value of the warrant. This problem can be resolved by starting the process off with an assumed value for the warrant (say, the exercise value or the current market price of the warrant). This will yield a value for the warrant and this estimated value can then be used as an input to re-estimate the warrant’s value until there is convergence.

**Conclusion**

An option is an asset with payoffs that are contingent on the value of an underlying asset. A call option provides its holder with the right to buy the underlying asset at a fixed price, whereas a put option provides its holder with the right to sell at a fixed price, any time before the expiration of the option. The value of an option is determined by six variables - the current value of the underlying asset, the variance in this value, the strike price, life of the option, the riskless interest rate and the expected dividends on the asset. This is illustrated in both the Binomial and the Black-Scholes models, which value options by creating replicating portfolios composed of the underlying asset and riskless lending or borrowing.