Review: Time Value of Money

- Last time, we learned
  - How to value single payoffs
  - How to value multiple payoffs
    - Just take them separately and add prices
    - Annuities
    - Perpetuities

Review: Time Value of Money

- Present Value/ Future Value of cash flows
- What's the point?

Review: Time Value of Money

- Money received/paid at different points in time
  - Need to compare $100 received in five years to $200 received in ten years
  - How much am I willing to pay today for an annuity of $20 starting next year and lasting ten years?

Review: Time Value of Money

- Rule: adding money at one year in the future and money at two years in the future is adding apples and oranges
- Need to convert everything to the same units: dollars *at the same point in time*
- Move the 5 year payoff to 10 years, or the 10 year one back to 5, or both back to time 0

Review: Time Value of Money

- Because securities are just streams of cash flows, you can compare the price you must pay for them today with the present value of their cash flows
- If price > PV(Cash flows), sell the security
- If price < PV(Cash flows), buy the security
- Arbitrage drives prices to equal PV(cash flow)!
- Only riskless CFs

Review: Quoted Rates and EARs

- $(1+EAR)=\{1+(APR/m)\}^m$
- How much is $1 worth in a year's time at 10%
  - With semiannual compounding?
- $(1+EAR)=\exp(APR)$
  - With continuous compounding?

Today's Class

- Return measures
- Statistics review
  - Random variables
  - Expected value, variance, standard deviation, covariance, correlation
  - Linear combination of random variables: portfolio
- Investors' preference: risk and return tradeoff
- Asset allocation with one risky and one riskless security
  - Riskless security
  - Asset allocation line

Next Class

- Asset allocation with 2 risky assets
- Asset allocation with 2 risky and a riskless asset
- Asset allocation with many risky assets and one riskless asset
- Diversifiable and non-diversifiable risk

Return Measures

Andre de Souza

Digression: Mutual Funds

- Financial intermediaries that pool funds from investors and buy assets
- Advantages:
  - Record keeping and administration
  - Diversification and divisibility
  - Professional management and analysis
  - Lower transactions costs

10

Single-Period Realized Return

- Holding period return (HPR):

$$HPR = \frac{\text{ending price} + \text{cash dividend}}{\text{beginning price}} - 1$$

- Annualized holding period return (HPR) for a holding period of *t* years:

$$(1 + \text{annualized } HPR) = (1 + HPR)^{1/t}$$

$$\text{ann.HPR} = \left(\frac{\text{ending price} + \text{cash dividend}}{\text{beginning price}}\right)^{1/t} - 1$$

11

Multiple-Period Realized Return

(1) Arithmetic Average:

$$\frac{1}{T}(r_1 + r_2 + r_3 + \ldots + r_T)$$

- Not equivalent per-period return because it neglects compounding
- Useful for forecasting the return next period

12

Multiple-Period Realized Return

(2) Geometric Average

- Gives the equivalent per-period return
- Answers the question: what per-period constant return would have given me the same terminal payoff?

$$[(1+r_1)(1+r_2)(1+r_3)\ldots(1+r_T)]^{1/T}-1$$

$$=\left[\frac{\text{accumulated value}_T}{\text{value}_0}\right]^{1/T}-1$$

13

Multiple-Period Realized Return

(3) Internal rate of return, IRR

- Return if one can re-invest cash-flows at this rate
- "Dollar-weighted average"
- IRR is the rate that makes:

Initial price = present value of future net profits

$$P(0)=\sum_{t=1}^{\infty}\frac{C(t)}{(1+IRR)^t}$$

14

Risk and Return

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15

Random Variables

- Return for a risky security is a random variable.
- Definition: a value representing an outcome of an uncertain event
- The outcome may be
  - Discrete:
    - Example: depends on the flip of a coin
      - ✓ {heads, tails}
    - Example: depends on business cycle conditions
      - ✓ {recession, normal, boom}
  - Continuous:
    - Example: the weight of a baby
  - Stock prices (returns)?
    - Discrete? Continuous?
    - Scenario analysis

16

Probability Distribution

- The likelihood of each possible event
- Discrete outcomes
  - Fair coin: 50% head, 50% tail
  - Rigged coin: 60% head, 40% tail
  - What about our scenarios?
    - Historical
    - Analysts
- For continuous outcomes
  - Normal distribution

17

Expected Value (for discrete distributions)

- The average outcome if the event was repeated infinitely often.
- The probability-weighted average of possible outcomes.
- If the return R_i on an asset *i* is equal to $R_i(s)$ with probability $p(s)$ for $s=1,\ldots,S$, then the expected return (or mean return) is:

$$E(R_i) = \sum_{s=1}^{S} R_i(s)\, p(s)$$

18

Variance and Standard Deviation

- The *variance* is the average *squared* deviation from the expected value:

$$\sigma_i^2 = E\left([R_i(s) - E(R_i)]^2\right)$$

$$= \sum_{s=1}^{S} [R_i(s) - E(R_i)]^2 \, p(s)$$

- The *standard deviation* (SD) is the square root of the variance:

$$\sigma_i = \sqrt{\sigma_i^2}$$

- 'volatility' is another word for 'standard deviation'.

So now we have measures of...

- How much return to expect on average
  - Mean
- How variable this return could be around its average
  - Variance
- We'd like a measure of how much two returns tend to "move together"
  - Why?

Covariance

- The *covariance* between two random variables is the average of the products of their deviations from the mean:

$$\text{cov}(R_i, R_j) = E\left([R_i - E(R_i)][R_j - E(R_j)]\right)$$

$$= \sum_{s=1}^{S} [R_i(s) - E(R_i)][R_j(s) - E(R_j)] \, p(s)$$

- The covariance is
  - **Positive** if the random variables tend to be above their mean *at the same time*
  - **Negative** if one variable tends to be above its mean when the other is below its mean

Correlation

- The *correlation* is the covariance between two random variables, divided by their standard deviations:

$$\rho_{ij} = \frac{\mathrm{cov}(R_i, R_j)}{\sigma_i \sigma_j}$$

- The correlation is scaled so that:

$$-1 \le \rho_{ij} \le 1$$

Estimating Mean, Variance, and Covariance from Historical Data

- Use the "sample counterpart" of the definition:

$$\hat{E}(R_i) = \frac{1}{T}\sum_{t=1}^{T} R_i(t)$$

$$\hat{\sigma}_i^2 = \frac{1}{T-1}\sum_{t=1}^{T} [R_i(t) - \hat{E}(R_i)]^2$$

$$\mathrm{c\hat{o}v}(R_i, R_j) = \frac{1}{T-1}\sum_{t=1}^{T} [R_i(t) - \hat{E}(R_i)][R_j(t) - \hat{E}(R_j)]$$

Portfolio

- A combination of N assets, with returns $R_1, \ldots, R_N$.
- Portfolio *p*, with portfolio weights $\omega_1, \ldots, \omega_N$:
  - ω_i is percentage of wealth invested in asset *i:*

$$\omega_i = \frac{\text{Dollar value of position in stock i}}{\text{Total dollar value of portfolio}}$$

  - Portfolio weights sum to one: $\omega_1 + \ldots + \omega_N = 1$.
    - What does it mean for the weights if I say that not more than 50% of your portfolio value can be in a single asset?
  - A negative weight indicates a short position.

Portfolio Return and Portfolio Expected Return

- The return on the portfolio is:

$$R_p = \sum_{i=1}^{N} \omega_i R_i$$

- The expected return on the portfolio is:

$$E(R_p) = \sum_{i=1}^{N} \omega_i E(R_i)$$

Portfolio Variance and Standard Deviation

- With 2 securities (N=2), the portfolio variance is:

$$\sigma_p^2 = \omega_1^2\sigma_1^2 + \omega_2^2\sigma_2^2 + 2\omega_1\omega_2\rho_{12}\sigma_1\sigma_2$$

- In general, the portfolio variance is:

$$\sigma_p^2 = \sum_{i=1}^{N} \omega_i^2\sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j>i}^{N} \omega_i\omega_j\rho_{ij}\sigma_i\sigma_j$$

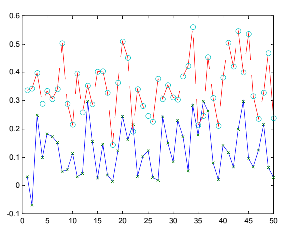
- The standard deviation of the portfolio is:

$$\sigma_p = \sqrt{\sigma_p^2}$$

Risk-Return Tradeoff

- Suppose you could only pick one of the following return patterns:

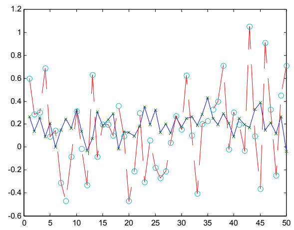
	Blue	Red
Mean	0.15	0.35
Stdev	0.10	0.10



Risk-Return Tradeoff

- Suppose you could only pick one of the following return patterns:

Mean	0.20	0.20
Stdev	0.10	0.40



Risk-Return Tradeoff

- Suppose you could only pick one of the following return patterns:

Mean	0.20	0.40
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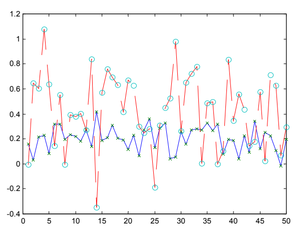
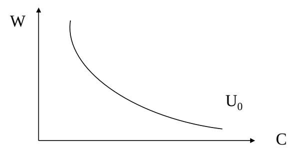


Risk-Return Tradeoff

- Recall two of the Axioms of Finance:
  - Investors prefer more to less
  - Investors are risk-averse
- This means that investors prefer an investment :
  - with a higher expected return $E(R_i)$
  - with a lower standard deviation (or volatility), σ_i
- Investors must trade off risk and return in order to <u>maximize their expected utility.</u>

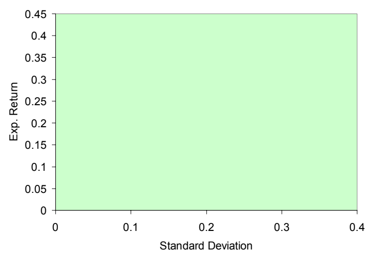
Capturing Utility: Indifference Curves (Review)

- A person likes 2 goods: wine (W) and cheese (C).
- An indifference curve gives all the combinations of W and C that give the same utility level $U_0 = U(W,C)$.
- People like to be on the highest possible indifference curve (people prefer more to less).



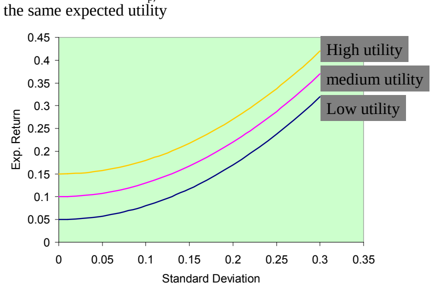
Indifference Curves in Finance

Indifference curve: A set of (σ_p, E(Rp)) combinations that give an investor the same expected utility



Indifference Curves in Finance

Indifference curve: A set of (σ_p, E(Rp)) combinations that give an investor the same expected utility



Capital Allocation Choice

- Goal: to give investor the highest possible utility
- We can think of an investor allocating capital between:
  - Risk free assets (using T-Bills as proxy), (or bank deposit)
  - Risky assets (stocks, bonds or some combination of both), (or pension/mutual fund) For now, we take the composition of the risky portfolio as given.
- The complete portfolio is the entire portfolio consisting of risky and risk-free assets.
- Two themes
  - 1) Risk / return tradeoff, 2) Effect of Risk aversion

Asset Allocation

- We wish to examine the most basic asset allocation choice:
  - How much should I invest in risk free securities?
  - How much should I invest in other, risky assets?
- Later, we will look at how to determine the optimal risky portfolio. Now it is taken as given.

Digression: Riskless Asset

- What is the risk-free rate?
  - The rate of return on the riskless security
  - Future payoff is known and guaranteed at the time of investment
  - Not random but can vary over time
- What determines the equilibrium risk-free rate?
  - Demand for credit (desire to borrow)
  - Supply of credit (willingness to lend)

How does the Fed lower rates?

- The Fed announces Federal Funds rate. (http://www.federalreserve.gov/FOMC/)
- Implemented by open market operation.
- Sales of bonds: P ↓ => r ↑

Properties of the Risk-Free Asset

- The risk-free return is denoted R_f
- The risk-free return is known for sure:
  $E[R_f] = R_f$
  □ $\sigma_f^2=0$
  $Cov(R_f,R_i) = \rho_{f,t} = 0$ for any other asset I

Q: In reality, is the risk-free asset really risk-free?

Now come back to asset allocation…

First, Asset allocation with one risky and one riskless security

Too simple?

John Bogle, founder of Vanguard Group

The fundamental decision of investing is the allocation of your assets: How much should you own in stock? How much should you own in bonds? How much should you own in cash reserves? That decision accounts for an astonishing 94% of the difference in total returns achieved by institutional investors. There is no reason to believe that the same relationship does not hold for individual investors

A Portfolio with one Risk-Free and one Risky Asset

- Let ω be the fraction of wealth invested in the risky asset (the rest is invested in the risk-free asset)
- Expected portfolio return:

$$E[R_p] = \omega \cdot E[R_r] + (1-\omega)R_f = R_f + \omega \cdot E[R_r - R_f]$$

- Variance of portfolio return:

$$\sigma_p^2 = \omega^2 \cdot \sigma_r^2$$

- The Standard deviation is:

$$\sigma_p = |\omega| \cdot \sigma_r$$

40

Investment Opportunity Set: the Capital Allocation Line

- Example:
  - Risky asset: US stock market : $E[R_{US}]=13.55\%$, $\sigma_{US}=15.35\%$
  - Risk-free : $R_f=7\%$

41

Investment Opportunity Set with a Risk-free and a Risky Asset

- Consider various portfolios *p* (which are long the risky asset and long or short the risk-free asset).
- What is the risk-return relationship ? **combine the expected return and volatility formulas**

$$E[R_p] = R_f + \frac{E[R_r] - R_f}{\sigma_r} \sigma_p$$

$$= R_f + (\text{Sharpe ratio of risky asset}) \ \sigma_p$$

$$= R_f + SR_r \ \sigma_p$$

42

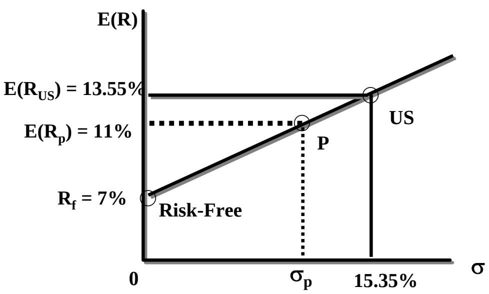
Investment Opportunity Set: the Capital Allocation Line

- Example:
  - Risky asset: US stock market : $E[R_{US}]=13.55\%$, $\sigma_{US}=15.35\%$
  - Risk-free : $R_f=7\%$
- Hence: $E[R_p]=0.07+SR\cdot\sigma_p$

$$SR=\frac{E[R_{US}]-R_f}{\sigma_{US}}=\frac{0.1355-0.07}{0.1535}=0.4267$$

- Sharpe Ratio (SR) = extra return per unit of risk

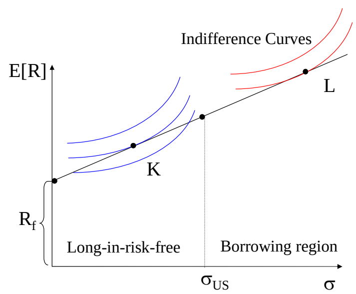
Investment Opportunity Set: Capital Allocation Line (CAL)



How to allocate assets: Effects of Risk Aversion

- Greater levels of risk aversion lead to greater proportions of the complete portfolio in the risk free asset.
- Lower levels of risk aversion lead to larger proportions of the complete portfolio in the risky asset. This could result in leveraged positions.

Optimal Portfolio Choice



Portfolio K:

optimal choice for risk-averse investor

long in the risky asset

long in risk free

Portfolio L:

optimal choice for risk-tolerant investor

borrow risk free asset to

invest in risky asset

Reading

- Today:
  BKM: 5.1*, 5.2*, 5.3*, 5.5*, RWJ: 5.3*
- Next class: BKM: 6.1*, 6.2*, 6.3*, 6.4*, CP: 9, 10*, 11*