

Review

- Statistical review
 - **Mean, variance, covariance, correlation**
- Portfolio: a linear combination of assets
 - **Portfolio weight**: percentage of wealth invested

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Returns example

State of economy	Probability of state	IBM	Risk-free
Boom	1/3	21%	5%
Normal	1/3	10%	5%
Bust	1/3	-1%	5%

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Returns example

- Suppose I invest only in IBM
- Expected return of IBM?
 - $(1/3)*21+(1/3)*10+(1/3)*(-1)=10\%$
 - This is just saying: If I invest \$200, how much do I make?
 - $(1/3)*42+(1/3)*20+(1/3)*(-2)=$20=0.10*200$
- Expected return on the risk-free asset?
 - $(1/3)*5+(1/3)*5+(1/3)*5=5\%$

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Returns example

- What is the variance of the return of IBM?

State	Probability	IBM	Deviation from mean	Squared deviation
Boom	1/3	21%	11	121% ²
Normal	1/3	10%	0	0% ²
Bust	1/3	-1%	-11	121% ²

- Variance = $(1/3) \cdot 121 + (1/3) \cdot 0 + (1/3) \cdot 121 = 80.667$ percent squared
- Standard deviation = Square root of variance = 8.98%

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- Now suppose you invest \$100, half in IBM, half in the risk-free asset

State	IBM	\$s from IBM	\$s from R _f	Total \$ return	Compare with 0.5*IBM+0.5*RF
Boom	21%	\$10.50	\$2.50	\$13	$0.5 \cdot 21 + 0.5 \cdot 5 = 13\%$
Normal	10%	\$5	\$2.50	\$7.5	$0.5 \cdot 10 + 0.5 \cdot 5 = 7.5\%$
Bust	-1%	-\$0.50	\$2.50	\$2	$0.5 \cdot (-1) + 0.5 \cdot 5 = 2\%$

- Expected return on $R_{port} = (1/3) \cdot 13 + (1/3) \cdot 7.5 + (1/3) \cdot 2 = 7.5\%$
- Compare with $0.5 \cdot (\text{Expected return on IBM}) + 0.5 \cdot (\text{expected return on rf})$

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- Now suppose you invest half in IBM, half in Rf

State	IBM	%age return on portfolio	Deviation from mean	Squared Deviation
Boom	21%	13%	$13 - 7.5 = 5.5$	30.25
Normal	10%	7.5%	$7.5 - 7.5 = 0$	0
Bust	-1%	2%	$2 - 7.5 = -5.5$	30.25

- Variance = $(1/3) \cdot 30.25 + (1/3) \cdot 0 + (1/3) \cdot 30.25 = 20.167$ %sqd
- Compare with $0.5 \cdot 0.5 \cdot 80.667 + 0.5 \cdot 0.5 \cdot 0 + 2 \cdot 0.5 \cdot 0.5 \cdot 8.98 \cdot 0 = 0$

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- How about shorting the risk-free asset? Invest \$50 in a portfolio, putting \$100 in IBM, and borrowing \$50

State	IBM	Ss from IBM	Ss from R _f	Total S return	Compare with 2*IBM- 1*RF (%)
Boom	21%	\$21	-\$2.5	\$18.5	2*21-1*5% = 37%
Normal	10%	\$10	-\$2.5	\$7.5	2*10-1*5% = 15%
Bust	-1%	-\$1	-\$2.5	-\$3.5	2*(-1)-1*5% = (-7)

- Expected return on $R_{port} = (1/3)*37 + (1/3)*15 + (1/3)*(-7) = 15\%$
- Compare with 2*(Expected return on IBM) - 0.5*(expected return on rf)
- Can easily work out variances

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Remember from last time

- Investors like return, don't like risk

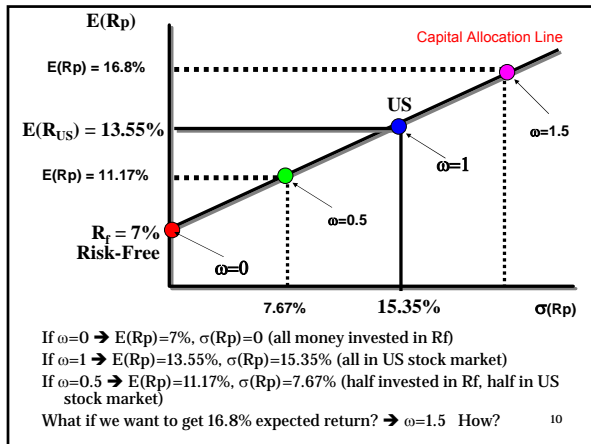
Stock A	Stock B	Pick?
High risk, low expected return	Low risk, high expected return	B
High risk, high expected return	Low risk, low expected return	Unclear, depends on your risk preferences

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Deciding which stock to pick

- Use Indifference Curves
- How do I use these?
 - First, draw all possible portfolios you can achieve
 - For one risky, one riskless asset, the set of all possible portfolios is a straight line, called the CAL
 - How do we get this line?

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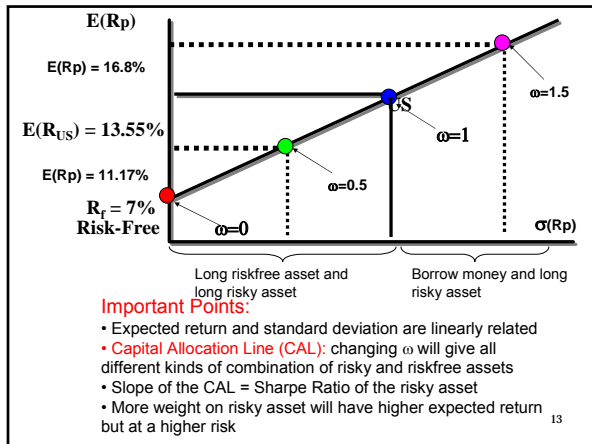


Using indifference curves

- In every case, higher return means higher risk, so our problem remains
- Second, draw indifference curves for your investor
 - Consider first a very risk-averse investor
 - Which curve would he prefer to lie on?
- What portfolio does he end up with?
 - What about a not-so-risk-averse guy?

Sharpe ratio

- The slope of the line = rise/run = $\frac{E[R_{risky}] - R_f}{\sigma_{risky}}$
- The Sharpe ratio
- Equation of the line is $E[R_p] = R_f + \frac{E[R_r] - R_f}{\sigma_r} \sigma_p$
 $= R_f + (\text{Sharpe ratio of risky asset}) \sigma_p$



Asset Allocation (cont.)

Andre de Souza

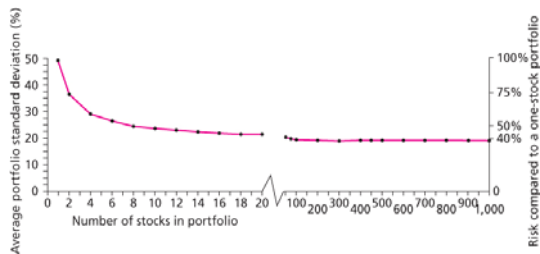
- ### Overview
- Diversification
 - Asset Allocation with 2 Risky Assets
 - Asset Allocation with 2 Risky and a Riskless Asset
 - Asset Allocation with Many Risky and a Riskless Asset
 - Diversifiable and non-diversifiable risk

Diversification

- Suppose I want to buy a share in Dell
- What kinds of risk do I worry about?
 - Michael Dell runs away with the office safe
 - The US economy goes into recession
- In general, two types of risk
 - Specific to the company
 - General economic conditions
- What if I bought ½ in Exxon, ½ in Dell?

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As the number of assets you hold increases ...



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Recall: expected return and standard deviation of portfolio

- The expected return on the portfolio of 2 securities is:

$$E(R_p) = \omega_1 E(R_1) + \omega_2 E(R_2)$$

- With 2 securities, the portfolio variance is:

$$\sigma_p^2 = \omega_1^2 \sigma_1^2 + \omega_2^2 \sigma_2^2 + 2\omega_1 \omega_2 \rho_{12} \sigma_1 \sigma_2$$

- The standard deviation is: $\sigma_p = \sqrt{\sigma_p^2}$

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Diversification with 2 assets: Example

- Suppose we have two assets, US and JP, with:

	mean	volatility
US	13.6%	15.4%
JP	15.0%	23.0%

and with correlation 27%.

- If an investor holds 60% in the US and 40% in JP what is the mean and volatility of the portfolio?

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Diversification with 2 assets: Example

- Portfolio mean:

$$E(R_p) = 0.6 \cdot 0.136 + 0.4 \cdot 0.150 = 14.2\%$$

- Portfolio variance:

$$\begin{aligned} \text{var}(R_p) &= (0.6)^2 \cdot (0.154)^2 + (0.4)^2 \cdot (0.230)^2 \\ &\quad + 2 \cdot 0.6 \cdot 0.4 \cdot 0.27 \cdot 0.154 \cdot 0.230 \\ &= 0.022 \end{aligned}$$

$$\sigma_p = 14.7\%$$

- This portfolio has higher expected return and lower risk than the US market alone!

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Risk and Return with Varying Weights

- Let ω be the weight in the US, and $1-\omega$ the weight in JP.

- The expected return of the portfolio is:

$$E(r_p) = \omega \cdot 0.136 + (1-\omega) \cdot 0.150$$

- The variance of the portfolio return is:

$$\begin{aligned} \text{var}(r_p) &= \omega^2 \cdot (0.154)^2 + (1-\omega)^2 \cdot (0.230)^2 \\ &\quad + 2 \cdot \omega \cdot (1-\omega) \cdot 0.27 \cdot 0.154 \cdot 0.230 \end{aligned}$$

- What happens when we vary ω ?

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Asset allocation with 2 risky assets

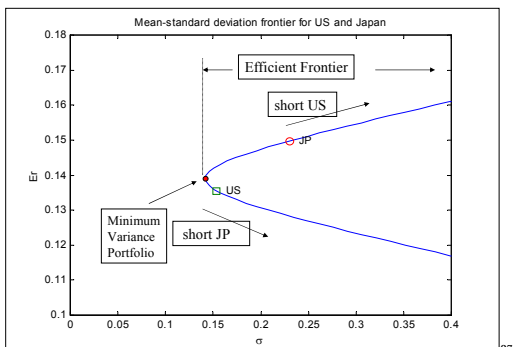
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Portfolio Terminology

- The **investment opportunity set** consists of all available risk-return combinations (by varying ω).
 - What's this for one-risky, one-riskless assets?
- The **minimum variance portfolio (MVP)** is the portfolio that provides the lowest variance (standard deviation) among all possible portfolios of risky assets
- The **minimum variance frontier** is the set of portfolios with minimum variance for a given expected return (same as investment opportunity set for 2 risky asset, different for many risky asset)
- An **efficient portfolio** is a portfolio that has the highest possible expected return for a given standard deviation
- The **efficient frontier** is the set of **efficient portfolios**. It is the upper portion of the **minimum variance frontier** starting at the minimum variance portfolio.

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Efficient Frontier



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Optimal Portfolio Choice with 2 Risky Assets

- Any investor should choose an **efficient portfolio** to benefit from **diversification**, i.e. always choose a portfolio from efficient frontier.
- The specific choice on the efficient frontier depends on the investor's **risk aversion**
- A more risk-averse investor should choose a portfolio with
 - lower risk
 - lower expected return

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Asset allocation with 2 risky assets and 1 riskfree asset

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Optimal Portfolio with 2 risky assets and 1 riskfree asset

- Putting together all our tools, we are now ready to construct **optimal** portfolios with 2 risky assets and 1 riskfree asset
- This is a three-step recipe:
 1. Use risky assets to construct the **efficient frontier**
 2. Introduce the risk-free assets and see what your new investment opportunity set is
 3. Choose the optimal portfolio

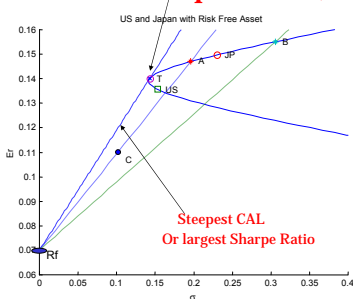
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STEP 1: Construct Efficient Frontier

- First, we want to construct the efficient frontier.
- **Efficient Frontier:** the set of efficient portfolios (the portfolios of risky assets that maximize expected return $E(r_p)$ for each level of portfolio risk σ_p .)
- We already did it in the case with 2 risky assets, so let's go straight to step 2

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Optimal Risky Portfolio (tangency portfolio)



Find a risky portfolio on efficient frontier that maximizes the Sharpe ratio or slope of CAL: tangency portfolio T

$$SR_i = \frac{E[R_i] - R_f}{\sigma_i}$$

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STEP 2: Bring in the risk-free asset

- Recall: Capital Allocation Line (CAL) is the line link the risky asset and riskfree asset.
- Once we bring in the risk free asset, we need to search for the CAL with the highest reward to variability ratio (Sharpe Ratio) that connects to one of the risky portfolios.

$$\text{SharpeRatio}_i (SR_i) = \frac{E[R_i] - R_f}{\sigma_i}$$

- This CAL will be tangent to one portfolio of risky assets on the efficient frontier. This portfolio is the **optimal risky portfolio**.

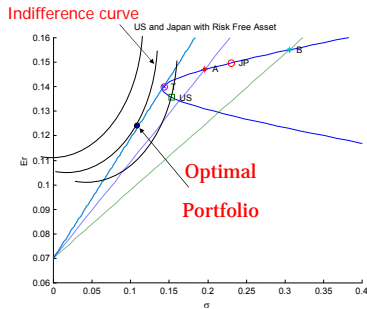
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STEP 3: Find your optimal portfolio

- The investor chooses the appropriate mix between the **optimal risky portfolio (tangency portfolio)** and the risk free asset based on his or her own risk aversion.
- **But notice that the optimal risky portfolio is the same for all investors (Separation principle).**

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Optimal Portfolio



Find a risky portfolio that maximizes the Sharpe ratio or slope of CAL: tangency portfolio T

$$SR_i = \frac{E[R_i] - R_f}{\sigma_i}$$

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Optimal Portfolio with 2 risky assets and 1 riskfree asset

- The three-step recipe:
 1. Use risky assets to construct the **efficient frontier**
 2. Find a portfolio on the **efficient frontier** that gives you the **steepest CAL (Capital Allocation Line)** or has the **largest Sharpe Ratio**
 3. **Optimal portfolio** is the portfolio by mixing the risk free asset with that risky portfolio found in step 2. The weight put in that risky portfolio and riskfree asset depends on the investor's **risk aversion**

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The Separation Property

- Asset allocation choice can be divided into two tasks
 - Find the optimal (tangency) portfolio and the optimal investment opportunity set (CAL)
 - Purely technical
 - Use information about investor's risk aversion to decide which risk-return combination to pick
- All investors hold the same portfolio!!!

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Asset Allocation with 2 Risky and 1 Risk-free asset - An Example

- We have \$1000 to invest in T-Bills (riskfree asset), US and JP. We want an expected return of 12%. Assume that the **tangency portfolio** is consisted of 60% US and 40% JP.
 - How much should we invest in the risk free asset?
 - How much should we invest in JP?
 - How much should we invest in US?
 - What would be the risk of this portfolio?
 - Will a combination of only US and JP that gives the same expected return have more or less risk than the portfolio composed of the risky portfolio and the risk free asset?

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Asset allocation with **many risky assets and 1 riskfree asset**

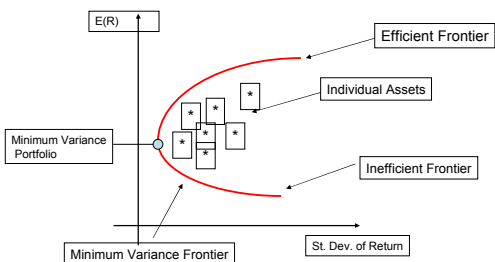
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Asset Allocation with Many Risky Assets and a Riskless Asset

- Calculations become more involved, but none of the fundamental ideas change
- **“Efficient frontier”**- the same general shape
- **“Best” CAL** - tangent to the efficient frontier
- New question: which risky assets are included in tangent portfolios?

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Investment Opportunity Set with Many Assets



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Optimal Portfolio Selection with Many Risky Assets and a Risk-Free

Same three steps as before:

- Construct the **efficient frontier** from different portfolios of risky assets

2. Find the **“tangency portfolio,”** that is, the portfolio with the highest **Sharpe ratio**:

$$SR_i = \frac{E[R_i] - R_f}{\sigma_i}$$

3. Choose the combination of the tangency portfolio and the risk-free asset to suit your risk-return preferences.

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Diversifiable and Non-diversifiable risks

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Diversification Revisited: Equally-Weighted Portfolios, **Independent Returns**

- Consider a simple example: suppose we have an equally weighted portfolio (holding weights $1/N$) of N **independent** stocks.
- The variance of the portfolio return is

$$\sigma_p^2 = \frac{1}{N^2} \sum_{i=1}^N \sigma_i^2$$

$$= \frac{1}{N} \left[\begin{array}{l} \text{average} \\ \text{variance} \end{array} \right]$$

- As the number of assets increase, the risk is diversified away.
- However, in reality most returns are positively correlated
 - **Not all risk can be diversified away**

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Risk in Equally-Weighted Portfolios: **The General Case**

- In reality, the stocks are **not independent**.
- The variance of the portfolio return is:

$$\sigma_p^2 = \frac{1}{N^2} \sum_{i=1}^N \sigma_i^2 + \frac{2}{N^2} \sum_{i=1}^N \sum_{j>i}^N \text{cov}(R_i, R_j)$$

$$= \frac{1}{N} \left[\begin{array}{l} \text{average} \\ \text{variance} \end{array} \right] + \left[1 - \frac{1}{N} \right] \left[\begin{array}{l} \text{average} \\ \text{covariance} \end{array} \right]$$

- What happens when $N \rightarrow \text{infinity}$?
- Variance of portfolio return
 - average covariance of returns (not zero)
- **Conclusion:** Not all risks can be diversified away.
- **Conclusion:** Only risk that matters is the covariance (common part)

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Classifications of Risk

- Part that cannot be diversified away:
 - ‘covariance risk’, ‘systematic risk’ or ‘non-diversifiable risk’
(E.g. market risk, macroeconomic risk, industry risk)
 - Part that can be diversified away (in a large portfolio):
 - ‘idiosyncratic risk’, ‘non-systematic risk’, ‘diversifiable risk’ or ‘unique risk’
(E.g. Individual company news)
- Investors only care about portfolio risk, not the individual risk of stocks

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Single Index Model

- How do we separate diversifiable from non-diversifiable risk for security i ?
 - How do we measure how much nondiversifiable risk a security has?
- Consider a single index model: $R_i = \alpha_i + \beta_i R_M + e_i$
- Just says
“All return comes from three parts: a constant, a part common to many securities, and a part particular to this security”

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Single Index Model

- What is the source of the common part?
 - We don’t know, but we can use a broad-based index as a proxy
- Market return: R_M
- Regression analysis: $R_i = \alpha_i + \beta_i R_M + e_i$
 - e_i : Idiosyncratic risk.
 - $\beta_i R_M$: Systematic risk.
 - $\beta_i = \text{Cov}(R_M, R_i) / \sigma_M^2$

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Reading

- Today: BKM: 6.1*, 6.2*, 6.3*, 6.4*, 6.5 , CP: 9, 10*, 11*
- Next class: BKM: 6.5*, 7.1*, 7.2, 7.3
- Problem Set 2 is due next class!

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