Supplementary problems with solutions: Arbitrage

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Note: A general way to deal with arbitrage problems is to first try to match payoffs at all dates other than the present.

1. A two year, \$1,000 face value, annual payment coupon bond with a 7% coupon rate sells for \$1025. I have access to a bank account which pays me 7% per year with annual compounding.

   (a) How much do I have to invest in the bank account today to replicate the payoffs on the bond?

   We need to find how much we should start with, investing at 7% a year, to create \$70 at the end of the first year, and \$1070 at the end of the second year. This is just the present value of this stream at 7% interest. In other words,

   $$PV = \frac{70}{1.07} + \frac{1070}{1.07^2} = \$1000$$

   Let's check this. Start with \$1,000 in the account. At the end of the first year, this has grown to \$1070. Withdraw \$70, leaving \$1000. At the end of the second year, this has grown to \$1070 again, which you can withdraw entirely, thus replicating the payoffs on the bond.

   (b) Construct an arbitrage. What is your riskless profit?

   We can replicate the payoffs on the bond starting with \$1000 in our bank account. To create the same payoffs with the bond would

cost us \$1025. To buy low, sell high, we "buy" the bank account and sell the bond.

Selling the bond gets us \$1025. We invest \$1,000 of this in the bank account, and pocket the difference.

Is this an arbitrage? At the end of year 1, the bank account has grown to \$1070. We withdraw \$70 of that amount, and pay the coupon we owe on the bond we sold short. This leaves us with \$1000 in the account, which grows to \$1070 at the end of the second year. We withdraw that amount and make the final payment on the bond.

(c) Now suppose I can borrow at 8%, and that the same bond costs \$970. Construct an arbitrage. What is your riskless profit?

In this case, the present value of the payoff stream of the bond, at 8%, is \$982.17. The bond costs \$970. Buy the bond for \$970, "sell" the bank account (i.e., borrow \$982.17 at 8%). This gives you an immediate payoff of \$12.17, and no payments afterwards.

(d) A bond price between \$982.17 and \$1,000 creates no arbitrage opportunities. Suppose the price is \$985. Suppose I buy the bond. If I borrow this money, I have to borrow at 8%. My loan grows to \$1063.8 in one year. I pay off \$70 of that amount with the first coupon on the bond, leaving me with \$993.8 as the balance on my loan. At the end of the second year, this has grown to \$1073.304, and I only get \$1070 from my bond, leaving me with a certain loss of \$3.304.

On the other hand, if I sell the bond, I get to invest the \$985 at 7%. This means that at the end of the first year, my account has grown to \$1053.95. I pay out the \$70 coupon, leaving me with \$983.95 in my account. At the end of the second year, this grows to \$1052.83, far short of what I need to make my final payment.

In this case, the borrowing rate is too high and the lending rate too low for me to be able to make any money by arbitrage. This is a form of transaction cost: we can get a band outside which the price cannot lie, but if it lies within the band it may lie anywhere.

2. A one-year zero coupon bond, with face value \$100, sells for \$96. The one year forward rate one year into the future is 5%. A two year zero coupon bond with face value \$100 trades at \$90.

(a) If I buy the one year zero coupon bond and, upon its maturing, invest the proceeds in a one year forward, how much will I have at the end of the second year?

I will have $100 * (1.05) = \$105$

(b) Construct an arbitrage.

To construct an arbitrage, first notice that using the strategy in part (a) means that I invest \$96 and get \$105 in two years. This means that if I am to end up with \$100, I must start with $\frac{100}{105}$ bonds, or 0.9523 bonds. This costs me \$91.42. This has the same payoff as the two year bond (i.e., a single payment at the end of two years of \$100). Therefore, we can now buy low, sell high: sell the strategy in part (a), buy the two year bond, for an immediate profit of \$1.42.