Problem Set 7 Solution: Fixed Income Valuation.

I. *Implied Yield Curve, Forward Rates and No Arbitrage:* Consider the following prices for U.S. treasury notes on 2/15/96.

Rate	Maturity	Price
4½	Aug 96	98:11
5¼	Feb 97	99:01
5¾	Aug 97	98:23
6	Feb 98	98:15

A. What is the implied yield curve (expressed in terms of APRs with semiannual compounding)?

The following formula relating the yield on a discount bond to the relevant discount factor is used throughout this question.Let $d_t(0)$ be the discount factor for a t-year discount bond, and $y_t(0)$ be the yield on a t year discount bond expressed as an APR with semiannual compounding:

$$y_t(0) = 2 \left\{ \left[\frac{1}{d_t(0)} \right]^{1/(2t)} - 1 \right\} \Leftrightarrow d_t(0) = \frac{1}{\left[1 + \frac{y_t(0)}{2}\right]^{2t}} .$$

To obtain the yield on a 6 month discount bond:

a. Can recover the discount factor on a 6-month discount bond using the Aug 96 note and the no-arbitrage formula for pricing coupon bonds:

98.3438 = (100 + [4½/2]) $d_{½}$ (Feb 96) ⇒ $d_{½}$ (Feb 96) = 0.96180.

b. Can convert the 6-month discount bond discount factor into a yield expressed as an APR with semi-annual compounding:

$y_{½}$ (Feb 96) = {[1/$d_{½}$ (Feb 96)] -1}x 2 = {[1/0.96180] -1}x 2 = 7.9440%.

To obtain the yield on a 1-year discount bond:

a. Can recover the discount factor on a 1-year discount bond using the Feb 97 note and the no-arbitrage formula for pricing coupon bonds:

99.03125 = (5¼/2) $d_{½}$ (Feb 96) + (100 + [5¼/2]) d_1 (Feb 96)

99.03125 = (5¼/2) 0.96180 + (100 + [5¼/2]) d_1 (Feb 96) ⇒ d_1 (Feb 96) = 0.94038

b. Can convert the 1-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):

y_1 (Feb 96) = {[1/d_1 (Feb 96)]$^{0.5}$ -1} x 2 = {[1/0.94038]$^{0.5}$ -1} x 2 = 6.2425%.

To obtain the yield on a 1½-year discount bond:

a. Can recover the discount factor on a 1½-year discount bond using the Aug 97 note and the no-arbitrage formula for pricing coupon bonds:

98.71875 = (5¾/2) $d_{½}$ (Feb 96) + (5¾/2) d_1 (Feb 96) + (100 + [5¾/2]) $d_{1½}$ (Feb 96)
98.71875 = (5¾/2) 0.96180 + (5¾/2) 0.94038 + (100 + [5¾/2]) $d_{1½}$ (Feb 96)
⇒ $d_{1½}$ (Feb 96) = 0.90644.

b. Can convert the 1½-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):

$y_{1½}$ (Feb 96) = {[1/$d_{1½}$ (Feb 96)]^{a} -1} x 2 = {[1/0.90644]^{a} -1} x 2 = 6.6571%.

To obtain the yield on a 2-year discount bond:

a. Can recover the discount factor on a 2-year discount bond using the Feb 98 note and the no-arbitrage formula for pricing coupon bonds:

98.46875 = (6/2) $d_{½}$ (Feb 96) + (6/2) d_1 (Feb 96) + (6/2) $d_{1½}$ (Feb 96) + (100 + [6/2]) d_2 (Feb 96)
98.46875 = 3 x 0.96180 + 3 x 0.94038 + 3 x 0.90644 + 103 d_2 (Feb 96)
⇒ d_2 (Feb 96) = 0.87420.

b. Can convert the 2-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):

y_2 (Feb 96) = {[1/d_2 (Feb 96)]^{¼} -1} x 2 = {[1/0.87420]^{¼} -1} x 2 = 6.8364%.

A. What are the implied forward rates for the 6 month periods starting in 6 months, in 1 year and in 18 months (expressed as APRs with semiannual compounding)?

First, calculate $d_{½,1}$(Feb 96), $d_{1,1½}$(Feb 96), and $d_{1½,2}$(Feb 96) using the formula $d_{t,t+\tau}(0) = d_{t+\tau}(0) / d_t(0)$:

a. $d_{½,1}$(Feb 96) = d_1(Feb 96)/$d_{½}$(Feb 96) =0.94038/0.96180 = 0.97773.
b. $d_{1,1½,}$(Feb 96) = $d_{1½}$(Feb 96)/d_1(Feb 96) = 0.90644/0.94038 = 0.96391.
c. $d_{1½,2}$(Feb 96) = d_2(Feb 96)/$d_{1½}$(Feb 96) = 0.87420/0.90644 = 0.96443.

Then calculate the associated forward rates expressed as APRs with semiannual compounding using the following formula

$$f_{t,t+\tau}(0) = 2 \left\{ \left[\frac{1}{d_{t,t+\tau}(0)} \right]^{1/(2\tau)} - 1 \right\} \quad \text{with } \tau = ½.$$

a. $f_{½,1}$(Feb 96) = 2 {[1/$d_{½,1}$(Feb 96)] - 1} = 2 {[1/0.97773] - 1} = 4.5554%.

b. $f_{1,1½,}$(Feb 96) = 2 {[1/$d_{1,1½}$(Feb 96)] - 1} = 2 {[1/0.96391] - 1} = 7.4883%.

c. $f_{1½,2}$(Feb 96) = 2 {[1/$d_{1½,2}$(Feb 96)] - 1} = 2{[1/0.96443] - 1} = 7.3764%.

C. If there are no arbitrage opportunities, what is the price of a Aug 97 U.S. Treasury strip?

Use the discount factor for a 18 month discount bond:
$P^{Aug\ 97\ strip}$(Feb 96) = $d_{1½}$(Feb 96) 100 = 0.906440 x 100 = 90.6440.

D. Suppose the price of a Feb 97 U.S. Treasury strip is 94. Is there an arbitrage opportunity? If so, describe a strategy which earns an arbitrage profit.

Use the implied yield on a 1 year discount bond (expressed as an APR with semiannual compounding):
$P^{Feb\ 97\ strip}$(Feb 96) = d_1(Feb 96) 100 = 0.940831 x 100 = 94.038.

Since the price of the Feb 97 strip implied by the coupon bonds is greater than the strip's actual price, you want to buy the Feb 97 strip and sell a synthetic Feb 97 strip created using the Aug 96 and Feb 97 coupon bonds.
Let *a* be the number of Feb 97 notes that you buy and *b* be the number of Aug 96 notes that you buy. Want to choose *a* and *b* so that the net cash flow at 2/15/97 is zero and the net cash flow at 8/15/96 is zero:

Position	2/15/96	8/15/96	2/15/97
Buy 1 Feb 97 strip	-94		100
Buy *a* x Feb 97 note	-*a* x 99.03125	*a* x 2.625	*a* x 102.625
Buy *b* x Aug 96 note	-*b* x 98.34374	*b* x 102.25	
Net	-94 - *a* x 99.03125 - *b* x 98.34374	0	0

So

a. 100 + a x 102.625 = 0 which implies *a* = -100/102.625 = -0.97442. Thus, the Feb 97 note is sold.

b. *a* x 2.625 + *b* x 102.25 = 0 which implies *b* = -*a* x 2.625/102.25 = 0.02502.

Thus,

Position	2/15/96	8/15/96	2/15/97
Buy 1 Feb 97 strip	-94		100
Sell 0.97442 Feb 97 notes	0.97442 x 99.03125 = 96.4980	-0.97442 x 2.625 = -2.558	-0.97442 x 102.625 = -100
Buy 0.02502 Aug 96 notes	-0.02502 x 98.34374 = -2.4606	0.02502 x 102.25 = 2.558	
Net	0.03744	0	0

E. Suppose the prices for U.S. Treasury notes on 8/15/96 are given by:

Rate	Maturity	Price
5¼	Feb 97	98:11
5¾	Aug 97	98:21
6	Feb 98	98:00

1. What is the return from holding the Aug 97 note from 2/15/96 to 8/15/96?

The return from holding the Aug 97 note from 2/15/96 to 8/15/96 is given by:
$\{P^{5\frac{3}{4}\text{ Aug 97}}(\text{Aug 96}) + C^{5\frac{3}{4}\text{ Aug 97}}(\text{Aug 96}) - P^{5\frac{3}{4}\text{ Aug 97}}(\text{Feb 96})\}/P^{5\frac{3}{4}\text{ Aug 97}}(\text{Feb 96})$
$= \{98.65625 + 2.875 - 98.71875\}/98.71875 = 2.849\%$.

2. What is the return from holding the Aug 96 note from 2/15/96 to 8/15/96?

On 2/15/96, the Aug 96 note has an identical payoff to that of a 6 month discount bond. The return from holding the Aug 96 note from 2/15/96 to 8/15/96 is given by the yield on a 6 month discount bond on 2/15/96 expressed as an effective semiannual rate:
$y_{½}(\text{Feb 96})/2 = 7.9440\%/2 = 3.9720\%$.

3. Calculate the implied yield curve (expressed in terms of APRs with semiannual compounding)?

Can recover the discount factor for and yield on a six month discount bond using the Feb 97 note since it has only one payment left on 2/15/97:
$d_{½}(\text{Aug 96}) = 98.34375/(100 + [5¼/2]) = 0.95828$; and,
$y_{½}(\text{Aug 96}) = \{[1/d_{½}(\text{Aug 96})] - 1\} \times 2 = \{[1/0.95828] - 1\} \times 2 = 8.7067\%$.

Can recover the yield on a 1 year discount bond by creating a synthetic 1 year discount bond:

(1) More specifically:
(a) let *c* be the number of 5¾ Aug 97 notes bought.
(b) let *b* be the number of 5¼ Feb 97 notes bought.

Position		8/15/96	2/15/97	8/15/97
Buy *b* 5¼ Feb 97 notes	BUY SYNTHETIC DISCOUNT BOND	-*b* 98.34375	*b* 102.625	
Buy *c* 5¾ Aug 97 notes		-*c* 98.65625	*c* 2.875	*c* 102.875
Net		-*c* 98.65625 -*b* 98.34375	0	100

(2) So $c\ 102.875 = 100$ implies $c = 0.97205$.
(3) So $c\ 2.875 + b\ 102.625 = 0$ implies $b = -0.02723$.
(4) Thus, the cost of the synthetic discount bond is
$0.97205 \times 98.65625 - 0.02723 \times 98.34375 = 93.221$.
(5) The discount factor for and yield on a 1 year discount bond can then be obtained:

d_1 (Aug 96) = 93.221/100 = 0.93221; and,
y_1 (Aug 96) = $\{[1/d_1 \text{ (Aug 96)}]^{0.5} - 1\} \times 2 = \{[1/0.93221]^{0.5} - 1\} \times 2$ = 7.1443%.

Can then recover the yield on a 1½ year discount bond by creating a synthetic 1½ year discount bond:

(1) More specifically:
- (a) let *d* be the number of 6 Feb 98 notes bought.
- (b) let *c* be the number of 5¾ Aug 97 notes bought.
- (c) let *b* be the number of 5¼ Feb 97 notes bought.

Position	8/15/96	2/15/97	8/15/97	2/15/98
Buy *b* 5¼ Feb 97 notes	-*b* 98.34375	*b* 102.625		
Buy *c* 5¾ Aug 97 notes	-*c* 98.65625	*c* 2.875	*c* 102.875	
Buy *d* 6 Feb 98 notes	-*d* 98	*d* 3	*d* 3	*d* 103
Net	-*d* 98 -c 98.65625 - *b* 98.34375	0	0	100

(2) So *d* 103 = 100 implies $d = 0.97087$

(3) So *d* 3 + *c* 102.875 = 0 implies $c = -0.02831$.

(4) So *d* 3 + *c* 2.875 + *b* 102.625 = 0 implies $b = -0.02759$.

(5) Thus, the cost of the synthetic discount bond is

0.97087 x 98 - *0.02831* x 98.65625 - *0.02759* x 98.34375 = 89.639.

(6) The discount factor for and yield on a 1½ year discount bond can then be obtained:

$d_{1½}$ (Aug 96) = 89.639/100 = 0.89639; and,
$y_{1½}$ (Aug 96) = $\{[1/0.89639]^{a} - 1\} \times 2$ = 7.4263%.

4. Consider 2/15/96's forward rate for the period 8/15/96 to 2/15/97. How does it compare to the 6 month interest rate on 8/15/96? If these two rates differ, discuss why.

$y_{½}$ (Aug 96) = 8.7067%; and $f_{½,1}$(Feb 96) = 4.5554%.

The two rates are different. But these rates need not be the same and generally will not be. Even if the expectations hypthesis holds and $f^*_{t,t+½}(0) = E_{at\ time\ 0}[y^*_{½}(t)]$, it need not be the case that $f^*_{t,t+½}(0) = y^*_{½}(t)$. The yield on a six month discount bond in Aug 96 depends on economic conditions at that time while the forward rate $f_{½,1}$(Feb 96) is set in Feb 96 and depends on expections in Feb 96 about economic conditions in Aug 96.

II. *Forward Rates and the Yield Curve.*

A. To determine the yield on a two year discount bond:

1. Use the Aug 95 strip to determine the ½-yeardiscount bond discount factor: $d_{½}(2/15/95) = 97.75/100 = 0.9775$.
2. Use the Feb 96 strip to determine the 1-year discount bond discount factor: $d_1(2/15/95) = 93.25/100 = 0.9325$.
3. Use the Aug 96 strip to determine the 1½-year discount bond discount factor: $d_{1½}(2/15/95) = 90/100 = 0.9$.
4. Then use the 5% Feb 97 government bond.
   a. The coupon of 2.5 paid in Aug 95 can be converted to a value today using $d_{½}(2/15/95)$:

$P^{½}(2/15/95) = 2.5 \times 0.9775 = 2.44375$.

5. The coupon of 2.5 paid in Feb 96 can be converted to a value today using $d_1(2/15/95)$:

$P^1(2/15/95) = 2.5 \times 0.9325 = 2.33125$.

6. The coupon of 2.5 paid in Aug 96 can be converted to a value today using $d_{1½}(2/15/95)$:

$P^{1½}(2/15/95) = 2.5 \times 0.90 = 2.25$.

7. The value today of the final cash flow of 102.5 paid in Feb 97 can be obtained by subtracting the values of the earlier coupons from the bond's price:

$P^2(2/15/95) = 98 - 2.44375 - 2.33125 - 2.25 = 90.975$.

8. The yield on a two year discount bond expressed as an APR with semiannual compounding can be obtained:

$y_2(2/15/95) = \{[102.5/90.975]^{¼} - 1\} \times 2 = 0.06054 = 6.054\%$.

B. Use the following formula to obtain the forward contract discount factor available today for the one year period starting in 6 months expressed as an EAR:

$d_{0.5,1.5}(2/15/95) = d_{1.5}(2/15/95)/d_{0.5}(2/15/95) = 0.9/0.9775 = 0.92072$.

Then express this forward contract discount factor as an APR with semiannual compounding by using:

$f_{0.5,1.5}(2/15/95) = \{[1/d_{0.5,1.5}(2/15/95)]^{0.5} - 1\} \times 2 = \{[1/0.92072]^{0.5} - 1\} \times 2 = 8.433\%$.

C. Now $d_{0.5,1.5}$(2/15/95) refers to the forward price at 2/15/95 for delivery of a 1 year-discount bond in 6 months. So $d_{0.5,1.5}$(2/15/95) refers to the forward price today of an Aug 96 strip to be delivered in 6 months time. The forward price at time 2/15/95 for a $100 face value strip can be calculated:

$d_{0.5,1.5}$(2/15/195) x 100 = 0.92072 x 100 = 92.0717.