Problem Set 3 Solution.

I.  Expected Return, Return Standard Deviation, Covariance and Portfolios (cont):

<table>
<thead>
<tr>
<th>State</th>
<th>Probability</th>
<th>Asset A</th>
<th>Asset B</th>
<th>Riskless Asset</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boom</td>
<td>0.25</td>
<td>24%</td>
<td>14%</td>
<td>7%</td>
</tr>
<tr>
<td>Normal Growth</td>
<td>0.5</td>
<td>18%</td>
<td>9%</td>
<td>7%</td>
</tr>
<tr>
<td>Recession</td>
<td>0.25</td>
<td>2%</td>
<td>5%</td>
<td>7%</td>
</tr>
</tbody>
</table>

A. What is the expected return and standard deviation of return of a portfolio consisting of $\omega$% invested in asset A and $(1-\omega)$% in the riskless asset when $\omega$% is

1. -20%?
2. 60%?
3. 120%?

As an illustration, for $\omega_{A,p} = -0.2$:

$$E[R_p] = \omega_{A,p} E[R_A] + (1 - \omega_{A,p}) R_f$$

$$= -0.2 \times 15.5\% + 1.2 \times 7\% = 5.3\%$$

and

$$\sigma[R_p] = |\omega_{A,p}| \sigma[R_A]$$

$$= |-0.2| \times 8.1701\% = 1.6340\%$$

<table>
<thead>
<tr>
<th>$\omega_{A,p}$</th>
<th>$E[R_p]$</th>
<th>$\sigma[R_p]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>5.3%</td>
<td>1.6340%</td>
</tr>
<tr>
<td>0.6</td>
<td>12.1%</td>
<td>4.9020%</td>
</tr>
<tr>
<td>1.2</td>
<td>17.2%</td>
<td>9.8041%</td>
</tr>
</tbody>
</table>

B. What is the expected return and standard deviation of return of a portfolio consisting of $\omega$% invested in asset B and $(1-\omega)$% in the riskless asset when $\omega$% is

1. -20%?
2. 60%?
3. 120%?

As an illustration, for $\omega_{B,p} = 1.2$:

$$E[R_p] = \omega_{B,p} E[R_B] + (1 - \omega_{B,p}) R_f$$

$$= 1.2 \times 9.25\% + -0.2 \times 7\% = 9.7\%$$

and

$$\sigma[R_p] = |\omega_{B,p}| \sigma[R_B]$$
\[ \omega_{B,p} = \frac{1.2}{3.1918} = 3.8301\% \]

<table>
<thead>
<tr>
<th>( \omega_{B,p} )</th>
<th>( E[R_p] )</th>
<th>( \sigma[R_p] )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>6.55%</td>
<td>0.6383%</td>
</tr>
<tr>
<td>0.6</td>
<td>8.35%</td>
<td>1.9151%</td>
</tr>
<tr>
<td>1.2</td>
<td>9.7%</td>
<td>3.8301%</td>
</tr>
</tbody>
</table>

C. If a risk-averse investor has to decide whether to hold either asset A with the riskless asset or asset B with the riskless asset, which asset would the investor prefer to hold in combination with the riskless asset? Explain why? Do you need more information about the investor’s preferences to answer the question?

Any risk averse individual prefers the risky asset whose CAL has the higher slope. The reason is that for any point on the lower sloped CAL, there exists a point on the higher sloped CAL with the same expected return but lower standard deviation.

slope-CAL(A) = \( \frac{E[R_A]-R_f}{\sigma[R_A]} = \frac{15.5\%-7\%}{8.1701\%} = 1.0404 \)
slope-CAL(B) = \( \frac{E[R_B]-R_f}{\sigma[R_B]} = \frac{9.25\%-7\%}{3.1918\%} = 0.7049 \)

So any risk averse individual prefers to hold asset A in combination with the riskless asset than asset B.

D. What is the expected return and standard deviation of return of a portfolio consisting of \( \omega\% \) invested in asset A and \((1-\omega)\% \) in asset B when \( \omega\% \) is
1. -20%?
2. 80%?
3. 120%?

As an illustration, for \( \omega_{A,p} = 0.8 \):

\[
E[R_p] = \omega_{A,p} E[R_A] + (1- \omega_{A,p} ) E[R_B] \\
= 0.8 \times 15.5\% + 0.2 \times 9.25\% = 14.25\%
\]

and

\[
\sigma[R_p]^2 = \omega_{A,p}^2 \sigma[R_A]^2 + \omega_{B,p}^2 \sigma[R_B]^2 + 2 \omega_{A,p} \omega_{B,p} \sigma[R_A, R_B] \\
= (0.8 \times 0.8) 66.75 + (0.2 \times 0.2) 10.1875 + 2 (0.8 \times 0.2) 24.125 \\
= 42.72 + 0.4075 + 7.72 = 50.8475
\]

\[
\sigma[R_p] = 7.1307\%.
\]
<table>
<thead>
<tr>
<th>$\omega_{A,p}$</th>
<th>$E[R_p]$</th>
<th>$\sigma[R_p]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.2</td>
<td>8%</td>
<td>2.4%</td>
</tr>
<tr>
<td>0.8</td>
<td>14.25%</td>
<td>7.1307%</td>
</tr>
<tr>
<td>1.2</td>
<td>16.75%</td>
<td>9.2167%</td>
</tr>
</tbody>
</table>

Problem IV: Asset A is given by $\times$ and Asset B is given by $+$

Diagram: Graph showing the relationship between $E[R]$ and $\sigma[R]$ with different combinations of assets.
II. Using Dividend Yield Information (cont): Suppose the following data is to be used by Ms Q (a risk-averse investor) to form a portfolio that consists of the small firm fund and T-bills.

\[ E[R_{\text{Small}}(t)] = 1.369\% \quad \sigma[R_{\text{Small}}(t)] = 8.779\% \]

\[ E[DP(\text{start t})] = 4.446\% \quad \sigma[DP(\text{start t})] = 1.513\% \]

\[ \sigma[DP(\text{start t}), R_{\text{Small}}(t)] = 1.967 \]

where \( DP(\text{start t}) \) is the dividend yield on the S&P 500 known at the start of month \( t \). \( R_{\text{Small}}(t) \) is the return on the small firm fund in month \( t \).

A. Suppose it is the end of March 1997, Ms Q does not know \( DP \) and the return on T-bills for April is 0.3%.

1. Will Ms Q short sell the small firm fund?

The expected April return on the small firm fund is:

\[ E[R_{\text{Small}}(t)] = 1.369\% \]

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. \( E[R_{\text{Small}}(t)] > R_f \). So Ms Q does not want to short sell.

2. Will Ms Q buy the small firm fund on margin?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. \( E[R_{\text{Small}}(t)] > R_f \). So Ms Q may want to buy on margin depending on how risk averse she is.

3. Will Ms Q buy a positive amount of both assets?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. \( E[R_{\text{Small}}(t)] > R_f \). So Ms Q may want to buy positive amounts of both depending on how risk averse she is.

B. Suppose it is the end of March 1997, Ms Q knows that \( DP \) is 2% and the return on T-bills for April is 0.3%.

1. Will Ms Q short sell the small firm fund?

Given Ms Q’s information, the expected April return on the small firm fund is:

\[ \mu_{\text{Small,DP}} + \varphi_{\text{Small,DP}} DP(\text{start Apr}) = -2.451 + 0.859 \times 2 = -0.733\% \]

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. \( E[R_{\text{Small}}(t)] < R_f \). So Ms Q does want to short sell.
2. Will Ms Q buy the small firm fund on margin?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{Small}(t)] < R_f$. So Ms Q does not want to buy on margin.

3. Will Ms Q buy a positive amount of both assets?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{Small}(t)] < R_f$. So Ms Q does not want to buy positive amounts of both.

C. Suppose it is the end of October 1997, Ms Q does not know DP and the return on T-bills for November is 0.4%.

1. Will Ms Q short sell the small firm fund?

2. Will Ms Q buy the small firm fund on margin?

3. Will Ms Q buy a positive amount of both assets?

The answer to this question is the same as for part A.

D. Suppose it is the end of October 1997, Ms Q knows that DP is 5% and the return on T-bills for November is 0.4%.

1. What is the expected November return on the small firm fund?

$\mu_{Small,DP} + \varphi_{Small,DP} \cdot DP\text{(start Nov)} = -2.451 + 0.859 \times 5 = 1.844\%$.

2. Will Ms Q short sell the small firm fund?

Given Ms Q’s information, the expected November return on the small firm fund is:

$\mu_{Small,DP} + \varphi_{Small,DP} \cdot DP\text{(start Nov)} = -2.451 + 0.859 \times 5 = 1.844\%$.

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{Small}(t)] > R_f$. So Ms Q does not want to short sell.

3. Will Ms Q buy the small firm fund on margin?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{Small}(t)] > R_f$. So Ms Q may want to buy on margin depending on how risk averse she is.

4. Will Ms Q buy a positive amount of both assets?

Ms Q wants to lie on the positive-sloped portion of the portfolio possibility curve. $E[R_{Small}(t)] > R_f$. So Ms Q may want to buy positive amounts of both depending on how risk averse she is.
III. *The Two Risky Asset Case:*

A. As an illustration, for $\omega_{S,p} = 0.6$:

$$E[R_p] = \omega_{S,p} E[R_S] + (1 - \omega_{S,p}) E[R_B]$$

$$= 0.6 \times 22\% + 0.4 \times 13\% = 18.4\%$$

and

$$\sigma[R_p]^2 = \omega_{S,p}^2 \sigma[R_S]^2 + \omega_{B,p}^2 \sigma[R_B]^2 + 2 \omega_{S,p} \omega_{B,p} \rho[R_S, R_B] \sigma[R_S] \sigma[R_B]$$

$$= (0.6 \times 0.6) (32 \times 32) + (0.4 \times 0.4) (23 \times 23) + 2 (0.6 \times 0.4) (0.15 \times 32 \times 23)$$

$$= 368.64 + 84.64 + 52.992 = 506.272$$

$$\sigma[R_p] = 22.5005\%.$$ 

<table>
<thead>
<tr>
<th>$\omega_{A,p}$</th>
<th>$E[R_p]$</th>
<th>$\sigma[R_p]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>13%</td>
<td>23%</td>
</tr>
<tr>
<td>0.2</td>
<td>14.8%</td>
<td>20.3678%</td>
</tr>
<tr>
<td>0.4</td>
<td>16.6%</td>
<td>20.1810%</td>
</tr>
<tr>
<td>0.6</td>
<td>18.4%</td>
<td>22.5005%</td>
</tr>
<tr>
<td>0.8</td>
<td>20.2%</td>
<td>26.6805%</td>
</tr>
<tr>
<td>1.0</td>
<td>22%</td>
<td>32%</td>
</tr>
</tbody>
</table>

B. Note that $\bullet$ denotes the tangency portfolio $T$ in the following graph.
C.

Use the following formula:

\[
\omega_{S,T} = \frac{\sigma[R_B] E[r_S] - \sigma[R_S, R_B] E[r_B]}{\{\sigma[R_B] E[r_S] - \sigma[R_S, R_B] E[r_B]\}^2 + \{\sigma[R_S] E[r_B] - \sigma[R_S, R_B] E[r_S]\}}
\]

where \( r_i = R_i - R_f \) is the excess return on asset \( i \) (in excess of the riskless rate).

Now
\[
E[r_S] = 22\% - 9\% = 13\%.
\]
\[
E[r_B] = 13\% - 9\% = 4\%.
\]
\[
\sigma[R_S]^2 = 32 \times 32 = 1024.
\]
\[ \sigma[R_B]^2 = 23 \times 23 = 529. \]
\[ \sigma[R_S, R_B] = 0.15 \times 32 \times 23 = 110.4. \]

So, the weight of S in the tangency portfolio T is given by
\[
\omega_{S,T} = \frac{529 \times 13 - 110.4 \times 4}{529 \times 13 - 110.4 \times 4 + 1024 \times 4 - 110.4 \times 13} = 0.7075,
\]
and the weight of B in the tangency portfolio T is
\[ \omega_{B,T} = (1 - \omega_{S,T}) = 0.2925. \]

Thus,
\[
E[R_T] = \omega_{S,T} E[R_S] + (1 - \omega_{S,T}) E[R_B] = 0.7075 \times 22\% + 0.2925 \times 13\% = 19.3671\% 
\]
and
\[
\sigma[R_T]^2 = \omega_{S,T}^2 \sigma[R_S]^2 + \omega_{B,T}^2 \sigma[R_B]^2 + 2 \omega_{S,T} \omega_{B,T} \sigma[R_S, R_B] = 0.7075 \times 0.7075 \times 1024 + 0.2925 \times 0.2925 \times 529 + 2 \times 0.7075 \times 0.2925 \times 110.4 = 512.57 + 45.26 + 45.69 = 603.52
\]
\[ \sigma[R_T] = 24.5667\%. \]

**D. The reward to variability ratio is just the slope of the CAL.**
\[ \text{slope-CAL(T)} = \frac{E[R_T] - R_f}{\sigma[R_T]} = \frac{19.3671\% - 9\%}{24.5667\%} = 0.4220. \]

**E.**

1. **E.1.**

   **a.** One Answer: Any portfolio on the CAL of the risky tangency portfolio T consists of T and the riskless asset. For fund p,
\[
E[R_p] = 15\% = \omega_{T,p} E[R_T] + (1 - \omega_{T,p}) R_f = R_f + \omega_{T,p} \{E[R_T] - R_f\} = 9\% + \omega_{T,p} \{19.3671\%-9\%\}
\]
which implies that the weight of the tangency portfolio T in fund p is
\[ \omega_{T,p} = \frac{6\%}{10.3671\%} = 0.5788. \]
Finally,
\[ \sigma[R_p] = |\omega_{T,p}| \sigma[R_T] = 0.5788 \times 24.5667\% = 14.22\%. \]

   **b.** Second Answer: The equation for the CAL (T) line is given by
\[ E[R_p] = R_f + \sigma[R_p] \{E[R_T] - R_f\}/\sigma[R_T]. \]
Thus,
\[ 15\% = 9\% + \sigma[R_p] \{19.3671\%-9\%\}/24.5667\%
\]
which implies
\[ \sigma[R_p] = \frac{6\%}{0.4220} = 14.22\%. \]
2. Know that  

\[ R_p = \omega_{T,p} R_T + (1 - \omega_{T,p}) R_f \]

where \( \omega_{T,p} = 0.5788 \) is the weight of the tangency portfolio \( T \) in the fund \( p \) and  

\[ R_T = \omega_{S,T} R_S + (1 - \omega_{S,T}) R_B \]

where \( \omega_{S,T} = 0.7075 \) is the weight of the stock fund \( S \) in the tangency portfolio \( T \).

So  

\[ R_p = \omega_{T,p} \{ \omega_{S,T} R_S + (1 - \omega_{S,T}) R_B \} + (1 - \omega_{T,p}) R_f \]

\[ = \omega_{T,p} \omega_{S,T} R_S + \omega_{T,p} (1 - \omega_{S,T}) R_B + (1 - \omega_{T,p}) R_f \]

giving  

\[ \omega_{S,p} = \omega_{T,p} \omega_{S,T} = 0.5788 \times 0.7075 = 0.4095 \] as the weight of the stock fund \( S \) in fund \( p \).  

\[ \omega_{B,p} = \omega_{T,p} (1 - \omega_{S,T}) = 0.5788 \times 0.2925 = 0.1693 \] as the weight of the bond fund \( B \) in fund \( p \).  

\[ \omega_{f,p} = (1 - \omega_{T,p}) = 0.4212 \] as the weight of the riskless asset in fund \( p \).

F. If the fund \( q \) consists only of the stock fund and the bond fund:  

\[ E[R_q] = 15\% = \omega_{S,q} E[R_S] + (1 - \omega_{S,q}) E[R_B] \]

\[ = E[R_S] + \omega_{S,q} \{ E[R_S] - E[R_B] \} \]

\[ = 13\% + \omega_{S,q} 9\% \]

which implies that the weight of the stock fund \( S \) in the fund \( q \) is given by  

\[ \omega_{S,q} = \frac{2\%}{9\%} = 0.2222 \]

and the weight of the bond fund \( B \) in \( q \) is  

\[ \omega_{B,q} = (1 - \omega_{S,q}) = 0.7778. \]

So fund \( q \)’s standard deviation is  

\[ \sigma[R_q]^2 = \omega_{S,q}^2 \sigma[R_S]^2 + \omega_{B,q}^2 \sigma[R_B]^2 + 2 \omega_{S,q} \omega_{B,q} \sigma[R_S, R_B] \]

\[ = (0.2222x0.2222) (1024) + (0.7778x0.7778) (529) + 2 (0.2222x0.7778) (110.4) \]

\[ = 50.558 + 320.031 + 38.160 = 408.749 \]

\[ \sigma[R_q] = 20.2175\%. \]

Can see that though fund \( p \) from the previous question and fund \( q \) both have the same expected return, fund \( p \) has the lower standard deviation. So any risk averse individual would prefer to hold fund \( p \).