**Solution Set 8**

**Problem Set 8 Solution: Fixed Income Valuation.**

I.  *Implied Yield Curve, Forward Rates and No Arbitrage:* Consider the following prices for U.S. treasury notes on 2/15/96.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4½</td>
<td>Aug 96</td>
<td>98:11</td>
</tr>
<tr>
<td>5¼</td>
<td>Feb 97</td>
<td>99:01</td>
</tr>
<tr>
<td>5¾</td>
<td>Aug 97</td>
<td>98:23</td>
</tr>
<tr>
<td>6</td>
<td>Feb 98</td>
<td>98:15</td>
</tr>
</tbody>
</table>

A. What is the implied yield curve (expressed in terms of APRs with semiannual compounding)?

The following formula relating the yield on a discount bond to the relevant discount factor is used throughout this question. Let $d_t(0)$ be the discount factor for a $t$-year discount bond, and $y_t(0)$ be the yield on a $t$ year discount bond expressed as an APR with semiannual compounding:

$$
y_t(0) = 2 \left\{ \left[ \frac{1}{d_t(0)} \right]^{1/(2t)} - 1 \right\} \Rightarrow d_t(0) = \frac{1}{\left[ 1 + \frac{y_t(0)}{2} \right]^{2t}}.
$$

To obtain the yield on a 6 month discount bond:

a. Can recover the discount factor on a 6-month discount bond using the Aug 96 note and the no-arbitrage formula for pricing coupon bonds:

$$98.3438 = (100 + \frac{4½}{2}) d_{6/2} (Feb 96) + (100 + \frac{5¼}{2}) d_{1} (Feb 96) \Rightarrow d_{6/2} (Feb 96) = 0.96180.$$  

b. Can convert the 6-month discount bond discount factor into a yield expressed as an APR with semi-annual compounding:

$$y_{6/2} (Feb 96) = \{[1/d_{6/2} (Feb 96)] -1\} x 2 = \{[1/0.96180] -1\} x 2 = 7.9440\%.$$  

To obtain the yield on a 1-year discount bond:

a. Can recover the discount factor on a 1-year discount bond using the Feb 97 note and the no-arbitrage formula for pricing coupon bonds:

$$99.03125 = (5¼/2) d_{6/2} (Feb 96) + (100 + \frac{5¼}{2}) d_{1} (Feb 96) \Rightarrow d_{6/2} (Feb 96) = 0.94038.$$  

b. Can convert the 1-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the
above stated formula):

\[ y_{1\,\text{(Feb 96)}} = \left\{ \frac{1}{d_{1\,\text{(Feb 96)}}} \right\}^{0.5} - 1 \times 2 = \left\{ \frac{1}{0.94038} \right\}^{0.5} - 1 \times 2 = 6.2425\% \]

To obtain the yield on a 1½-year discount bond:

a. Can recover the discount factor on a 1½-year discount bond using the Aug 97 note and the no-arbitrage formula for pricing coupon bonds:

\[
98.71875 = \left( \frac{5\frac{3}{4}}{2} \right) d_{\frac{1}{2}}(\text{Feb 96}) + \left( \frac{5\frac{3}{4}}{2} \right) d_1(\text{Feb 96}) + \left( 100 + \frac{5\frac{3}{4}}{2} \right) d_{1\frac{1}{2}}(\text{Feb 96})
\]

\[
98.71875 = \left( \frac{5\frac{3}{4}}{2} \right) 0.96180 + \left( \frac{5\frac{3}{4}}{2} \right) 0.94038 + \left( 100 + \frac{5\frac{3}{4}}{2} \right) d_{1\frac{1}{2}}(\text{Feb 96})
\]

\[ d_{1\frac{1}{2}}(\text{Feb 96}) = 0.90644. \]

b. Can convert the 1½-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):

\[ y_{1\frac{1}{2}}(\text{Feb 96}) = \left\{ \frac{1}{d_{1\frac{1}{2}}(\text{Feb 96})} \right\}^{0.5} - 1 \times 2 = \left\{ \frac{1}{0.90644} \right\}^{0.5} - 1 \times 2 = 6.6571\% \]

To obtain the yield on a 2-year discount bond:

a. Can recover the discount factor on a 2-year discount bond using the Feb 98 note and the no-arbitrage formula for pricing coupon bonds:

\[
98.46875 = \left( \frac{6}{2} \right) d_{\frac{1}{2}}(\text{Feb 96}) + \left( \frac{6}{2} \right) d_1(\text{Feb 96}) + \left( \frac{6}{2} \right) d_{1\frac{1}{2}}(\text{Feb 96}) + \left( 100 + \frac{6}{2} \right) d_2(\text{Feb 96})
\]

\[
98.46875 = 3 \times 0.96180 + 3 \times 0.94038 + 3 \times 0.90644 + 103 d_2(\text{Feb 96})
\]

\[ d_2(\text{Feb 96}) = 0.87420. \]

b. Can convert the 2-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):

\[ y_2(\text{Feb 96}) = \left\{ \frac{1}{d_2(\text{Feb 96})} \right\}^{\frac{1}{2}} - 1 \times 2 = \left\{ \frac{1}{0.87420} \right\}^{\frac{1}{2}} - 1 \times 2 = 6.8364\% \]

B. What are the implied forward rates for the 6 month periods starting in 6 months, in 1 year and in 18 months (expressed as APRs with semiannual compounding)?

First, calculate \( d_{\frac{\tau}{2}}(\text{Feb 96}) \), \( d_{1,\frac{\tau}{2}}(\text{Feb 96}) \), and \( d_{1\frac{\tau}{2},2}(\text{Feb 96}) \) using the formula

\[ d_{t,\tau}(0) = \frac{d_t(0)}{d_0(0)}: \]

a. \( d_{\frac{\tau}{2}}(\text{Feb 96}) = d_{\frac{\tau}{2}}(\text{Feb 96})/d_{\frac{\tau}{2}}(\text{Feb 96}) = 0.94038/0.96180 = 0.97773. \)

b. \( d_{1,\frac{\tau}{2}}(\text{Feb 96}) = d_{1,\frac{\tau}{2}}(\text{Feb 96})/d_1(\text{Feb 96}) = 0.90644/0.94038 = 0.96391. \)

c. \( d_{1\frac{\tau}{2},2}(\text{Feb 96}) = d_{1\frac{\tau}{2},2}(\text{Feb 96})/d_{1\frac{\tau}{2}}(\text{Feb 96}) = 0.87420/0.90644 = 0.96443. \)

Then calculate the associated forward rates expressed as APRs with semiannual compounding using the following formula

\[ f_{t,\tau}(0) = 2 \left\{ \frac{1}{d_{t,\tau}(0)} \right\}^{\frac{1}{2\tau}} - 1 \] with \( \tau = \frac{1}{2}. \)
a. \( f_{1/3,1}^{\text{Feb 96}} = 2 \left\{ \left[ 1/d_{1/3,1}^{\text{Feb 96}} \right] - 1 \right\} = 2 \left\{ [1/0.97773] - 1 \right\} = 4.5554\% . \\
b. \( f_{1,1/2}^{\text{Feb 96}} = 2 \left\{ \left[ 1/d_{1,1/2}^{\text{Feb 96}} \right] - 1 \right\} = 2 \left\{ [1/0.96391] - 1 \right\} = 7.4883\% . \\
c. \( f_{1/2,2}^{\text{Feb 96}} = 2 \left\{ \left[ 1/d_{1/2,2}^{\text{Feb 96}} \right] - 1 \right\} = 2 \left\{ [1/0.96443] - 1 \right\} = 7.3764\% . \\

C. If there are no arbitrage opportunities, what is the price of a Aug 97 U.S. Treasury strip?

Use the discount factor for a 18 month discount bond:
\[ P_{\text{Aug 97 strip}}^{\text{Feb 96}} = d_{1.5}^{\text{Feb 96}} \times 100 = 0.906440 \times 100 = 90.6440. \]

D. Suppose the price of a Feb 97 U.S. Treasury strip is 94. Is there an arbitrage opportunity? If so, describe a strategy which earns an arbitrage profit.

Use the implied yield on a 1 year discount bond (expressed as an APR with semiannual compounding):
\[ P_{\text{Feb 97 strip}}^{\text{Feb 96}} = d_{1}^{\text{Feb 96}} \times 100 = 0.940831 \times 100 = 94.038. \]

Since the price of the Feb 97 strip implied by the coupon bonds is greater than the strip’s actual price, you want to buy the Feb 97 strip and sell a synthetic Feb 97 strip created using the Aug 96 and Feb 97 coupon bonds.

Let \( a \) be the number of Feb 97 notes that you buy and \( b \) be the number of Aug 96 notes that you buy. Want to choose \( a \) and \( b \) so that the net cash flow at 2/15/97 is zero and the net cash flow at 8/15/96 is zero:
### Solution Set 8

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/96</th>
<th>8/15/96</th>
<th>2/15/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1 Feb 97 strip</td>
<td>-94</td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>Buy a x Feb 97 note</td>
<td>-a x 99.03125</td>
<td>a x 2.625</td>
<td>a x 102.625</td>
</tr>
<tr>
<td>Buy b x Aug 96 note</td>
<td>-b x 98.34374</td>
<td>b x 102.25</td>
<td></td>
</tr>
<tr>
<td>Net</td>
<td>-94 - a x 99.03125</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

So

a. $100 + a \times 102.625 = 0$ which implies $a = -100/102.625 = -0.97442$. Thus, the Feb 97 note is sold.

b. $a \times 2.625 + b \times 102.25 = 0$ which implies $b = -a \times 2.625/102.25 = 0.02502$.

Thus,

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/96</th>
<th>8/15/96</th>
<th>2/15/97</th>
</tr>
</thead>
<tbody>
<tr>
<td>Buy 1 Feb 97 strip</td>
<td>-94</td>
<td></td>
<td>100</td>
</tr>
<tr>
<td>Sell 0.97442 Feb 97 notes</td>
<td>0.97442 x 99.03125</td>
<td>-0.97442 x 2.625</td>
<td>-0.97442 x 102.625</td>
</tr>
<tr>
<td></td>
<td>= 96.4980</td>
<td>= -2.558</td>
<td>= -100</td>
</tr>
<tr>
<td>Buy 0.02502 Aug 96 notes</td>
<td>-0.02502 x 98.34374</td>
<td>0.02502 x 102.25</td>
<td></td>
</tr>
<tr>
<td></td>
<td>= -2.4606</td>
<td>= 2.558</td>
<td></td>
</tr>
<tr>
<td>Net</td>
<td>0.03744</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

E. Suppose the prices for U.S. Treasury notes on 8/15/96 are given by:

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 ¼</td>
<td>Feb 97</td>
<td>98:11</td>
</tr>
<tr>
<td>5 ¾</td>
<td>Aug 97</td>
<td>98:21</td>
</tr>
<tr>
<td>6</td>
<td>Feb 98</td>
<td>98:00</td>
</tr>
</tbody>
</table>

1. What is the return from holding the Aug 97 note from 2/15/96 to 8/15/96?
The return from holding the Aug 97 note from 2/15/96 to 8/15/96 is given by:
\[
\{P^{5\%} \text{Aug 97 (Aug 96)} + C^{5\%} \text{Aug 97 (Aug 96)} - P^{5\%} \text{Aug 97 (Feb 96)}\}/P^{5\%} \text{Aug 97 (Feb 96)}
\]
\[
= \{98.65625 + 2.875 - 98.71875\}/98.71875 = 2.849\%.
\]

2. What is the return from holding the Aug 96 note from 2/15/96 to 8/15/96?

On 2/15/96, the Aug 96 note has an identical payoff to that of a 6 month discount bond. The return from holding the Aug 96 note from 2/15/96 to 8/15/96 is given by the yield on a 6 month discount bond on 2/15/96 expressed as an effective semiannual rate:
\[
y^{\frac{1\%}{2}}(\text{Feb 96})/2 = 7.9440\%/2 = 3.9720\%.
\]

3. Calculate the implied yield curve (expressed in terms of APRs with semiannual compounding)?

To obtain the yield on a 6-month discount bond as of Aug 96:

a. Can recover the discount factor on a 6-month discount bond using the Feb 97 note and the no-arbitrage formula for pricing coupon bonds:
\[
98.34375 = (100 + \frac{5}{4}/2) \cdot d^{\frac{1\%}{2}}(\text{Aug 96}) \Rightarrow d^{\frac{1\%}{2}}(\text{Aug 96}) = 0.95828.
\]

b. Can convert the 6-month discount bond discount factor into a yield expressed as an APR with semi-annual compounding:
\[
y^{\frac{1\%}{2}}(\text{Aug 96}) = \left\{\frac{1}{d^{\frac{1\%}{2}}(\text{Aug 96})}\right\} - 1 \times 2 = \left\{1/0.95828\right\} - 1 \times 2 = 8.7067\%.
\]

To obtain the yield on a 1-year discount bond as of Aug 96:

a. Can recover the discount factor on a 1-year discount bond using the Aug 97 note and the no-arbitrage formula for pricing coupon bonds:
\[
98.65625 = (5\frac{1}{4}/2) \cdot d^{\frac{1\%}{2}}(\text{Aug 96}) + (100 + \frac{5\frac{1}{4}/2}) \cdot d(\text{Aug 96})
\]
\[
98.65625 = (5\frac{1}{4}/2) \cdot 0.95828 + (100 + \frac{5\frac{1}{4}/2}) \cdot d(\text{Aug 96}) \Rightarrow d(\text{Aug 96}) = 0.93221.
\]

b. Can convert the 1-year discount bond discount factor into a yield expressed as an APR with semi-annual compounding (using the above stated formula):
\[
y(\text{Aug 96}) = \left\{\frac{1}{d(\text{Aug 96})}\right\}^{0.5} - 1 \times 2 = \left\{1/0.93221\right\}^{0.5} - 1 \times 2 = 7.1443\%.
\]

To obtain the yield on a 1\(\frac{1}{2}\)-year discount bond as of Aug 96:

a. Can recover the discount factor on a 1\(\frac{1}{2}\)-year discount bond using the Feb 98 note and the no-arbitrage formula for pricing coupon bonds:
\[
98 = (6/2) \cdot d^{\frac{1\%}{2}}(\text{Aug 96}) + (6/2) \cdot d(\text{Aug 96}) + (100 + [6/2]) \cdot d_{\frac{1}{2}}(\text{Aug 96})
\]
\[
98 = (6/2) \cdot 0.95828 + (6/2) \cdot 0.93221 + (100 + [6/2]) \cdot d_{\frac{1}{2}}(\text{Aug 96})
\]
\[
= d_{\frac{1}{2}}(\text{Aug 96}) = 0.89639.
\]

b. Can convert the 1\(\frac{1}{2}\)-year discount bond discount factor into a yield
expressed as an APR with semi-annual compounding (using the above stated formula):

\[ y_{1/2} \ (\text{Aug} \ 96) = \{[1/d_{1/2} \ (\text{Aug} \ 96)]^{a} - 1\} \times 2 = \{[1/0.89639]^{a} - 1\} \times 2 = 7.4263\%. \]

4. Consider 2/15/96's forward rate for the period 8/15/96 to 2/15/97. How does it compare to the 6 month interest rate on 8/15/96? If these two rates differ, discuss why.

\[ y_{1/2} \ (\text{Aug} \ 96) = 8.7067\%; \text{ and } f_{1/2,1} \ (\text{Feb} \ 96) = 4.5554\%. \]

The two rates are different. But these rates need not be the same and generally will not be. Even if the expectations hypothesis holds and \( f_{t+t/2} \ (0) = E_{\text{time} \ 0} [y_{t+1/2} (t)] \), it need not be the case that \( f_{t+t/2} \ (0) = y_{t+1/2} (t) \). The yield on a six month discount bond in Aug 96 depends on economic conditions at that time while the forward rate \( f_{1/2,1} \ (\text{Feb} \ 96) \) is set in Feb 96 and depends on expectations in Feb 96 about economic conditions in Aug 96.

II. \textit{Forward Rates and the Yield Curve.}

A. To determine the yield on a two year discount bond:

1. Use the Aug 95 strip to determine the \( 1/2 \)-yr round discount bond discount factor: \( d_{1/2} \ (2/15/95) = 97.75/100 = 0.9775 \).

2. Use the Feb 96 strip to determine the 1-yr discount bond discount factor: \( d_{1} \ (2/15/95) = 93.25/100 = 0.9325 \).

3. Use the Aug 96 strip to determine the 11/2-yr discount bond discount factor: \( d_{11/2} \ (2/15/95) = 90/100 = 0.9 \).

4. Then use the 5% Feb 97 government bond.

a. The coupon of 2.5 paid in Aug 95 can be converted to a value today using \( d_{1/2} \ (2/15/95) \):

\[ P_{1/2} \ (2/15/95) = 2.5 \times 0.9775 = 2.44375. \]

b. The coupon of 2.5 paid in Feb 96 can be converted to a value today using \( d_{1} \ (2/15/95) \):

\[ P_{1} \ (2/15/95) = 2.5 \times 0.9325 = 2.33125. \]

c. The coupon of 2.5 paid in Aug 96 can be converted to a value today using \( d_{11/2} \ (2/15/95) \):

\[ P_{11/2} \ (2/15/95) = 2.5 \times 0.90 = 2.25. \]

7. The value today of the final cash flow of 102.5 paid in Feb 97 can be obtained by subtracting the values of the earlier coupons from the bond’s price:

\[ P_{2} \ (2/15/95) = 98 - 2.44375 - 2.33125 - 2.25 = 90.975. \]

8. The yield on a two year discount bond expressed as an APR with
semiannual compounding can be obtained:
\[ y_2(2/15/95) = \left\{ \left[ \frac{102.5}{90.975} \right]^{1/4} - 1 \right\} \times 2 = 0.06054 = 6.054\%. \]

B. Use the following formula to obtain the forward contract discount factor available today for the one year period starting in 6 months expressed as an EAR:
\[ d_{0.5,1.5}(2/15/95) = d_{1.5}(2/15/95)/d_{0.5}(2/15/95) = 0.9/0.9775 = 0.92072. \]
Then express this forward contract discount factor as an APR with semiannual compounding by using:
\[ f_{0.5,1.5}(2/15/95) = \left\{ \left[ \frac{1}{d_{0.5,1.5}(2/15/95)} \right]^{0.5} - 1 \right\} \times 2 = \left\{ \left[ \frac{1}{0.92072} \right]^{0.5} - 1 \right\} \times 2 = 8.433\%. \]

C. Now \( d_{0.5,1.5}(2/15/95) \) refers to the forward price at 2/15/95 for delivery of a 1 year-discount bond in 6 months. So \( d_{0.5,1.5}(2/15/95) \) refers to the forward price today of an Aug 96 strip to be delivered in 6 months time. The forward price at time 2/15/95 for a $100 face value strip can be calculated:
\[ d_{0.5,1.5}(2/15/95) \times 100 = 0.92072 \times 100 = 92.0717. \]