Lecture 1: Overview

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II. Asset Classes
III. Financial System.
IV. Financial Markets
V. Financial Intermediaries
VI. Issues addressed by Finance Theory
VII. Key Concepts.

Lecture 1-2: Time Value of Money

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Lecture 1: Overview

I. Reading
   A. BKM Chapter 1.
   B. Skim BKM Chapters 2 and 4.

II. Asset Classes
   A. Real Assets
      1. natural resources.
      2. physical capital.
      3. human capital.
   B. Financial Assets (referred to as securities)
      1. Money (as a medium of exchange)
         a. is held to allow the completion of transactions.
      2. Debt
         a. a claim to a predetermined payment stream secured on a set of real
            or financial assets.
         b. maturity is time from issue to expiration.
      3. Equity
         a. residual claim to a set of real or financial assets (usually of a
            corporation) usually coupled with corporate control.
      4. Derivatives
         a. payoff is dependant on the value of some other (usually financial)
            asset.
C. Illustration.
   1. Debt vs. Equity.
      a. Suppose XYZ Co’s assets pay off a random amount CF in 1 year’s time and XYZ has issued debt with a promised payment of $100 in 1 year’s time, and equity.

<table>
<thead>
<tr>
<th>CF</th>
<th>&lt;100</th>
<th>60</th>
<th>80</th>
<th>100</th>
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</table>

b. If CF is uncertain, XYZ’s equity is riskier than XYZ’s debt.

2. Derivatives.
   a. A call option gives its holder the right (but not the obligation) to buy an asset by paying a prespecified price (the strike price).
   b. Consider a call option on XYZ Co’s equity with a strike price of $40 that can be exercised in 1 year’s time.

<table>
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<th>Equity</th>
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<td></td>
<td>max{0, Equity-40}</td>
<td></td>
<td></td>
<td></td>
</tr>
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</table>

c. Notice that the call option’s payoff at the exercise date is never less than 0.

Example: IBM Corporation.
   A. Real Assets: plant used to build Thinkpads.
   B. Claims on the Real Assets:
      1. Equity: IBM stock.
      2. Debt: IBM bonds.
      3. IBM stock is much more volatile than IBM bonds.
   C. Derivatives: Claims on IBM stock.
      1. A call option on IBM stock gives the holder the right (but not the obligation) to buy the stock at a given exercise price.
III. Financial System.
   A. The financial system refers to the collection of institutions by which financial assets are created and traded.
   B. Purposes (which allow the financial system to create wealth)
      1. transfer capital from savers (investors) to capital users (usually corporations).
      2. discipline investment decisions by firms.

Example: A firm may want to expand by going public; i.e., by issuing equity to the public in return for cash. Since the public knows that the firm is going to use cash to expand, an investor will only subscribe to the IPO if she thinks expansion is a value enhancing strategy.

3. allow investors to smooth consumption intertemporally.

Example: An MBA student has low income now but high future income. A student loan allows the student to smooth her consumption through time relative to her income through time.

4. facilitate the reduction of riskbearing by repackaging risks.
5. disseminate information.

C. Institutions
   1. Government
   2. Financial Markets: institutions which trade financial assets.
   3. Financial Intermediaries: entities which operate within or outside financial markets to facilitate the trading of financial assets.
IV. Financial Markets

A. Primary vs Secondary Markets
   1. Primary Market: new issues of a security are sold to initial buyers.

Example: An IPO is an initial public offering of equity by a privately-held firm.

2. Secondary Market: previously issued securities are traded in a secondary market.

Example: NYSE and the National Association of Securities Dealers Automated Quotation System (NASDAQ) are examples of secondary markets for equities.

B. Exchange vs Over-the-Counter Market
   1. Exchange: Buyers and sellers of securities meet in one central location to conduct trades.

Examples: 1) NYSE (stocks); 2) Chicago Board of Trade (futures).

2. Over-the Counter Market: Dealers at different locations stand ready to buy and sell securities "over the counter" to anyone that accepts their prices.

Examples: 1) government bonds are traded over the counter through primary and secondary dealers; 2) NASDAQ for stocks.

C. Money vs Capital Markets
   2. Capital: long term debt instruments (>1 year maturity) and equity.
V. Financial Intermediaries
   A. Services Provided
      1. reduce search costs associated with finding saving or investment opportunities.
      2. generate information needed by investors.
      3. provide risk and portfolio management services.
      4. issue financial assets that repackage risks.
      5. take advantage of the economies of scale associated with buying and selling financial assets.
   B. Types
      1. Depository Institutions
         a. Commercial Banks.
         b. Savings and Loan Associations, Mutual Savings Banks.
         c. Credit Unions.
      2. Contractual Savings Institutions
         a. Life Insurance Companies.
         b. Fire and Casualty Insurance Companies.
         c. Pension Funds.
      3. Investment Intermediaries
         a. Brokers.
         b. Mutual Funds.
         c. Money Market Mutual Funds.
         d. Finance Companies.
VI. Issues addressed by Finance Theory
   A. Financial decision-making by corporations. How do corporations decide whether to undertake an investment project? (Corporate Finance)
   B. Financial decision-making by individuals. How do individuals invest their savings?
   C. Valuation of assets both real and financial. Why do expected returns vary across assets?

Examples: 1) CAPM is a asset pricing model; 2) Black-Scholes model values call options; 3) Cox Ingersoll Ross model values fixed income assets.

Why is this an important question? Because expected returns vary greatly across assets.

Mean Nominal 1 Month Return on U.S. Stock and Bond Portfolios: 7/26–12/95
Lecture 1 Foundations of Finance

TB - 1 mth Treasury Bills
GB - Long-term U.S. Government Bonds
CB - Corporate Bonds
S&P - S&P 500 Index
SMV - Portfolio of Small Market Value Stocks (Small Caps)
HBM - Portfolio of High Book-to-market Stocks (Value Stocks)

Mean Nominal 5 Year Return on U.S. Stock and Bond Portfolios:
7/26/631 - 1/91/12/95
VII. Key Concepts.
   A. Time value of money: a dollar today is worth more than a dollar later.

Example: Treasury strips pay a face value of $100 at the maturity date. Strips always trade for less than $100. See for example Friday’s Wall Street Journal for Thursday May 11.
B. Diversification: don’t put all your eggs in one basket.
   1. Portfolios of assets and individual assets have similar average returns; but
   2. Portfolios volatility declines as the number of assets in the portfolio increases.


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**Figure 6. Excess standard deviation against time and number of stocks.** The excess standard deviation of a portfolio is the difference between the portfolio’s standard deviation and the standard deviation of an equally weighted index. The top panel plots annualized excess standard deviation against time. Excess standard deviation is calculated each year from daily data within the year, for randomly selected portfolios containing two stocks (solid line), five stocks (top dashed line), 20 stocks (long dashed line), and 50 stocks (bottom dashed line). The bottom panel plots annualized excess standard deviation against the number of stocks in the portfolio, for sample periods 1963 to 1973 (solid line), 1974 to 1985 (bottom dashed line), and 1986 to 1997 (top dashed line).
C. Risk-adjustment: riskier assets offer higher expected returns
   1. Assets offer different average returns because they have different risk levels. The flip side is that investors require different returns on different investments depending on their risk levels.
   2. Need to quantify what we mean by risk.

Example: Above discussion of differences in average return across risk classes. Stocks offer higher average return than government and corporate debt but risk (however defined) is also higher.
D. No arbitrage: 2 assets with the same cash flows must have the same price.
   1. An investment that does not require any cash outflows and generates a strictly positive cash inflow with some probability is known as an arbitrage opportunity.
   2. Well-functioning market: arbitrage opportunities can not exist since any individual who prefers more to less wants to invest as much as possible in the arbitrage opportunity.
   3. The absence of arbitrage implies that any two assets with the same stream of cash flows must have the same price. This implication is known as the law of one price. Even small price differences represent a riskless profit.

Example: Friday May 12 WSJ
U.S. Treasury Strips - Prices Thursday May 11

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<th>Chg</th>
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<td>88:12</td>
<td>...</td>
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<td>np</td>
<td>88:11</td>
<td>88:12</td>
<td>...</td>
<td>4.99</td>
</tr>
</tbody>
</table>

4. Price differences of assets with identical cash flows must be due to tax and liquidity differences plus transaction costs.
E. Option value: a right (without obligation) to do any action in the future must have a non-negative value today
   1. An option gives the holder the right but not the obligation to take a certain action in the future.
   2. An option always has non-negative value for the holder.

Example: A firm has two bond issues outstanding:
1) 5% semi-annual coupon bonds maturing in 11/08.
2) 5% semi-annual coupon bonds maturing in 11/08 but the holder has the option to convert the bonds to stock at any time prior to 11/08.

For a par value of $100, which bond is more expensive? Bond 2) is more expensive since the option to convert is valuable: bond 2) cannot be less valuable than the bond 1).
F. Market Efficiency: price is an unbiased estimate of value
   1. In an efficient market: the price of a security is an unbiased estimate of its value.
   2. U.S. stock market is probably efficient with respect to publicly available information. So cannot use publicly available information to earn higher than average risk-adjusted returns on average.
   3. Market efficiency is one of the big warnings issued by the course.

Examples:
A. Mutual funds:
   1. Cross-sectionally annual fund return varies inversely 1 for 1 with the expense ratio.
   2. So if a fund increases its expense ratio from 0.5% p.a. to 1.5% p.a., its annual performance can be expected to decline by about 1%.

B. Individuals:
   2. Account data for over 60,000 households from a large discount brokerage firm from Feb 1991 through Dec 1996.
   3. Average annualized returns:
      a. Value-weighted market index: 17.1%.
      b. Households before trading costs: 17.7%.
      c. Households, net: 15.3%.
      d. 20% of households that trade the most, net: 10%.
   4. The central message is that trading is hazardous to your wealth.
Lectures 1-2: Time Value of Money

I. Reading
   A. RWJ Chapter 4 and 5.

II. Time Line
   A. $1 received today is not the same as a $1 received in one period's time.
   B. The timing of a cash flow affects its value.
   C. When valuing cash flow streams: a good idea is to draw a time line.

   $100

   is not the same as

   $100
III. Interest Rate: Discrete Compounding

A. Example:
1. Question: Today is the start of 2006. Suppose I can invest $100 at an effective annual interest rate of 12%. What is my $100 worth at the end of the year?

\[
\begin{array}{c|c|c}
\text{end 05} & \text{end 06} \\
\$100 & V_1 \\
\end{array}
\]

\[
V_1 = V_0 + \text{Interest} = \$100 + \$12 = \$100(1+0.12) = \$112.
\]

B. Definition:
1. The effective interest rate \( r \) (expressed as a decimal) over any period tells what $x will be worth at the end of the period using the following formula: $x (1+r)$.
2. The effective interest rate \( r \) (expressed as a decimal) over any period from \( t \) to \( (t+1) \) satisfies:

\[
V_{t+1} = V_t (1+r)
\]

where \( V_t \) is the value at time \( t \) and \( V_{t+1} \) is the value at time \( t+1 \).

C. Example:
1. Question: Today is the start of 2006. Suppose I can invest $100 at the start of 2007 at an effective annual interest rate of 12%. What is my $100 worth at the end of 2007?

\[
\begin{array}{c|c|c}
\text{end 05} & \text{end 06} & \text{end 07} \\
\$100 & \text{ } & V_2 \\
\end{array}
\]

\[
V_2 = V_1 (1+0.12) = \$100(1+0.12) = \$112.
\]
IV. Single Sums: Multiple Periods and Future Values

A. Example (cont):

1. Question: Today is the start of 2006. Suppose I can invest $100 at an effective annual interest rate of 12%. What is my $100 worth at the end of 3 years?

   \[
   \begin{array}{c|c}
   \text{end 05} & \text{end 08} \\
   \hline
   \$100 & V_3 \\
   \end{array}
   \]

2. One Answer: Obtain $V_3$ in three steps

\[
V_0 = \$100 \\
V_i = V_{i-1}(1+r) = \$100(1+0.12) = \$112 \\
V_2 = V_1(1+r) = \$112(1+0.12) = \$125.44 \\
V_3 = V_2(1+r) = \$125.44(1+0.12) = \$140.49
\]

\[
\begin{array}{c|c|c|c}
\text{end 05} & \text{end 06} & \text{end 07} & \text{end 08} \\
\hline
\$100 & \$112 & \$125.44 & \$140.49 \\
\end{array}
\]

3. Another Answer: Can see that the 3 steps could be combined into 1

\[
V_0 = \$100 \\
V_3 = V_0(1+r)^3 = \$100(1+0.12)^3 = \$140.49
\]

4. Notice how the answer is the initial investment $V_0$ times a multiplier that only depends on the effective interest rate and the investment interval. This multiplier is known as the future value interest factor.
B. The future value formula answers the following question
   1. If we invest some money at a given effective interest rate, how much
      money would we have at some future time?
   2. If we invest \( V_0 \) today at a given effective interest rate per period of \( r \)
      (expressed as a decimal), how much money would we have in \( n \) periods
      time; i.e., what is \( V_n \)?
   3. If we invest \( V_t \) in \( t \) periods time at a given effective interest rate per period
      of \( r \) (expressed as a decimal), how much money would we have after \( n \)
      periods from investing the money; i.e., what is \( V_{t+n} \)?

C. The general future value formula:
   1. \( V_{t+n} = V_t (1+r)^n \) where \([(1+r)^n]=FVIF_{r,n}\) is the future value interest factor.
   2. Notice that the future value interest factor does not depend on when the
      money is invested.

D. Example:
   1. Question: Today is the start of 2006. Suppose I can invest $100 at the end
      of 2006 at an effective annual interest rate of 12%. What is my $100
      worth after being invested for 3 years?

<table>
<thead>
<tr>
<th>end 06</th>
<th>end 09</th>
</tr>
</thead>
<tbody>
<tr>
<td>$100</td>
<td>$V_4</td>
</tr>
</tbody>
</table>

   2. Answer: Use the future value formula. \( V_1 = $100 \)

   \[
   V_4 = V_1 \times FVIF_{0.12,3} = V_1 (1+0.12)^3 = $100(1+0.12)^3 = $140.49
   \]
V. Single Sums: Multiple Periods and Present Values
   A. Example cont:
      1. Question: Today is the start of 2006. Suppose I can invest at an effective annual interest rate of 12%. How much do I need to invest today to have $140.49 at the end of 3 years?

      
      \[
      V_0 = \frac{V_3}{(1+r)^3} = \frac{140.49}{(1+0.12)^3} = 100
      \]

      3. Notice how the answer is the final value \( V_3 \) times a multiplier that only depends on the effective interest rate and the investment interval. This multiplier is known as the present value interest factor.
B. The present value formula answers the following question
1. If we can invest money at a given effective interest rate, how much money do we need to invest today to have a given sum at some future time?
2. If we can invest money at a given effective interest rate $r$ (expressed as a decimal), how much money do we need to invest today $V_0$ to have a given sum $V_n$ in $n$ periods time?
3. If we can invest money at a given effective interest rate $r$ (expressed as a decimal), how much money do we need to invest in $t$ periods time $V_t$ to have a given sum $V_{t+n}$ in $(t+n)$ periods from today?

C. The general present value formula:
1. $V_t = V_{t+n} \left[1/(1+r)^n\right]$ where $[1/(1+r)^n]=(1+r)^{-n}$ = PVIF$_{r,n}$ is the present value interest factor.

D. Example:
1. Question: Today is the start of 2006. Suppose I can invest at an effective annual interest rate of 12%. How much do I need to invest at the start of 2007 to have $140.49 at the end of 4 years from today?

\[
\begin{array}{c|c|c}
\text{end 06} & \text{end 09} & \\
V_1 & V_4 & $140.49 \\
\end{array}
\]

2. Answer: Use the present value formula:

\[
V_1 = V_4 \times PVIF_{0.12,3} = V_4/(1+r)^3 = $140.49/(1+0.12)^3 = $100
\]
VI. Equivalent Effective Interest Rates Over Different Compounding Periods.

A. General relation between the effective rates for compounding periods of different lengths:
   1. The effective \( n \)-period rate \( r_n \) (expressed as a decimal) is related to the effective one period rate:

   \[(1+r_n) = (1+r)^n.\]

   2. Note \( r_n \neq n \cdot r \).

B. Example:
   1. Question: Suppose the effective annual interest rate is 12%. What is the effective 3-year interest rate?
   2. Answer: Using the effective rate formula

   \[1+r_3 = (1+r)^3 = 1.00123 = 1.4049\]

   and so \( r_3 = 0.4049 \) and the effective 3-year rate is 40.49%.

C. Example
   1. Question: Suppose the effective monthly rate is 0.94888%. What is the effective annual rate?
   2. Answer: Here one period is a month. \( r = 0.0094888 \). Using the effective rate formula

   \[1+r_{12} = (1+r)^{12} = 1.0094888^{12} = 1.12\]

   and so \( r_{12} = 0.12 \) and the effective annual rate is 12%.
D. The effective rate formula also applies for $n$ a fraction; e.g., if the effective 1-year rate is known, the monthly effective rate can be calculated using the formula with $n = 1/12$.

E. Example

1. Question: Suppose the effective annual rate is 12% What is the effective monthly rate?

2. Answer: Here one period is a year. $r = 0.12$. Using the effective rate formula

$$1 + r_{1/12} = (1 + r)^{1/12} = 1.12^{1/12} = 1.009488$$

and so $r_{1/12} = 0.009488$ and the effective monthly rate is 0.9488%.

F. EAR

1. Definition: EAR is the effective annual rate.
VII. Alternate Interest Rate Concepts

A. Nominal Rate or APR

1. Definition: when the compounding period is some fraction of a year \( \frac{1}{m} \), the nominal rate \( i_{\text{nom}} \) (expressed as a decimal) equals \( m \, r_{\frac{1}{m}} \) where \( r_{\frac{1}{m}} \) is the effective \( \frac{1}{m} \)-year rate (expressed as a decimal).

2. Example: if the compound period is a month \( (m=12) \) and the effective monthly rate is 0.94888\% then \( r_{\frac{1}{12}} = 0.0094888 \), \( i_{\text{nom}} = 12 \times 0.0094888 = 0.11387 \) and the nominal rate is 11.387\%.

3. Fact: the nominal rate only equals the effective annual rate when the compound period is a year.

4. Example (cont): with a compound period of a month, the nominal rate is 11.387\% while the effective annual rate is 12\%.

5. Periodic rate

a. Definition: when the compounding period is some fraction of a year \( \frac{1}{m} \), the periodic rate (expressed as a decimal) equals the effective \( \frac{1}{m} \)-year rate \( r_{\frac{1}{m}} \).

B. Formulas.

1. From Nominal Rate to EAR (both expressed as decimals)

\[
r = \left(1 + \frac{i_{\text{nom}}}{m}\right)^m - 1
\]

2. From EAR to Nominal Rate (both expressed as decimals)

\[
i_{\text{nom}} = \left[ \left(1 + r\right)^{\frac{1}{m}} - 1 \right] \, m
\]

Example:

<table>
<thead>
<tr>
<th>Compound Period</th>
<th>EAR</th>
<th>Nominal Rate</th>
<th>EAR</th>
<th>Nominal Rate</th>
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<td>12%</td>
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<tr>
<td>6 months</td>
<td>12%</td>
<td>11.6601%</td>
<td>15.5625%</td>
<td>15%</td>
</tr>
<tr>
<td>3 months</td>
<td>12%</td>
<td>11.4949%</td>
<td>15.8650%</td>
<td>15%</td>
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<tr>
<td>1 month</td>
<td>12%</td>
<td>11.3866%</td>
<td>16.0755%</td>
<td>15%</td>
</tr>
<tr>
<td>1 day (365 days=1year)</td>
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<td>11.3346%</td>
<td>16.1798%</td>
<td>15%</td>
</tr>
<tr>
<td>continuous</td>
<td>12%</td>
<td>11.3329%</td>
<td>16.1834%</td>
<td>15%</td>
</tr>
</tbody>
</table>
C. Continuous Compounding.

1. Definition:
   a. Let \( r \) be the effective interest rate for one period and define \( r' \), the continuous interest rate for a period as follows:
   \[
   \exp\{r'\} = (1+r)
   \]
   b. Thus, the continuous rate can be obtained from the effective rate as follows:
   \[
   r' = \ln (1+r).
   \]

2. Example: The effective annual rate is 12%.
   a. Question: What is the continuously compounded annual rate?
   b. Answer: EAR is \( r = 0.12 \). The continuously compounded annual rate (expressed as a decimal) is \( r' = \ln(1+r) = \ln(1.12) = 0.1133 \).

3. Rules for using continuous interest rates.
   a. The continuous interest rate for \( n \) periods (where \( n \) can be a fraction) is given by:
   \[
   r'_n = nr'.
   \]
   C.f. effective interest rates for which this is not true: \( r_n \neq nr \).
   This additivity is a major advantage of continuous compounding.
   b. To obtain future values, use the following formula:
   \[
   V_{t+n} = V_t \exp \{nr'\}.
   \]
   c. To obtain present values, use the following formula:
   \[
   V_t = V_{t+n} \exp \{-nr'\}.
   \]

4. Example (cont): The effective annual rate is 12%.
   a. Question: What is the continuously compounded semiannual rate?
   b. Answer: The continuously compounded annual rate is 11.33%; i.e., \( r' = 0.1133 \). So the continuously compounded semiannual rate is \((11.33/2)\% = 5.666\%\) since
   \[
   r'_{1/2} = (1/2) r' = 0.5 \times 0.1133 = 0.05666.
   \]
   c. Question: How much will $500 invested today be worth in 6 months?
   d. Answer: Two approaches to obtaining what $500 will be worth in 6 months:
      (1) Continuous Compounding:
      \[
      V_{1/2} = $500 \exp \{(1/2) r'\} = $500 \exp \{0.05666\} = $500 \times 1.0583005 = $529.15.
      \]
      (2) Discrete Compounding:
      \[
      V_{1/2} = $500 \times 1.12^{1/2} = $500 \times 1.0583005 = $529.15.
      \]

5. Note: \( r' \) can be interpreted as the nominal interest rate associated with an infinitesimally small compound period.
VIII. Multiple Cash Flows.

A. Rules

1. Once each of a set of cash flows at different points in time has been converted to a cash flow at the same point in time, those cash flows can be added to get the value of the set of cash flows at that point.

\[
\begin{array}{cccccccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots & N-1 & N \\
C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 & C_{10} & \ldots & C_{N-1} & C_N \\
\end{array}
\]

\[
V_0^1 = C_1 \cdot PVIF_{r,1} = C_1 (1+r)^{-1} \\
V_0^2 = C_2 \cdot PVIF_{r,2} = C_2 (1+r)^{-2} \\
V_0^3 = C_3 \cdot PVIF_{r,3} = C_3 (1+r)^{-3} \\
V_0^4 = C_4 \cdot PVIF_{r,4} = C_4 (1+r)^{-4} \\
\ldots \\
V_0^{N-1} = C_{N-1} \cdot PVIF_{r,N-1} = C_{N-1} (1+r)^{-(N-1)} \\
V_0^N = C_N \cdot PVIF_{r,N} = C_N (1+r)^{-N}
\]

\[
V_0 = V_0^1 + V_0^2 + V_0^3 + V_0^4 + \ldots + V_0^{N-1} + V_0^N
\]

2. The value of the stream of cash flows at any other point can then be ascertained using the present or future value formulas:

\[
V_n = V_0 \cdot FVIF_{r,n} = V_0 (1+r)^n.
\]

3. So cash flows occurring at different points in time can not be added but cash flows which occur at the same time can be added.
B. Example:

1. Question: Today is the start of 2006. i. How much do I need to invest today at an effective annual rate of 10% to meet a $500 obligation in 2 years and a $800 payment in 3 years? ii. How much would I have to invest if I delay my investment date 1 year?

2. end 05                      end 06                                                                                 end 09  
   V_0                            V_1                        $500                      $800

3. Answer i.:
   a. using present value formula, amount needed to be invested today to meet the $500 obligation in 2 years:

   end 05                      end 06                                                                                 end 09  
   V^2_0                                                                                   $500

   V^2_0 = $500 \ PVIF_{0.1,2} = $500 (1+0.1)^{-2} = $413.22

   b. using present value formula, amount needed to be invested today to meet the $800 obligation in 3 years:

   end 05                      end 06                                                                                 end 09  
   V^3_0                                                                                   $800

   V^3_0 = $800 \ PVIF_{0.1,3} = $800 (1+0.1)^{-3} = $601.05

   c. so total amount needed to be invested today is

   V_0 = V^2_0 + V^3_0 = $413.22 + $601.05 = $1014.27.

4. Answer ii.:
   a. once the stream of cash flows has been converted to a single sum at a certain point in time (here time 0), can use the present and future value formulas to convert the stream to a single sum at any other time.

   b. so total amount that would have to be invested in one year using the future value formula:

   \[ V_1 = 1014.27 \ FVIF_{0.1,1} = 1014.27 (1+0.1) = 1115.70 \]
IX. Particular Cash Flow Pattern: Annuity

A. Definition:
1. N equal payments made at equal intervals (which can be more or less than one period).

\[
\begin{array}{cccccccc}
  & t-1 & t & t+1 & t+2 & t+3 & t+4 & \ldots & t+N-1 & t+N \\
C & C & C & C & C & C & C & & C \\
\end{array}
\]

B. Converting to a single sum:
1. One Approach: convert each cash flow to a single sum at a given point in time using the present or future value formula and then add up these sums to give the annuity’s value at that point in time.
2. Another Approach:
   a. choose the period so that it is equal to the interval between cash flows.
   b. the annuity's first cash flow occurs at (t+1) and its last at (t+N).
   c. use the present value annuity factor (PVAF_{r,N}) to obtain the value at time t, one period before the first cash flow:

\[
V_t = C \times PVAF_{r,N}
\]
where

1. the effective interest rate over a period is \( r \).
2. the formula gives the single sum equivalent at the point in time one period before the first cash flow.

\[
PVAF_{r,N} = \frac{1 - (1+r)^{-N}}{r}.
\]

d. or use the future value annuity factor (FVAF_{r,N}) to obtain the value at time \( t+N \), one period before the first cash flow

\[
V_{t+N} = C \times FVAF_{r,N}
\]
where

1. the effective interest rate over a period is \( r \).
2. the formula gives the single sum equivalent at the point in time corresponding to the last cash flow.

\[
FVAF_{r,N} = \frac{(1+r)^N - 1}{r}.
\]
e. notice that

1. \( FVAF_{r,N} = PVAF_{r,N} (1+r)^N \); and so
2. \( V_{t+N} = V_t (1+r)^N \) which is consistent with the future value formula.
C. Example:

1. Question: Today is the start of 2006. Suppose I receive $1000 at the end of each year for the next 3 years. If I can invest at an effective annual rate of 10%, how much would I have in 4 years time?

   \[
   \begin{array}{c|c|c|c}
   \text{end 05} & \text{end 06} & \text{end 09} \\
   \$1000 & \$1000 & \$1000 & V_4 \\
   \end{array}
   \]

2. One Answer: using the future value formula

\[
\begin{align*}
V_1^4 &= \$1000 \times FVIF_{0.1,3} = \$1000 \times (1+0.1)^3 = \$1331 \\
V_2^4 &= \$1000 \times FVIF_{0.1,2} = \$1000 \times (1+0.1)^2 = \$1210 \\
V_3^4 &= \$1000 \times FVIF_{0.1,1} = \$1000 \times (1+0.1)^1 = \$1100 \\
\hline
V_4 &= V_1^4 + V_2^4 + V_3^4 = \$1331 + \$1210 + \$1100 = \$3641
\end{align*}
\]

3. Another Answer: using the future value annuity formula gives \( V_3 \) and then can use the future value formula to get \( V_4 \)

\[
\begin{align*}
V_3 &= C \times FVAF_{0.1,3} = \$1000 \times \frac{(1+0.1)^3-1}{0.1} = $3310 \\
V_4 &= V_3 \times FVIF_{0.1,1} = V_3 \times (1+0.1) = $3310 \times 1.1 = $3641.
\end{align*}
\]

4. Another Answer: using the present value annuity formula gives \( V_0 \) and then can use the future value formula to get \( V_4 \)

\[
\begin{align*}
V_0 &= C \times PVAF_{0.1,3} = \$1000 \times \frac{1-(1+0.1)^{-3}}{0.1} = $2486.852 \\
V_4 &= V_0 \times FVIF_{0.1,4} = V_0 \times (1+0.1)^4 = $2486.852 \times 1.4641 = $3641.
\end{align*}
\]
D. A More Difficult Example:
1. Question: Suppose I receive $100 at the end of the month and three further $100 payments in three, five and seven months from today. The effective monthly interest rate is 1%. i. What amount could I borrow today using these four $100 payments to repay the loan? ii. What is the APR with monthly compounding? iii. What is the APR with compounding every 2 months?

\[
\begin{array}{ccccccc}
-1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
$100 & $100 & $100 & $100 & \\
\end{array}
\]

\[r=0.01\]

\[1+r_2 = (1+r)^2\] and so \[r_2 = 1.01^2 - 1 = 0.0201.\]

2. Answer to i.: Since payments are made every 2 months, the appropriate interest rate to use for the present value annuity factor is the effective 2 month rate \[2.01\%;\] i.e., use \[r_2 = 0.0201.\] And since payments are made every 2 months, the present value annuity factor gives the single sum equivalent 2 months before the first payment: here one month before today. So

\[
V_{-1} = 100 \ PVAF_{0.0201,4} = 100 \left\{1 - \frac{1.0201^{-4}}{0.0201}\right\} = 100 \times 3.8068 = 380.68.\]

is the amount that could have been borrowed one month earlier. To get the amount that could be borrowed today, use the future value interest factor:

\[
V_0 = V_{-1} \ FVIF_{0.01,1} = 380.68 \times 1.01 = 384.49.
\]

3. Answer to ii. With monthly compounding the APR is \[12\times1\%=12\%\]

4. Answer to iii. With compounding every 2 months, the APR is \[6\times2.01\% = 12.06\%.\]
E. Amortization:

1. Question: Suppose I borrow $5000 on 1st January 2006 at an APR of 18% compounded monthly. i. If I have a three year loan and I make loan repayments at the end of each month, what is my monthly payment? ii. What is the loan balance outstanding after the first loan payment? iii. How much interest accumulates in the first month of the loan?

<table>
<thead>
<tr>
<th>1/1/06</th>
<th>2/1/06</th>
<th>3/1/06</th>
<th>12/1/06</th>
<th>1/1/07</th>
<th>2/1/07</th>
<th>12/1/08</th>
<th>1/1/09</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>35</td>
<td>36</td>
</tr>
<tr>
<td>$5000</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
<td>C</td>
</tr>
</tbody>
</table>

2. Effective Interest Rate: APR is 18% so the effective monthly rate is (18%/12)=1.5%. Thus, r=0.015.

3. Answer i.

\[ V_0 = 5000 = C \times PVAF_{0.015,36} = C \left[ \frac{1-(1+0.015)^{-36}}{0.015} \right] = C \times 27.660684; \] and so

\[ C = \frac{5000}{27.660684} = 180.762. \]

4. Answer ii.
   a. Loan balance outstanding at time 1 (after the first payment is made):

   \[ 5000 \times 1.015 - C = 5075 - 180.76 = 4894.24. \]

   b. But the loan balance outstanding at time 1 (after the first payment is made) is equal to the single sum equivalent at time 1 of the remaining 35 payments:

   \[ V_{2-36}^1 = 180.762 \times PVAF_{0.015,35} = 180.762 \left[ \frac{1-(1+0.015)^{35}}{0.015} \right] = 4894.24. \]

5. Answer iii.
   a. Interest for 1/06 = $5000 \times 0.015 = $75.
6. Question (cont): iv. What is the loan balance outstanding after the 12th loan payment?
7. Answer iv.

<table>
<thead>
<tr>
<th>Time</th>
<th>Interest</th>
<th>Balance prior to Payment</th>
<th>Payment</th>
<th>Balance after Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td>5000</td>
</tr>
<tr>
<td>1</td>
<td>75</td>
<td>5075</td>
<td>180.76198</td>
<td>4894.238</td>
</tr>
<tr>
<td>2</td>
<td>73.41357</td>
<td>4967.6516</td>
<td>180.76198</td>
<td>4786.8896</td>
</tr>
<tr>
<td>3</td>
<td>71.803344</td>
<td>4858.693</td>
<td>180.76198</td>
<td>4677.931</td>
</tr>
<tr>
<td>4</td>
<td>70.168965</td>
<td>4748.0999</td>
<td>180.76198</td>
<td>4567.338</td>
</tr>
<tr>
<td>5</td>
<td>68.51007</td>
<td>4635.848</td>
<td>180.76198</td>
<td>4455.0861</td>
</tr>
<tr>
<td>6</td>
<td>66.826291</td>
<td>4521.9124</td>
<td>180.76198</td>
<td>4341.1504</td>
</tr>
<tr>
<td>7</td>
<td>65.117256</td>
<td>4406.2676</td>
<td>180.76198</td>
<td>4225.5057</td>
</tr>
<tr>
<td>8</td>
<td>63.382585</td>
<td>4288.8882</td>
<td>180.76198</td>
<td>4108.1263</td>
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<tr>
<td>9</td>
<td>61.621894</td>
<td>4169.7482</td>
<td>180.76198</td>
<td>3988.9862</td>
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<tr>
<td>10</td>
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<td>4048.821</td>
<td>180.76198</td>
<td>3868.059</td>
</tr>
<tr>
<td>11</td>
<td>58.020885</td>
<td>3926.0799</td>
<td>180.76198</td>
<td>3745.3179</td>
</tr>
<tr>
<td>12</td>
<td>56.179768</td>
<td>3801.4977</td>
<td>180.76198</td>
<td>3620.736877</td>
</tr>
</tbody>
</table>

b. Alternatively, the loan balance outstanding after 12 payments is the single sum equivalent at time 12 of the last 24 payments:

\[ V_{13-36}^{12} = \$180.762 \times PVAF_{0.015,24} = \$180.762 \left[ \frac{1-(1+0.015)^{-24}}{0.015} \right] = \$3620.736. \]

8. Question (cont): v. How much interest accumulates in the 13th month of the loan?
9. Answer v. From question iv., the loan balance outstanding after 12 payments is \( V_{13-36}^{12} = \$3620.74 \). So the amount of interest accumulating during the 13th month is \( \$3620.74 \times 0.015 = \$54.311 \).
X. Particular Cash Flow Pattern: Perpetuity

A. Definition:
   1. equal payments made at equal intervals forever.

   \[
   \begin{array}{cccccc}
   t-1 & t & t+1 & t+2 & t+3 & t+4 \\
   \hline
   C & C & C & C & ... \\
   \end{array}
   \]

B. Converting to a single sum:
   1. can not convert each cash flow to a single sum at a given point in time using the present or future value formulas and then add up these sums to give the value of the annuity at that point in time.
   2. Approach:
      a. choose the period so that it is equal to the interval between cash flows.
      b. the perpetuity's first cash flow occurs at \( (t+1) \).
      c. use the present value perpetuity factor (PVPF_r) which satisfies
      d. the effective interest rate over a period is \( r \).
      e. the formula gives the single sum equivalent at the point in time one period before the first cash flow.

\[
(1) \quad PVPF_r = \frac{1}{r} .
\]

\[
(2) \quad \text{note that } PVPF_r = \lim_{N \to \infty} \frac{1 - (1+r)^{-N}}{r}.
\]

\[
V_i = C \times PVPF_r \quad \text{and}
\]

\[
PVPF_r = \frac{1}{r} = \lim_{N \to \infty} PVAF_{r,N} = \lim_{N \to \infty} \frac{1 - (1+r)^{-N}}{r}.
\]
C. Example.

1. Question: Mr X wants to set aside an amount of money today that will pay his son and his descendants $10000 at the end of each year forever, with the first payment to be made at the end of 2007. If Mr X can invest at an effective annual rate of 10%, how much would he have to invest today (the 31st December 2005)?

<table>
<thead>
<tr>
<th>end 05</th>
<th>end 06</th>
<th>end 07</th>
<th>end 09</th>
</tr>
</thead>
</table>
| $0    | $10000| $10000| $10000| ...

2. Answer:
   a. First, calculate how much Mr X needs to have at the end of 2006 using the present value annuity formula; can use this formula because the first payment is at the end of 2007 and the payments are made annually.

   \[ V_1 = $10000 \times PV_{P,0.1} = $10000/0.1 = $100000. \]

   b. Second, calculate the amount that Mr X must invest at the end of 2005 using the present value formula:

   \[ V_0 = V_1 \times PV_{I,0.1} = $100000/(1+0.1) = $90909. \]