Lecture 8: Bond Pricing and Forward Rates.

I. Reading.
II. Discount Bond Yields and Prices.
III. Fixed-income Prices and No Arbitrage
IV. The Yield Curve.
V. Other Bond Pricing Issues.
VI. Holding Period Return.
VII. Forward Rates.
Lecture 8: Bond Pricing and Forward Rates.

I. Reading.
   A. BKM, Chapter 15.1-15.2, 15.5.

II. Discount Bond Yields and Prices.
   A. Relation between Prices and Yields for Discount Bonds.
      1. Yields are usually quoted in the industry as APRs with semiannual
         compounding; i.e. as bond equivalent yields.
      2. Let $p_{\tau}(t)$ be the price at time $t$ on a $\tau$-year discount bond with face value
         $C_{\tau}(t+\tau)$.
      3. For discount bonds, yield (expressed as an APR with semiannual
         compounding) is related to price in the following way:

         $$p_{\tau}(t) = \frac{C_{\tau}(t+\tau)}{[1 + \frac{y_{\tau}(t)}{2}]^{2\tau}} \Rightarrow y_{\tau}(t) = 2 \left\{ \left[ \frac{C_{\tau}(t+\tau)}{p_{\tau}(t)} \right]^{\frac{1}{2\tau}} - 1 \right\}.$$

      4. Example:
         a. Government note and strip prices for 2/15/06.
         b. One period is a year.

   Government Bonds and Notes.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Aug 06</td>
<td>?</td>
</tr>
<tr>
<td>4</td>
<td>Feb 07</td>
<td>?</td>
</tr>
<tr>
<td>6</td>
<td>Aug 07</td>
<td>?</td>
</tr>
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</table>

U.S. Treasury Strips.

<table>
<thead>
<tr>
<th>Type</th>
<th>Maturity</th>
<th>Ask Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>ci</td>
<td>Aug 06</td>
<td>97</td>
</tr>
<tr>
<td>ci</td>
<td>Feb 07</td>
<td>94</td>
</tr>
<tr>
<td>ci</td>
<td>Aug 07</td>
<td>90</td>
</tr>
</tbody>
</table>
c. Can calculate the yield on a six month discount bond (expressed as an APR with semi-annual compounding) using the price of the Aug 06 strip:

\[ y_{\frac{1}{2}} \text{ (Feb 06)} = 2 \times \{\frac{100}{97} - 1\} = 6.186\%. \]

d. Can calculate the yield on a one year discount bond (expressed as an APR with semi-annual compounding) using the price of the Feb 07 strip:

\[ y_1 \text{ (Feb 06)} = 2 \times \left\{\frac{100}{94}\right\}^{\frac{1}{2}} - 1 = 6.284\%. \]

e. Can calculate the yield on a 1.5 year discount bond (expressed as an APR with semi-annual compounding) using the price of the Aug 07 strip:

\[ y_{\frac{3}{2}} \text{ (Feb 06)} = 2 \times \left\{\frac{100}{90}\right\}^{\frac{1}{3}} - 1 = 7.149\%. \]
B. Yields on Discount Bonds expressed as Discount Factors.

1. The $\tau$-period discount factor at time $t$, denoted $d_\tau(t)$, is the price at $t$ of $1$ received for certain at time $t+\tau$.

2. The $\tau$-period discount factor at time $t$ is analogous to the PVIF discussed in the time value of money.

\[
d_\tau(t) = \frac{1}{\frac{y_\tau(t)}{2}} \Rightarrow y_\tau(t) = 2 \left\{ \frac{1}{d_\tau(t)} \right\}^{1/(2\tau)} - 1 \]

3. These formulas are used below to calculate the yield on a discount bond when the relevant discount factor is known and to calculate the relevant discount factor when the discount bond’s yield is known.

4. Example (cont):
   a. Can calculate the $\frac{1}{2}$ year discount factor on 2/15/06 using the yield of the Aug 06 strip (expressed as an APR with semi-annual compounding):

\[
d_{\frac{1}{2}}(\text{Feb 06}) = \frac{1}{\left[1 + \frac{0.06186}{2}\right]^{1}} = 0.97.
\]

   b. Can calculate the 1 year discount factor on 2/15/06 using the yield of the Feb 07 strip (expressed as an APR with semi-annual compounding):

\[
d_1(\text{Feb 06}) = \frac{1}{\left[1 + \frac{0.06284}{2}\right]^{2}} = 0.94.
\]

   c. Can calculate the 1½ year discount factor on 2/15/06 using the yield of the Aug 07 strip (expressed as an APR with semi-annual compounding):

\[
d_{1\frac{1}{2}}(\text{Feb 06}) = \frac{1}{\left[1 + \frac{0.07149}{2}\right]^{3}} = 0.90.
\]
5. If the price of a discount bond paying \( C_t(t+\tau) \) in \( \tau \) periods is \( p_t(t) \), then \( d_t(t) \) is given by \( p_t(t)/C_t(t+\tau) \).

a. Can calculate \( d_{\frac{1}{2}}(\text{Feb 06}) \) as follows using the Aug 06 strip:
\[
d_{\frac{1}{2}}(\text{Feb 06}) = p_{\frac{1}{2}}(\text{Feb 06})/100 = 97/100 = 0.97.
\]

b. Can calculate \( d_1(\text{Feb 06}) \) as follows using the Feb 07 strip:
\[
d_1(\text{Feb 06}) = p_1(\text{Feb 06})/100 = 94/100 = 0.94.
\]

c. Can calculate \( d_{1\frac{1}{2}}(\text{Feb 06}) \) as follows using the Aug 07 strip:
\[
d_{1\frac{1}{2}}(\text{Feb 06}) = p_{1\frac{1}{2}}(\text{Feb 06})/100 = 90/100 = 0.90.
\]
III. Fixed-income Prices and No Arbitrage
A. An Arbitrage Opportunity.
   1. Definition: An investment that does not require any cash outflows and generates a strictly positive cash inflow with some probability is known as an arbitrage opportunity.
   2. In well functioning markets arbitrage opportunities cannot exist since any individual who prefers more to less wants to invest as much as possible in the arbitrage opportunity.

   1. The absence of arbitrage implies that any two assets with the same stream of riskless cash flows must have the same price.
   2. This implication is known as the law of one price.
   3. Otherwise, could buy the lower priced asset and sell the higher priced asset and earn an arbitrage profit.
      a. Zero cash flows in the future.
      b. Positive cash flow today.
   4. Example: WSJ for 5/12/06. Compare U.S. Treasury Strips maturing on the same date. The differences in price are very small.

1. Any bond paying the certain cash flow stream, $C_i(t+\frac{1}{2}), C_i(t+1),..., C_i(t+N)$, must have the following price at time $t$ for there to be no arbitrage:

$$P^i(t) = d_{\frac{1}{2}}(t) C_i(t+\frac{1}{2}) + d_1(t) C_i(t+1) + ... + d_N(t) C_i(t+N).$$

2. Alternatively, the price of any bond can be obtained by discounting back each certain cash flow at the yield on the discount bond with the same maturity as the cash flow:

$$P^i(t) = \frac{1}{[1 + \frac{y_{\frac{1}{2}}(0)}{2}]^1} C_i(t+\frac{1}{2}) + \frac{1}{[1 + \frac{y_1(0)}{2}]^2} C_i(t+1) + ... + \frac{1}{[1 + \frac{y_N(0)}{2}]^{2N}} C_i(t+N).$$

3. Example (cont): Since above we calculated the discount factors at Feb 06 for Aug 06, Feb 07 and Aug 07, we can determine the value each of the three bonds/notes in the absence of arbitrage using this formula:

a. 4 Aug 06:

$$P^{4Aug\,06}(Feb\,06) = d_{\frac{1}{2}}(Feb\,06) \times [100 + 4/2] = 0.97 \times 102 = 98.94.$$  

b. 4 Feb 07:

$$P^{4Feb\,07}(Feb\,06) = d_{\frac{1}{2}}(Feb\,06) \times [4/2] + d_1(Feb\,06) \times [100 + 4/2] = 0.97 \times 2 + 0.94 \times 102 = 97.82.$$  

c. 6 Aug 07:

$$P^{6Aug\,07}(Feb\,06) = d_{\frac{1}{2}}(Feb\,06) \times [6/2] + d_1(Feb\,06) \times [6/2] + d_{\frac{1}{2}}(Feb\,06) \times [100 + 6/2] = 0.97 \times 3 + 0.94 \times 3 + 0.90 \times 103 = 98.43.$$
D. How to Earn an Arbitrage Profit when the Law of One Price is Violated.

1. Basic Idea.
   a. Buy the undervalued assets and sell the overvalued assets.
   b. Need to choose the weights to ensure that all cash flows are zero or positive.
   c. Easiest way is to choose the weights so that all future cash flows are zero; then today’s cash flow will be positive if there is an arbitrage opportunity.

2. How to use coupon bonds and a mispriced strip to create an arbitrage position.
   a. Example (cont): Suppose the 4 Feb 07 note is priced at 98 on 2/15/06 which is too high relative to the prices of the Aug 06 and Feb 07 strips.
      (1) Must sell the 4 Feb 07 note.
      (2) The idea is to
         (a) sell the 4 Feb 07 note (overpriced); and
         (b) buy a “synthetic” 4 Feb07 note created using the Aug 06 and Feb 07 strips.
      (3) Must buy the Aug 06 and Feb 07 strips.
      (4) More specifically:
         (a) let \( a \) be the number of Feb 07 strips bought.
         (b) let \( b \) be the number of Aug 06 strips bought.

<table>
<thead>
<tr>
<th>Position</th>
<th>2/15/06</th>
<th>8/15/06</th>
<th>2/15/07</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell 1 4 Feb 07 note</td>
<td>SELL NOTE</td>
<td>1 x 98 = 98</td>
<td>-1 x 4/2 = -2</td>
</tr>
<tr>
<td>Buy ( a ) Feb 07 strips</td>
<td>BUY SYNTHETIC NOTE</td>
<td>(-a x 94)</td>
<td>(a 100 = 102)</td>
</tr>
<tr>
<td>Buy ( b ) Aug 06 strips</td>
<td>-b x 97</td>
<td>(b 100 = 2)</td>
<td></td>
</tr>
<tr>
<td>Net</td>
<td>98 - a 94 - b 97</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

(5) So \( a 100 = 102 \) implies \( a = 1.02 \).
(6) So \( b 100 = 2 \) implies \( b = 0.02 \).
(7) His position earns an arbitrage profit of:

\[ 98 - 1.02 \times 94 - 0.02 \times 97 = 0.18 \text{ today}. \]

(8) It seems like a lot of trouble for 18 cents. But sell 1M of the 4 Feb 07 note and the profit becomes $180000.

IV. The Yield Curve.
A. Yield Curve Definition.
1. Discount bonds of differing maturities can have different yields to maturity.
2. The yield curve is just the yield to maturity (YTM) on a n-year discount bond graphed as a function of n.
3. Note that the YTM on the shortest maturity discount bond of interest is known as the spot rate.
4. It is not correct to use the yield to maturity on a n-year coupon bond as the yield on a n-year zero coupon bond: these are not the same.
5. Example (cont): 2/15/06
   a. Yield on a 1½ year discount bond (expressed as an APR with semi-annual compounding) is 7.149%.
   b. Price of the 6 Aug 07 note can be calculated using law of one price:

   \[
   P^{6 \text{ Aug 96 (Feb 06)}} = \frac{1}{[1 + \frac{0.06186}{2}]^1} \cdot 103 + \frac{1}{[1 + \frac{0.06284}{2}]^2} \cdot 103 + \frac{1}{[1 + \frac{0.07149}{2}]^3} \cdot 103
   \]

   \[
   = 98.43.
   \]
   c. YTM on the 6 Aug 07 note is 7.122% since

   \[
   98.43 = \frac{1}{[1 + \frac{0.07122}{2}]^1} \cdot 103 + \frac{1}{[1 + \frac{0.07122}{2}]^2} \cdot 103 + \frac{1}{[1 + \frac{0.07122}{2}]^3} \cdot 103
   \]

   which is lower than the yield on a 1½ year discount bond.
   d. The reason is that the note has coupon cash flows prior to Aug 07 and the yields on ½ year and 1 year discount bonds are less than 7.149%.
   e. YTM on the 10 Aug 07 note is 7.106% which is even lower than for the 6 Aug 07 note since a greater portion on the 10 Aug 07's cash flows occur prior to Aug 07 than the 6 Aug 07 note.
B. Yield Curve Calculation.

1. Short end can be obtained from U.S. T-bill yields.

2. Long end can be obtained from U.S. Treasury strips.

3. It is also possible to obtain the yield curve using U.S. Treasury notes and bonds despite the presence of coupon payments.
   a. It is not correct to use the yield to maturity on a n-year coupon bond as the yield on a n-year zero coupon bond: these are not the same.
   b. The concept of no arbitrage in well functioning markets can be used.

4. Generally, to replicate a T-period discount bond using coupon bonds:
   a. Use the T-period coupon bond to generate the time T cash flow; so buy the T-period coupon bond.
   b. Use the (T-1) period bond to offset the (T-1)th coupon from the T-period coupon bond; so sell the (T-1)-period coupon bond.
   c. Use the (T-2)-period bond to offset the sum of time (T-2) coupons for the T-period and (T-1)-period coupon bond.
   d. Continue until the cash flows for all points in time between now and time T are zero.

5. Once a discount bond has been replicated, its price and cashflow can be used to calculate its yield.
6. Example (cont): 2/15/06

Government Bonds and Notes.

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>Aug 06</td>
<td>98.94</td>
</tr>
<tr>
<td>4</td>
<td>Feb 07</td>
<td>97.82</td>
</tr>
<tr>
<td>6</td>
<td>Aug 07</td>
<td>98.43</td>
</tr>
</tbody>
</table>

a. Can recover the six month discount factor at 2/15/06 using the Aug 06 note since it has only one payment left on 8/15/06:

\[ d_{\frac{1}{2}} (Feb 06) = \frac{98.94}{100 + \left[\frac{4}{2}\right]} = 0.97. \]

b. Can recover the 1 year discount factor at 2/15/06 bond the 4 Feb 07 notes and \( d_{\frac{1}{2}} (Feb 06) \):

\[
P_{4 Feb 07}^{Feb 06} = 2 \times d_{\frac{1}{2}} (Feb 06) + 102
\]
\[97.82 = 2 \times 0.97 + 102 \times d_{1} (Feb 06) \Rightarrow d_{1} (Feb 06) = 0.94.

(2) So \( a \) implies \( a = 0.9804 \).

(3) So \( a + b \) implies \( b = 0.0192 \). Since \( b \) is negative, this means that strategy involves selling 0.0192 of the 4 Aug 06 notes.

(4) Thus, the cost of the synthetic discount bond is

\[ 0.9804 \times 97.82 - 0.0192 \times 98.94 = 94 \]
\[ \Rightarrow d_{1} (Feb 06) = \frac{94}{100} = 0.94. \]
V. Other Bond Pricing Issues.
   A. Introduction.
      1. Sometimes prices appear to violate the law of one price.
      2. Usually there is a reason for an apparent violation.
      3. Many fixed income traders are constantly looking for arbitrage opportunities. They are in a position to act on any such opportunity immediately. Thus, it is unlikely that an arbitrage opportunity could be identified by looking at the WSJ.
   
   B. Reasons for deviations from the law of one price.
      1. Taxes.
         a. If two fixed income instruments have the same pretax cash flows but these cash flows are taxed differently, then the two instruments may have different prices
      2. Liquidity.
         a. If the liquidity of two fixed income instruments differ, so too may their prices.
      3. Transaction Costs.
         a. When there are transaction costs in the form of a bid ask spread, no arbitrage implies the ask price for a given instrument must be greater than some cutoff and its bid price must be less than another cutoff.
C. Default Risk.

1. Fixed income instruments with the same promised cash flows may have different prices if they have differing probabilities of default.

2. If a fixed income instrument has some probability of default:
   a. expected cash flows from the instrument are less than the promised cash flows.
   b. there is uncertainty or risk associated with the cash flows; so investors require a higher expected return to hold a corporate compared to an equivalent treasury.

3. Hence, the yield on a discount bond with default risk (e.g., corporate debt) will be greater than the yield on a discount bond of the same maturity without any default risk (e.g., government bills).

<table>
<thead>
<tr>
<th>1 Period Strip</th>
<th>Treasury</th>
<th>Corporate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Promised Payment P</td>
<td>110</td>
<td>110</td>
</tr>
<tr>
<td>Prob of Default</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>Expected Payment E[CF]</td>
<td>110</td>
<td>55</td>
</tr>
<tr>
<td>Expected Return E[R]</td>
<td>10%</td>
<td>10%</td>
</tr>
<tr>
<td>Price</td>
<td>100</td>
<td>50</td>
</tr>
<tr>
<td>YTM</td>
<td>10%</td>
<td>120%</td>
</tr>
</tbody>
</table>
VI. Holding Period Return.
   A. Definition.
      1. Let $h_i(t+\frac{1}{2})$ denote the $\frac{1}{2}$ year holding period return on bond $i$ over the $\frac{1}{2}$ year ending at $t+\frac{1}{2}$.
      2. The $\frac{1}{2}$ year holding period return is defined as follows:

$$h_i(t) = \frac{p_i(t+\frac{1}{2}) + C_i(t+\frac{1}{2}) - p_i(t)}{p_i(t)}$$

where

$p_i(t)$ is the price of the bond at time $t$; if the bond matures at time $t$, $p_i(t)$ is the face value of the bond;
$C_i(t)$ is the coupon payment at time $t$ on the bond.

3. The holding period return on a bond is analogous to the return on an asset (which has been defined already).
4. **Example (cont):** Government strip, note and bond prices 2/15/06

**Government Bonds and Notes.**

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Aug 07</td>
<td>98.43</td>
<td>7.122%</td>
</tr>
</tbody>
</table>

**U.S. Treasury Strips.**

<table>
<thead>
<tr>
<th>Type</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>ci</td>
<td>Aug 06</td>
<td>97</td>
<td>6.186%</td>
</tr>
<tr>
<td>ci</td>
<td>Feb 07</td>
<td>94</td>
<td>6.284%</td>
</tr>
<tr>
<td>ci</td>
<td>Aug 07</td>
<td>90</td>
<td>7.149%</td>
</tr>
</tbody>
</table>

Government strip, note and bond prices 8/15/06

**Government Bonds and Notes.**

<table>
<thead>
<tr>
<th>Rate</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>Aug 07</td>
<td>97.7</td>
<td>8.446%</td>
</tr>
</tbody>
</table>

**U.S. Treasury Strips.**

<table>
<thead>
<tr>
<th>Type</th>
<th>Maturity</th>
<th>Ask Price</th>
<th>Ask Yield</th>
</tr>
</thead>
<tbody>
<tr>
<td>ci</td>
<td>Aug 06</td>
<td>100</td>
<td>-</td>
</tr>
<tr>
<td>ci</td>
<td>Feb 07</td>
<td>98</td>
<td>4.082%</td>
</tr>
<tr>
<td>ci</td>
<td>Aug 07</td>
<td>92</td>
<td>8.514%</td>
</tr>
</tbody>
</table>
a. What is the 6 month holding period return on Aug 06 strips from 2/15/06 to 8/15/06?

\[ h_{Aug\ 06\ strip\ (8/15/06)} = (100-97)/97 = 3.093\% \]

(which is 2 x 3.093\% = 6.186\% when expressed as an APR with semi-annual compounding).

b. What is the 6 month holding period return on Feb 07 strips from 2/15/06 to 8/15/06?

\[ h_{Feb\ 07\ strip\ (8/15/06)} = (98-94)/94 = 4.255\% \]

(which is 2 x 4.255\% = 8.510\% when expressed as an APR with semi-annual compounding).

c. What is the 6 month holding period return on the 6 Aug 07 notes from 2/15/06 to 8/15/06?

\[ h_{6\ Aug\ 07\ note\ (8/15/06)} = (97.7+3-98.43)/98.43 = 2.306\% \]

(which is 2 x 2.306\% = 4.612\% when expressed as an APR with semi-annual compounding).
B. Note.
1. The ½ year holding period return on a ½ year discount bond equals its yield to maturity.
   a. Example. YTM on Aug 06 strip at 2/15/06 is 6.186% which equals its holding period return (expressed as an APR with 6 month compounding) over the ½ year period 2/15/06 to 8/15/06.
2. For any bond of longer maturity:
   a. its ½ year holding period return need not equal either its own yield to maturity or the YTM on a ½ year discount bond.
   b. if its YTM at the end of the holding period is greater than its YTM at the start of the holding period, then its holding period return will be less than its YTM at the start of the holding period.
   c. if its YTM at the end of the holding period is less than its YTM at the start of the holding period, then its holding period return will be greater than its YTM at the start of the holding period.
   d. Example (cont):

<table>
<thead>
<tr>
<th>Instrument</th>
<th>Holding Period Return (2/15/06-8/15/06)</th>
<th>YTM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Start (2/15/06)</td>
</tr>
<tr>
<td>6% Aug 07 Note</td>
<td>4.612%</td>
<td>7.122%</td>
</tr>
<tr>
<td>Feb 07 Strip</td>
<td>8.510%</td>
<td>6.284%</td>
</tr>
</tbody>
</table>

3. When a bond pays coupons within the holding period, there is a question how these should be incorporated into the holding period return calculation.
VII. Forward Rates.

A. Definition.

1. Example (cont): It is 2/15/06. You want to lock in an interest rate for the period 8/15/06 to 2/15/07. So agree today to invest an amount on 8/15/06 for 6 months at an interest rate (expressed as an APR with semi-annual compounding) of 6.38%.
   a. This rate is known as the 6 month forward rate available on 2/15/06 for the period starting on 8/15/06.
   b. So if I agree on 2/15/06 to invest $1 on 8/15/06, I will receive $1 + 0.0638/2 = $1.0319 on 2/15/07.

2. Let $f_{t,t+\alpha}(0)$ be the $\alpha$-period forward rate (expressed as an APR with semiannual compounding) available at time 0 for the period starting $t$ periods after time 0.
   a. $f_{t,t+\alpha}(0)$ is also known as the $t \times (t+\alpha)$ FRA (forward rate agreement) rate available at time 0.
   b. Example. You want to lock in an interest rate for the period 8/15/06 to 2/15/07. The relevant forward rate is $f_{\frac{1}{2},1}(Feb\ 06)$.
   c. If an investor knew at time 0 that she would have $1 M available to invest at time $t$ and wanted a certain rate of return on a $\alpha$ period investment, then $f_{t,t+\alpha}(0)$ is the applicable yield expressed as an APR with semiannual compounding.

3. Why is there a need for forward contracts?
   a. Because the yield curve shifts through time in a manner that is not totally predictable.
   b. Example (cont).
      (1) Using a forward contract on 2/15/06, an investor can lock in an APR of 6.38% over the period 8/15/06 to 2/15/07.
      (2) If the investor waits until 8/15/06 to invest for the six month period, she may get a higher or lower rate.
      (3) Given the yield curve on 8/15/06 above, the investor would only have received an APR of 4.082% (YTM on the Feb 07 strip as of 8/15/06).
      (4) Investor may not want this uncertainty.
B. Forward rates expressed as forward contract discount factors.

1. Let $d_{t,t^\alpha}(0)$ be the price paid at $t$ for $1$ delivered at $(t^\alpha)$ which can be locked in at time $0$.
   a. Example. You want to lock in an interest rate for the period 8/15/06 to 2/15/07. The relevant discount factor is $d_{8,1}$(Feb 06).
   b. Now $d_{t,t^\alpha}(0)$ is referred to as the $\alpha$-period forward contract discount factor available at time $0$ for the period starting at time $t$.

2. It is possible to go from $d_{t,t^\alpha}(0)$ to $f_{t,t^\alpha}(0)$ and vice versa using analogous formulas to those for discount bonds:

\[
d_{t,t^\alpha}(0) = \frac{1}{1 + \left(\frac{f_{t,t^\alpha}(0)}{2}\right)^{2\alpha}} \Leftrightarrow f_{t,t^\alpha}(0) = 2 \left\{ \frac{1}{d_{t,t^\alpha}(0)} \right\}^{1/(2\alpha)} - 1.
\]

a. Example (cont):
   (1) Can calculate the discount factor associated with the forward rate available on 2/15/06 for the period from 8/15/06 to 2/15/07 using the forward rate $f_{8,1}$(Feb 06) = 6.38%:

\[
d_{8,1}$(Feb 06) = \frac{1}{[1 + \left(\frac{0.0638}{2}\right)]^{1}} = 0.9691.
\]

   (2) Can calculate the forward rate (expressed as an APR with semi-annual compounding) available on 2/15/06 for the period from 8/15/06 to 2/15/07 using the relevant discount factor for that forward rate $d_{8,1}$(Feb 06) = 0.9691:

\[
f_{8,1}$(Feb 06) = 2 \times \{1/0.9691 - 1\} = 6.38%.
C. Relation between Forward Rates and Discount Bond Yields.

1. Example (cont):
   a. On 2/15/06, we can use Aug 06 discount bonds and the 6 month forward rate available for the period starting on 8/15/06 to replicate a Feb 07 discount bond.
   b. Thus, should be able to use the discount bond yields

   \[ y_{\frac{1}{2}} \text{ (Feb 06)} = 6.186\%; \text{ and} \]
   \[ y_{1} \text{ (Feb 06)} = 6.284\%. \]

   and the law of one price to get the forward rate \( f_{\frac{1}{2},1} \text{ (Feb 06)} \).

   c. What are the cash flows?

   \[
   \begin{array}{ccc}
   \text{2/15/06} & \text{8/15/06} & \text{2/15/07} \\
   0 & \frac{1}{2} & 1 \\
   \end{array}
   \]

   BUY $1 worth of Feb 07 strips
   -1 \hspace{1cm} \left[1 + \frac{y_{1}(0)}{2}\right]^{2} \hspace{1cm} \text{Feb 07 strips}

   BUY $1 worth of Aug 06 strips and roll proceeds into a 6 month forward contract for the period starting 8/15/06
   -1 \hspace{1cm} \left[1 + \frac{y_{\frac{1}{2}}(0)}{2}\right] \hspace{1cm} \text{Aug 06 strips}
   -1 \hspace{1cm} \left[1 + \frac{y_{\frac{1}{2}}(0)}{2}\right] \left[1 + \frac{f_{\frac{1}{2},1}(0)}{2}\right] \hspace{1cm} \text{forward contract}
   -1 \hspace{1cm} 0 \hspace{1cm} \left[1 + \frac{y_{\frac{1}{2}}(0)}{2}\right] \left[1 + \frac{f_{\frac{1}{2},1}(0)}{2}\right]

   d. Since both investments generate a certain cash flow on 2/15/07, the law of one price implies that

   \[ \left[1 + \frac{y_{1}(0)}{2}\right]^{2} = \left[1 + \frac{y_{\frac{1}{2}}(0)}{2}\right] \left[1 + \frac{f_{\frac{1}{2},1}(0)}{2}\right] \]

   \[ \left[1 + \frac{0.06284}{2}\right]^{2} = \left[1 + \frac{0.06186}{2}\right] \left[1 + \frac{f_{\frac{1}{2},1}(0)}{2}\right] \]

   \[ f_{\frac{1}{2},1}(0) = 6.38\%. \]

2. In general,

   \[ \left[1 + \frac{\gamma_{t+\alpha}(0)}{2}\right]^{2(\tau+\alpha)} = \left[1 + \frac{\gamma_{t}(0)}{2}\right]^{2\tau} \left[1 + \frac{f_{t+\alpha}(0)}{2}\right]^{2\alpha} \]

   which implies

   \[ d_{t+\alpha}(0) = d_{t}(0) d_{t+\alpha}(0). \]
3. Example (cont): What is the 1-year forward rate expressed as an APR with semiannual compounding available on 2/15/06 for the period starting 8/15/06?

a. First calculate \( d_{1/2,1/2} \) (Feb 06). Using the formula,

\[
d_{1/2,1/2} \text{ (Feb 06)} = d_{1/2} \text{ (Feb 06)} / d_{1/2} \text{ (Feb 06)} = 0.90 / 0.97 = 0.9278. 
\]

b. Then get the forward rate \( f_{1/2,1/2} \) (Feb 06) as follows:

\[
f_{1/2,1/2} \text{ (Feb 06)} = 2 \times \left\{ \frac{1}{0.9278} \right\}^{1/2} - 1 \approx 7.63\%. 
\]