Portfolio choice and equity characteristics: characterizing the hedging demands induced by return predictability

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Abstract

This paper examines portfolio allocation across equity portfolios formed on the basis of characteristics like size and book-to-market. In particular, the paper assesses the impact of return predictability on portfolio choice for a multi-period investor with a coefficient of relative risk aversion of 4. Compared to the investor’s allocation in her last period, return predictability with dividend yield causes the investor early in life to tilt her risky-asset portfolio away from high book-to-market stocks and away from small stocks. These results are explained using Merton’s (Econometrica 41 (1973) 867) characterization of portfolio allocation by a multi-period investor in a continuous time setting. Abnormal returns relative to the investor’s optimal early life portfolio are also calculated. These abnormal returns are found to exhibit the same cross-sectional patterns as abnormal returns calculated relative to the market portfolio: higher for small rather than large firms, and higher for high rather than low book-to-market firms. Thus, hedging demand may be a partial explanation for the high expected returns documented for small firms and high book-to-market firms. However, even with this hedging demand, the investor wants to sell short the low book-to-market portfolio to hold the

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1. Introduction

Fama and French (1992) find that size and book-to-market ratio explain cross-sectional variation in expected return over and above the beta coefficient derived from the capital asset pricing model (CAPM). One explanation for this result is that investors care about more than just portfolio mean and standard deviation when choosing their portfolios. Merton (1973) and Fama (1970) describe conditions under which investors also care about the covariance of their portfolios with a set of state variables. If returns can be predicted using a set of lagged instruments and investors are multi-period optimizers, then investors care about the covariance of their portfolios with those instruments. Empirical research indicates that U.S. equity returns are predictable and that investor horizons extend beyond a single month.1 However, for these multi-period considerations to affect expected returns in equilibrium, they must have a large impact on portfolio choice. This paper examines how an investor's multi-period horizon affects portfolio choice when returns are predictable and calibrated to U.S. data. The calibrated equity portfolios are chosen to exhibit cross-sectional variation in firm size and book-to-market ratio. The investor has constant relative risk aversion utility with a coefficient of 4.

Several recent papers have considered portfolio allocations by a multi-period investor confronted with return predictability that is calibrated to U.S. data. In these papers, however, the only equity portfolio available to the investor is the market portfolio. My paper is the first to consider portfolio allocation by a multi-period investor who has access to more than one domestic equity portfolio. Consequently, hedging demand can affect not just the amount allocated to equities by the investor, but the composition of the equity portfolio as well. Since empirical evidence shows that expected returns vary with size and

1Campbell (1987) and Fama and French (1989), among others, find that stock return variation can be explained by the one-month Treasury bill rate, the term premium, and the dividend yield.
book-to-market, I give the investor access to portfolios formed on the basis of these two firm characteristics. It follows that the paper can potentially contribute information about hedging demand as an explanation for the size and book-to-market effects. Thus, this paper is the first portfolio choice paper to consider whether the size and book-to-market effects can be explained by the hedging demand induced by predictability.

The paper has several interesting results. Compared to the investor’s allocation in her last period, return predictability with dividend yield causes the investor, early in life, to tilt her risky-asset portfolio away from high book-to-market stocks and away from small stocks. Further, while the investor’s optimal last-period portfolio is indistinguishable from the conditional minimum-variance frontier in conditional mean–standard deviation space, her optimal early in-life portfolio has a larger negative covariance with dividend yield than the conditional minimum-variance portfolio with the same expected return.

The paper also provides intuition for the direction of the tilt in the risky-asset portfolio induced by the hedging demand. This intuition builds on Merton’s (1973) characterization of the portfolio allocation by a multi-period investor in a continuous time setting. The idea is that the investor’s risky-asset portfolio is a combination of two portfolios. One is the tangency portfolio in mean–standard deviation space: the mean–variance optimal (MVO) portfolio. The other is the portfolio that is maximally correlated with the state variable: the covariance–variance optimal (CVO) portfolio. Stevens (1998) shows that the weights in the MVO portfolio depend, in part, on the pattern of expected excess returns across the risky assets. His argument can also be used to show that the weights in the CVO portfolio depend, in part, on the pattern of conditional covariances with the state variable across the risky assets. Thus, the tilt in the risky-asset portfolio induced by hedging demands can be characterized once the weight of CVO in the investor’s optimal portfolio has been determined.

This intuition is used to better understand the tilts in the risky-asset portfolio for the investor with access to the three size portfolios or the three book-to-market portfolios. For example, the young investor holds a portfolio exhibiting a more negative covariance with dividend yield than the combination of MVO and the riskless asset with the same conditional expected return. To obtain this portfolio, the investor tilts away from the high book-to-market portfolio, even though this portfolio has the most negative covariance with dividend yield of the three portfolios. This tilting occurs in part because the pattern of increasingly negative covariances going from the low to the high book-to-market portfolio is less pronounced than the increase in expected excess return going from the low to the high book-to-market portfolio. The high expected excess return on the high book-to-market portfolio gives it a large weight in the MVO portfolio, which is reduced in the investor’s risky-asset portfolio once the
investor starts to care about negative covariance with dividend yield. Interestingly, the weights in the CVO portfolio are all positive using the three size portfolios or the three book-to-market portfolios. Since the young investor holds positive amounts of the CVO portfolio, this explains why the hedging demand induced by dividend yield as a predictor has the effect of making the investor’s allocations less extreme.

Treating the young investor as the representative agent, CAPM abnormal returns are calculated using the investor’s optimal early life portfolio as the market portfolio. Since hedging demand produces an optimal early in-life portfolio that lies inside the conditional mean–variance frontier, these abnormal returns must be non-zero, though the size and direction of any cross-sectional variation are both ambiguous ex ante (see Kandel and Stambaugh, 1995). In the calibrations, these abnormal returns are found to exhibit cross-sectional patterns consistent with those for abnormal returns calculated relative to a market proxy (the value-weighted NYSE): that is, higher for small than large firms, and higher for high than low book-to-market firms. The cross-sectional dispersion in abnormal return obtained using the investor’s optimal early life portfolio, as a fraction of the dispersion obtained using the market proxy, is about 15% for both the size portfolios and for the book-to-market portfolios. Increasing the investor’s risk aversion from 4 to 10 makes the dispersion in abnormal return (calculated relative to the investor’s optimal early-life portfolio) even bigger. Thus, hedging demand may be a partial explanation for the high expected returns on small and high book-to-market stocks.

However, even with this hedging demand, the investor wants to sell short the low book-to-market portfolio to hold the high book-to-market portfolio. For this reason, treating this paper’s investor as the representative agent is problematic, and one must be careful not to read too much into the abnormal return results discussed above. Instead, it is probably better to think of this investor as just a single individual or group in the economy who takes advantage of the time-variation in investment opportunities created by other investors with different preferences. These other investors might possess habit preferences as studied in Campbell and Cochrane (1999), or they might conform to a behavioral model as presented by Barberis et al. (2000).

The utility costs of using a value-weighted equity index or of ignoring predictability are also calculated. An investor using a value-weighted equity index would give up a much larger fraction of her wealth to have access to book-to-market portfolios rather than size portfolios. Further, an investor would give up a much larger fraction of her wealth to have access to dividend yield information than term spread information. The paper also performs a sensitivity analysis to determine which predictability parameters drive hedging demands. Both the persistence of the predictive variable and its correlation
with asset returns are important for generating hedging demands. This sensitivity analysis can help explain why dividend yield as a predictive variable generates large hedging demands but term spread does not, since the former has a larger persistence parameter and return correlations of a much larger magnitude than term spread. A robustness check indicates that the investor's allocation decision is insensitive to reasonable variation in the investor's impatience parameter. Thus, it appears that the value chosen for this parameter is not driving the results.


However, these papers rarely allow the investor to hold multiple risky assets, and none allow for the multiple risky assets to be portfolios of U.S. stocks. Brennan et al. (1997) and Campbell and Viceira (2000) allow investors to hold long-term bonds in addition to stocks, while Ang and Bekaert (1999) consider the portfolio allocation problem when investors can invest in international funds. My paper is the first to consider portfolio choice when portfolios formed on the basis of size and book-to-market ratios are available to a multi-period investor.

In another related paper, Campbell (1996) examines whether cross-sectional variation in expected returns can be explained using the Euler equation from the multi-period investor's problem. Campbell assumes the existence of a representative agent and uses log-linear approximations to substitute for consumption in the Euler equation. He uses size and industry stock portfolios and bond portfolios and finds that stock market risk is the main factor determining excess returns. In part, this result follows from the high cross-sectional correlation between asset covariance with the stock market and asset covariance with news about future opportunity sets. The current paper complements Campbell's work by quantifying the direction and magnitude of the hedging demands induced by dividend yield and term spread as return predictors. It extends Campbell by considering stock portfolios formed on the basis of the book-to-market ratio.

A number of papers have made recent contributions to our understanding of the book-to-market and size effects. Pursuing a risk-based explanation, Fama
and French (1993) create mimicking portfolios formed on the basis of size and book-to-market ratios, and show that the factor loadings with respect to these portfolios can substantially reduce the abnormal returns of extreme size and book-to-market portfolios. Ferson et al. (1999) show that this result does not always imply a risk-based explanation for the effects. Daniel and Titman (1997) present evidence that the firm characteristic, whether size or book-to-market ratio, is still related to expected return after controlling for the stock’s loading with respect to the mimicking portfolio. Fama and French (1995) examine whether size and book-to-market factors in fundamentals like earnings and sales can explain cross-sectional variation in expected returns. However, using portfolios formed on the basis of size and book-to-market characteristics, they find that the loadings of portfolio returns on the book-to-market factor are close to zero and do not exhibit reliable cross-sectional variation related to portfolio book-to-market results. Lakonishok et al. (1994) argue that the high expected returns earned by high book-to-market value stocks are due to market inefficiency or suboptimal investor behavior. They show that a portfolio of value stocks is no riskier than a portfolio of glamour stocks along a number of dimensions. Jagannathan and Wang (1996) test a conditional CAPM with time-varying Beta coefficients and a market proxy that includes the return on human capital, and show that this model explains the cross-section of expected returns for size-sorted and Beta-sorted portfolios better than the static CAPM.

Recently, Ferson and Harvey (1999) use a set of predictive variables, and reject a conditional version of the Fama-French three factor model by showing that it produces abnormal returns that vary with the predictive variables. Liew and Vassalou (2000) examine data for 10 countries, and find that the book-to-market and size mimicking portfolios have incremental ability to forecast future economic growth over and above that of the market portfolio. And Lamont (1999) constructs portfolios of assets designed to track economic variables, and shows that such portfolios can be useful for hedging economic risk. None of these papers examine how return predictability and a multi-period horizon effect an investor’s portfolio allocation across size and book-to-market portfolios, which is the focus of the current paper.

The paper is organized as follows. Section 2 describes the investor’s problem and its solution. Section 3 calibrates asset returns to the U.S. economy. Section 4 presents the results and explanations, while Section 5 concludes.

2. The framework

This section describes the investor’s problem, the technique used to solve the investor’s problem, and the calculation of utility cost.
2.1. The investor’s problem

The paper considers situations in which there are $N$ risky assets and a single riskless asset available for investment. The $N \times 1$ vector of risky-asset returns from time $t$ to $t+1$, $R_{t+1}$, is either independent and identically distributed (i.i.d.) for all time periods $t$, or predictable using a $K \times 1$ vector, $Z_t$, of instruments available at $t$. The risk-free rate $R_f$ is assumed to be constant. The paper considers the optimal portfolio problem of an investor with a finite life of $T$ periods.

The investor has a time-separable utility function with constant relative risk aversion (CRRA) and a rate of time preference equal to $\beta$. Expected lifetime utility is given by

$$E \left[ \sum_{t=1}^{T} \frac{b^{t+1}}{1-\gamma} \left| Z_t \right| \right],$$

where $Z_t$ is the vector of state variables for the investor at time $t$, $c_t$ is investor’s consumption at time $t$, and $\gamma$ is the investor’s relative-risk-aversion coefficient.

When transaction costs are zero and returns are predictable, $Z_t$ equals the set of $K \times 1$ predictive variables. The formulation in Eq. (1) allows the investor to live until the terminal date with probability 1.

The law of motion of the investor’s wealth, $W_t$, is given by

$$W_{t+1} = (W_t - c_t) \left[ \alpha_t \left( R_{t+1} - R_f i_N \right) + R_f \right] = W_t(1 - \kappa_t)R_{W,t+1},$$

where $\alpha_t$ is the $N \times 1$ vector of portfolio weights chosen for the risky assets at $t$, $R_{W,t+1}$ is the portfolio return from $t$ to $t+1$, and $\kappa_t$ is the fraction of wealth consumed at $t$. Two problems are solved: the first allows short selling, while the second rules it out.

Given my parametric assumptions, the Bellman equation faced by the investor is given by

$$\frac{a(Z_t, t)W_t^{1-\gamma}}{1-\gamma} = \max_{\kappa_t, \alpha_t} \left\{ \frac{\kappa_t^{1-\gamma}W_t^{1-\gamma}}{1-\gamma} + \beta(1 - \kappa_t)^{1-\gamma}W_t^{1-\gamma}\frac{1}{1-\gamma}E \left[ a(Z_{t+1}, t+1)R_{W,t+1}^{1-\gamma} \left| Z_t \right| \right] \right\},$$

$$t = 1, \ldots, T - 1.$$
be rewritten as

\[
\frac{a(Z_t, t)}{1 - \gamma} = \max_{k_t, \pi_t} \left\{ \kappa_t^{1-\gamma} + \beta (1 - \kappa_t)^{1-\gamma} \frac{1}{1 - \gamma} \mathbb{E}\left[ a(Z_{t+1}, t + 1) R_{W, t+1|Z_t}^{1-\gamma} \right] \right\},
\]

\[t = 1, \ldots, T - 1.\]  \hspace{1cm} (4)

The Bellman equation in Eq. (4) is solved by backward iteration, starting with \(t = T - 1\) and \(a(Z_T, T) = 1\). Thus, \(a(Z_t, t)\) is obtained by solving the optimization problem in Eq. (4) using \(a(Z_{t+1}, t + 1)\) from the previous iteration.

2.2. Utility cost calculation

For a given set of available portfolios, each set of instruments is associated with a return generating process for the return vector that reflects the predictive ability of the instruments, also known as the C process. The marginal distribution for returns under this return generating process need not be i.i.d. However, the unconditional distribution for the return vector can be calculated. Also of interest is the return generating process, or process U, that is i.i.d. with a covariance matrix equal to that of the unconditional distribution of the C process. Two investor problems are considered for each set of instruments. First, the investor can use the set of instruments \(Z_t\) when making decisions at time \(t\), which is referred to the conditional (C) problem for this set of available portfolios. Alternatively, the investor can assume that the return vector follows the i.i.d. U process which is referred to as the unconditional (U) problem for this set of available portfolios.

A number of utility cost calculations are performed. When calculating the cost of using a policy and process pair \(a\) rather a policy and process pair \(b\), the calculated cost represents the fraction of wealth that an investor using \(a\) would be prepared to give up to be given access to \(b\). If one or both of the processes is a given conditional process, then an average cost is calculated using the unconditional distribution for the set of instruments.

3. Return calibration

This section describes the return and instrument data that are used, together with the quadrature approximation.
3.1. Data

The investor’s portfolio choice problem is solved allowing the investor access to several different sets of risky assets in addition to a riskless asset. The first is the VM set in which the only available risky asset is the value-weighted portfolio of all assets on the NYSE. The data for the VM set is obtained from the Center for Research in Security Prices (CRSP) Index Files. The second is the 3M set which consists of three risky portfolios formed on the basis of firm size. The three risky portfolios, M1, M2, and M3, are value-weighted portfolios of, respectively, the smallest three, the middle four and the largest three size deciles available from the CRSP Capitalization File of NYSE stocks. The third set is the 3B set which consists of three risky portfolios formed on the basis of firm book-to-market ratio. These portfolios, denoted B1, B2, and B3, are formed from the six value-weighted portfolios, SL, SM, SH, BL, BM, and BH, from Fama and French (1993) and Davis et al. (2000). The notation S indicates that the firms in the portfolio are smaller than 50% of NYSE stocks, and the notation B indicates that the portfolio firms are larger that 50% of NYSE stocks. The notation L indicates that the firms in the portfolio have book-to-market ratios that place them in the bottom three deciles for all stocks. Analogously, M indicates the middle four deciles, and H indicates the top three deciles. The high book-to-market portfolio, B3, is an equally weighted portfolio of SH and BH. Portfolio B2 is an equally weighted portfolio of SM and BM, and B1 is an equally weighted portfolio of SL and BL. The final set of risky assets available to the investor is the 3B2M set which consists of six risky portfolios formed on the basis of firm size and book-to-market ratio. These portfolios are the six value-weighted portfolios SL, SM, SH, BL, BM, and BH from Fama, and French (1993) and Davis et al. (2000).

The choice of firm characteristics used to form portfolios is predicated by the aim of achieving a wide dispersion of expected returns across the portfolios. Work by Berk (1995) provides a theoretical rationale for using variables that depend on price. Both size and book-to-market ratio satisfy this criterion. Further, empirical work by Banz (1981), Stattman (1980), and Fama and French (1992, 1993), among others, finds that average return depends on both these variables, even after controlling for market Beta.

The investor is allowed various sets of predictive variables when making portfolio choices. With the first set, U, the investor assumes that returns are i.i.d. With the second set, D, the only predictive variable that the investor uses is the continuously compounded 12-month dividend yield on the value-weighted NYSE, which is taken from CRSP. With the third set, S, the only predictive variable that the investor uses is the yield spread between 20-year

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*I would like to thank Gene Fama and Ken French for making this data available to me.*
and one-month Treasury securities, from Ibbotson data service. Finally, with set \( F \), the investor uses two predictive variables, \( D \) and \( S \).

Predictive variables are chosen to have parsimonious predictability. Fama and French (1989) find that dividend yield and term spread predict distinct return components for a cross-section of asset classes. The other criterion for choosing predictive variables is some economic justification for the predictive relation. Both dividend yield and term spread move with the business cycle, making them natural predictors in a setting in which expected return moves over the business cycle.

All asset returns, including the risk-free rate, are deflated using monthly consumer price index (CPI) inflation indicators from CITIBASE data service. The data period used is from July 1927 to November 1996. The continuously compounded risk-free rate is estimated to be the mean of the continuously compounded one-month Treasury bill rate from CRSP over this period, which gives a value for \( R_f \) the directly compounded risk-free-rate, of 0.042%.

Following Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000), a vector auto regression (VAR) is estimated using ordinary least squared (OLS) regression for each combination of assets and predictive variables \( (Z) \). The asset returns \( (R) \) are converted to a continuously compounded basis for the VAR. Hence, \( R \) is replaced by 
\[
    r_t = \ln(1 + R_t)
\]
where \( r_t \) is an \( N \times 1 \) vector of coefficients from a regression of \( e_t \) on \( v_t \), and \( u_t \) is a i.i.d. disturbance vector which has mean zero and covariance matrix \( \Sigma_u \), and is uncorrelated with \( v_t \). The disturbance vector \( [u_t \; v_t] \) is assumed to be multivariate and normally distributed, but with truncation for extreme realizations. Truncation is assumed so that short selling is not ruled out by extreme realizations of \( e_t \) that have positive probability under the normal distribution, but which are, in fact, implausible.
3.2. Quadrature approximation

The data VAR is approximated using a variation of the Gaussian quadrature method described by Tauchen and Hussey (1991). First, Tauchen and Hussey’s method is used to discretize the predictive variable vector, \( Z_t \), treating it as a first-order autoregressive process as in Eq. (6). The quadrature method is then used to calibrate a discrete distribution for the innovation, \( u_t \). I can then calculate a discrete distribution for \( r_{t+1} \) for each \( \{Z_{t+1}, Z_t\} \) pair from the transformation of \( Z \), since \( v_{t+1} = Z_{t+1} - aZ - bZ_t \). This approach ensures that \( Z \) is the only state vector. I chose a specification with 19 quadrature points for the dividend yield \( D \), 7 for the term spread \( S \); and 3 points for the innovations in stock returns. Balduzzi and Lynch (1999) also use this basic approach, and find that the approximation is able to capture important dimensions of the predictability in the data. However, in an improvement over Balduzzi and Lynch (1999), this study implements the discretization in a manner that produces exact matches for important moments for portfolio choice. In particular, the procedure matches both the conditional mean vector and the covariance matrix for log returns at all grid points of the predictive variables, as well as the unconditional volatilities of the predictive variables and the correlations of log returns with the predictive variables. Finally, the data values for \( \Sigma_{ev} \) are taken to be the covariance matrix for the associated untruncated normal distributions when performing the quadrature approximation. Because the truncation typically uses extreme cutoffs, the misstatement of \( \Sigma_{ev} \) by the approximation is likely to be small.

I find that increasing the number of grid points for stock returns from 3 to 15 has virtually no effect on the optimal portfolio weights chosen by the investor. Using 15 grid points for returns, the largest realization for \( u_{t+1} \) is more than six standard deviations from zero. Further, this number of return grid points, in conjunction with 19 grid points for dividend yield as the predictive variable, results in both the following events having positive probabilities for all three book-to-market portfolios: a one-month return that is less than \(-70\%\); and a one-month return that is greater than \(240\%\). At the same time, the smallest one-month return in the data across the three book-to-market portfolios is \(-36\%\), while the largest is only \(61\%\). In addition, the investor’s optimal unconditional portfolio never realizes a one-month return of less than \(-38\%\), while for the old investor’s optimal conditional portfolio, the minimum one-month return is \(-50\%\), which is still far away from \(-100\%\).

There are two implications of these results. First, implausibly large deviations from the mean are needed for the possibility of negative wealth to affect the investor’s portfolio choice. Second, the investor’s optimal portfolio is largely unaffected by the severity of a symmetric truncation that is sufficient to ensure that the possibility of negative wealth does not drive the investor’s portfolio choice. This implication follows from the insensitivity of the
Table 1
Calibration of the three size portfolios

Table 1 reports moments and parameters for three size portfolios (the 3M asset set) estimated from U.S. data and calculated for two quadrature approximations (Quads) based on VARs that use log dividend yield \((D)\) or term spread \((S)\) as the only state variable. The large firm portfolio is denoted M3 and the small firm portfolio is denoted M1. The data period is July 1927 to November 1996. Panel A reports unconditional sample moments for the data including abnormal return estimates from regressing excess asset returns on the excess return of the value-weighted portfolio of NYSE stocks. Panel A also reports data and quadrature results for the two VARs: \(b\) is the vector of VAR slopes and \(R^2\) denotes the regression \(R^2\). Panel B reports the unconditional covariance matrix for the data and for the two quadrature approximations. Panels C and D report the conditional covariance matrices for the data VAR and the quadrature VAR using, respectively, the log dividend yield and the term spread as the only state variable. All results are for continuously compounded returns except the abnormal return, which is calculated using discrete returns. Returns are expressed per month and in percent.

<table>
<thead>
<tr>
<th>Asset/variable</th>
<th>Unconditional mean</th>
<th>Average abnormal return</th>
<th>(D) as the only predictive variable</th>
<th>(S) as the only predictive variable</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td></td>
<td>Data (b) (R^2) Quad (b) (R^2)</td>
<td>Data (b) (R^2) Quad (b) (R^2)</td>
</tr>
<tr>
<td>M3</td>
<td>0.53</td>
<td>-0.002</td>
<td>0.28 0.26 0.28 0.26 0.38 0.50 0.38 0.50</td>
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<tr>
<td>M2</td>
<td>0.67</td>
<td>0.068</td>
<td>0.49 0.49 0.49 0.49 0.58 0.69 0.58 0.69</td>
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</tr>
<tr>
<td>M1</td>
<td>0.72</td>
<td>0.129</td>
<td>0.61 0.52 0.61 0.52 0.72 0.70 0.72 0.70</td>
<td></td>
</tr>
<tr>
<td>(D)</td>
<td>0.00</td>
<td>0.129</td>
<td>0.97 94.67 0.96 92.58 0.87 75.49 0.85 72.34</td>
<td></td>
</tr>
<tr>
<td>(S)</td>
<td>0.00</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Asset/variable</td>
<td>Data</td>
<td>Quad: D as the only predictive variable</td>
<td>Quad: S as the only predictive variable</td>
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<tr>
<td></td>
<td>M3</td>
<td>M2</td>
<td>M1</td>
<td>D</td>
</tr>
<tr>
<td>Panel B: Unconditional standard deviations, covariances (above diagonal), and correlations (below)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M3</td>
<td>5.38</td>
<td>35.46</td>
<td>39.27</td>
<td>−0.88</td>
</tr>
<tr>
<td>M2</td>
<td>0.94</td>
<td>7.00</td>
<td>57.42</td>
<td>−0.97</td>
</tr>
<tr>
<td>M1</td>
<td>0.85</td>
<td>0.96</td>
<td>8.54</td>
<td>−1.03</td>
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<tr>
<td>D</td>
<td>−0.16</td>
<td>−0.14</td>
<td>−0.12</td>
<td>1.00</td>
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<tr>
<td>S</td>
<td>0.06</td>
<td>0.08</td>
<td>0.09</td>
<td>0.06</td>
</tr>
<tr>
<td>Panel C: Conditional standard deviations, covariances (above diagonal), and correlations (below) for the VAR with D as the only predictive variable</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M3</td>
<td>5.38</td>
<td>35.32</td>
<td>39.10</td>
<td>−1.15</td>
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<tr>
<td>M2</td>
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<tr>
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<td>0.96</td>
<td>8.52</td>
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<tr>
<td>D</td>
<td>−0.93</td>
<td>−0.89</td>
<td>−0.83</td>
<td>0.23</td>
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<tr>
<td>Panel D: Conditional standard deviations, covariances (above diagonal), and correlations (below) for the VAR with S as the only predictive variable</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M3</td>
<td>5.37</td>
<td>35.24</td>
<td>38.99</td>
<td>−0.03</td>
</tr>
<tr>
<td>M2</td>
<td>0.94</td>
<td>6.98</td>
<td>57.01</td>
<td>0.08</td>
</tr>
<tr>
<td>M1</td>
<td>0.85</td>
<td>0.96</td>
<td>8.51</td>
<td>0.18</td>
</tr>
<tr>
<td>S</td>
<td>−0.01</td>
<td>0.02</td>
<td>0.04</td>
<td>0.50</td>
</tr>
</tbody>
</table>
Table 2
Calibration of the three book-to-market portfolios

Table 2 reports moments and parameters for three book-to-market portfolios (the 3B asset set) estimated from U.S. data and calculated for two quadrature approximations (Quads) based on VARs that use log dividend yield \((D)\) or term spread \((S)\) as the only state variable. The high book-to-market portfolio is denoted B3 and the low book-to-market portfolio is denoted B1. The data period is July 1927 to November 1996. Panel A reports unconditional sample moments for the data including abnormal return estimates from regressing excess asset returns on the excess return of the value-weighted portfolio of NYSE stocks. Panel A also reports data and quadrature results for the two VARs: \(b\) is the vector of VAR slopes and \(R^2\) denotes the regression \(R^2\). Panel B reports the unconditional covariance matrix for the data and for the two quadrature approximations. Panels C and D report the conditional covariance matrices for the data VAR and the quadrature VAR using, respectively, the log dividend yield and the term spread as the only state variable. All results are for continuously compounded returns except the abnormal return, which is calculated using discrete returns. Returns are expressed per month and in percent.

<table>
<thead>
<tr>
<th>Asset/variable</th>
<th>Unconditional mean</th>
<th>Average abnormal return</th>
<th>(D) as the only predictive variable</th>
<th>(S) as the only predictive variable</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Data (b) (R^2) Quad (b) (R^2)</td>
<td>Data (b) (R^2) Quad (b) (R^2)</td>
</tr>
<tr>
<td>B3</td>
<td>0.91</td>
<td>0.310</td>
<td>0.54 0.52 0.54 0.52</td>
<td>0.51 0.47 0.51 0.47</td>
</tr>
<tr>
<td>B2</td>
<td>0.70</td>
<td>0.123</td>
<td>0.41 0.40 0.41 0.40</td>
<td>0.50 0.60 0.50 0.60</td>
</tr>
<tr>
<td>B1</td>
<td>0.53</td>
<td>−0.043</td>
<td>0.41 0.40 0.41 0.40</td>
<td>0.56 0.73 0.56 0.73</td>
</tr>
<tr>
<td>(D)</td>
<td>0.00</td>
<td></td>
<td>0.97 94.67 0.96 92.58</td>
<td></td>
</tr>
<tr>
<td>(S)</td>
<td>0.00</td>
<td></td>
<td></td>
<td>0.87 75.49 0.85 72.34</td>
</tr>
<tr>
<td>Asset/variable</td>
<td>Data</td>
<td>Quad: $D$ as the only predictive variable</td>
<td>Quad: $S$ as the only predictive variable</td>
<td></td>
</tr>
<tr>
<td>----------------</td>
<td>------</td>
<td>---------------------------------</td>
<td>---------------------------------</td>
<td></td>
</tr>
<tr>
<td></td>
<td>B3</td>
<td>B2</td>
<td>B1</td>
<td>$D$</td>
</tr>
<tr>
<td>Panel B: Unconditional standard deviations, covariances (above diagonal), and correlations (below)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B3</td>
<td>7.41</td>
<td>46.67</td>
<td>44.16</td>
<td>−0.99</td>
</tr>
<tr>
<td>B2</td>
<td>0.97</td>
<td>6.47</td>
<td>40.25</td>
<td>−0.94</td>
</tr>
<tr>
<td>B1</td>
<td>0.92</td>
<td>0.96</td>
<td>6.49</td>
<td>−0.91</td>
</tr>
<tr>
<td>$D$</td>
<td>−0.13</td>
<td>−0.15</td>
<td>−0.14</td>
<td>1.00</td>
</tr>
<tr>
<td>$S$</td>
<td>0.08</td>
<td>0.08</td>
<td>0.07</td>
<td>0.06</td>
</tr>
<tr>
<td>Panel C: Conditional standard deviations, covariances (above diagonal), and correlations (below) for the VAR with $D$ as the only predictive variable</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B3</td>
<td>7.39</td>
<td>46.45</td>
<td>43.94</td>
<td>−1.51</td>
</tr>
<tr>
<td>B2</td>
<td>0.97</td>
<td>6.46</td>
<td>40.08</td>
<td>−1.34</td>
</tr>
<tr>
<td>B1</td>
<td>0.92</td>
<td>0.96</td>
<td>6.48</td>
<td>−1.31</td>
</tr>
<tr>
<td>$D$</td>
<td>−0.88</td>
<td>−0.90</td>
<td>−0.88</td>
<td>0.23</td>
</tr>
<tr>
<td>$S$</td>
<td>0.08</td>
<td>0.08</td>
<td>0.07</td>
<td>0.06</td>
</tr>
<tr>
<td>Panel D: Conditional standard deviations, covariances (above diagonal), and correlations (below) for the VAR with $S$ as the only predictive variable</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B3</td>
<td>7.39</td>
<td>46.41</td>
<td>43.88</td>
<td>0.13</td>
</tr>
<tr>
<td>B2</td>
<td>0.97</td>
<td>6.45</td>
<td>39.97</td>
<td>0.06</td>
</tr>
<tr>
<td>B1</td>
<td>0.92</td>
<td>0.96</td>
<td>6.46</td>
<td>−0.02</td>
</tr>
<tr>
<td>$S$</td>
<td>0.04</td>
<td>0.02</td>
<td>−0.01</td>
<td>0.50</td>
</tr>
</tbody>
</table>
investor’s portfolio choice to increases in the number of grid points, so long as portfolio value remains positive. Consequently, my results are likely to be informative of optimal portfolio choice by a CRRA investor with access to equity portfolios formed on the basis of size and book-to-market ratio.

4. Results

This section discusses the results of the paper, starting with a comparison of moments and parameters for the data and the quadrature approximations. Then, portfolio allocations are presented for an investor who lives for 20 years or 240 months, has a time preference parameter $\beta$ of $1/R_f$, and a relative risk aversion coefficient $\gamma$ of 4. Utility cost and consumption results are also presented. Finally, several sensitivity analyses are reported and discussed.

4.1. Data and quadrature VAR

Table 1 reports data and quadrature parameters for the 3M asset set which consists of the three size portfolios. Panel A reports unconditional means, average abnormal returns and coefficients for two VARs, one using dividend yield, $D$, as the state variable, and the other using term spread, $S$, as the state variable. While means and abnormal returns are reported only for the data, VAR coefficients are reported for both the data and the relevant quadrature approximation. Abnormal returns are estimated by regressing excess asset returns on the excess return of the value-weighted NYSE, VM, taking the riskless rate, $R_f$, to be constant and equal to the sample average of 0.042%. Panel B reports unconditional standard deviations, covariances, and correlations for returns and state variables in the data and in each quadrature approximation. Panel C reports conditional standard deviations, covariances, and correlations for returns and dividend yield, $D$, based on the VAR with $D$ as the predictive variable. Results are reported for the data and the quadrature approximation that uses dividend yield, $D$, as the state variable. Panel D reports analogous results to Panel C but with term spread, $S$, as the state variable rather than $D$. Table 2 reports the same data and quadrature results as Table 1 for the 3B asset set, which consists of the three book-to-market portfolios. All results in Tables 1 and 2 are for continuously compounded returns, except the abnormal return, which is calculated using discrete returns.

Panel A of Table 1 shows that the mean return is decreasing in firm size, ranging from 0.53% for M3 up to 0.72% for the small firm portfolio M1. Abnormal return ranges from $-0.002\%$ for M3 up to 0.129% for the small firm portfolio M1. Turning to the book-to-market portfolios in Table 2, Panel A shows that the mean return is increasing in book-to-market, ranging from 0.91% for B3 down to 0.53% for the low book-to-market portfolio B1.
Abnormal return is also increasing in book-to-market, ranging from $-0.043\%$ for the low book-to-market portfolio B1 up to $0.31\%$ for B3. These results are consistent with earlier work documenting a small firm effect and a book-to-market effect.

Consistent with earlier studies, Panel A of Tables 1 and 2 also show that neither dividend yield nor term spread explain much of the empirical variation in monthly equity returns for the 3M or 3B asset sets. While the $R^2$'s are larger for term spread, the autoregressive parameter is larger for dividend yield.

Panels C and D of the two tables show that the covariances between asset shocks and the predictive variable shock are much larger for dividend yield. For the size portfolios, the covariance between asset shock and dividend yield shock is negative, and its magnitude is decreasing in firm size. However, the magnitude of the correlation is increasing in firm size, reflecting the higher volatility of small firms. Turning to the book-to-market portfolios, the covariance between asset shock and dividend yield shock is also negative, and its magnitude is increasing in the book-to-market ratio. However, the magnitude of the correlation is not monotonic in the book-to-market ratio.

Importantly, comparing the data and quadrature VAR parameters for either asset set, it appears that the approximation incorporates the predictability of the data. Both the VAR regression coefficients for the returns and the covariance matrix for the return residuals are very similar for the approximation and for the data, regardless of whether $D$ or $S$ is being used as $Z$. While the conditional covariances between returns and dividend yield are higher for the approximation than the data, a close inspection reveals that this result is driven by the higher conditional volatility of $D$ in the approximation than in the data. This interpretation is confirmed by the conditional correlations between return and $D$, which are similar in both the approximation and the data. As discussed above, the approximation is designed to produce predictive variables with exactly unit variance, and the tables confirm that this is the case. Similarly, in unreported results, the conditional means and covariances for the approximation’s log asset returns exactly match those implied by the VAR calculated for the data. Finally, the persistence parameter, $h_Z$, for the approximation is typically close to, but lower than, that for the data. Consequently, the approximation provides conservative estimates of both the magnitude of the hedging demands and of the utility costs associated with ignoring the predictive variables.

4.2. Portfolio allocation

Portfolio allocations are reported for the VM, 3M, and 3B asset sets in Figs. 1–3. For each set of two graphs, the left graph shows the allocations without short selling, while the right graph shows allocations with short selling. Each graph shows the investor’s allocation as a function of the investor’s age $t$. 
No Short selling

Short selling

Av. weight in VM

Age in months (t)

0 40 80 120 160 200 240

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

φ U

△ D

▽ S

Av. weight in VM

Age in months (t)

0 40 80 120 160 200 240

0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0

φ U

△ D

▽ S
where \( t = 1 \) is her first month of investing and \( t = 239 \) is her last. Allocations for three sets of predictive variables are plotted in each graph. The i.i.d. portfolio allocation \( U \) is plotted together with the average allocation when the investor uses dividend yield \( D \) or term spread \( S \). Averaging is performed using the unconditional distribution for the predictive variable.

### 4.2.1. Value-weighted market portfolio (VM)

Fig. 1 presents allocation results when the investor has access to the value-weighted index of NYSE stocks, VM, in addition to the riskless asset. Comparing the \( U \) allocation which uses the i.i.d. distribution to the \( D \) allocation which uses dividend yield, the left graph of Fig. 1 shows that the average allocation to VM in the last period using \( D \) is virtually identical to the \( U \) allocation of 53%. However, as the investor gets younger, the average allocation to VM increases from 53% to 68%. This difference of 15% is the hedging demand induced by using dividend yield as a predictive variable in the absence of short selling. The magnitude of this demand is comparable to earlier studies (see, for example, Barberis, 2000). When short selling is allowed, the direction of the hedging demand is the same, but its magnitude increases to 16%. As seen in the bottom graph of Fig. 1, the average allocation to VM increases from 53% in the last period to 69% early in life.

In contrast, when the investor uses \( S \), the yield spread variable, her average allocation to VM is virtually unchanged over her life, remaining close to the unconditional allocation. This effect occurs regardless of whether short selling is allowed. Thus, the hedging demand induced by \( S \) is small.

### 4.2.2. Size portfolios (3M set)

Fig. 2 presents allocation results when the investor has access to the 3 size portfolios of NYSE stocks, the 3M set, in addition to the riskless asset. The investor’s allocation decision can be broken into two parts: the allocation to the risky-asset portfolio (which consists of the 3 size portfolios), and the composition of the risky-asset portfolio. Panel A of Fig. 2 reports the average allocation to the risky-asset portfolio, while Panel B reports the composition of the risky-asset portfolio. In particular, Panel B contains three sets of graphs,
each plotting the average allocation to a size portfolio scaled by the average allocation to the risky-asset portfolio.

Interestingly, Panel A of Fig. 2 shows that the average allocations to the risky-asset portfolio, formed using assets M3, M2, and M1, are slightly lower than the average allocations to VM in Fig. 1. For example, having access to size portfolios rather than the value weighted index when using dividend yield $D$, reduces the old investor’s average allocation to stocks from 53% to 45%. Moreover, the greater flexibility afforded by the size portfolios increases slightly the investor’s overall hedging demand for stocks. When the investor uses dividend yield, that demand is large, positive, and of a similar magnitude to the VM case. In the absence of short selling, the average hedging demand is 20%, as compared to 15% for VM, while relaxing the short selling constraint results in a hedging demand of 21%, as compared to 16% for VM. As with VM, hedging demand is negligible when the investor uses term spread.

Turning to Panel B, the composition of the risky-asset portfolio when the investor uses $U$, the unconditional distribution, is 23% in M3, 47% in M2, and 30% in the smallest stock portfolio M1. Thus, it appears that the short selling restriction does not bind the investor. In the absence of return predictability, the investor places a greater fraction of her wealth in the bottom three size deciles M1, than the top three deciles M3.

Allowing the investor to use the term spread variable has almost no impact on the average allocations to the size portfolios, irrespective of whether short selling is allowed or not. However, the availability of the dividend yield variable causes the investor, early in life, to increase her average allocation to asset M3, and reduce her average allocation to M1, relative to the $U$ case. This result is robust to whether or not short selling is available.

How the investor’s allocations to the size portfolios change over her life cycle depends on whether short selling is allowed. I focus on the allocations when
Fig. 2. (continued)
Medium firm portfolio (M2): No short selling

Medium firm portfolio (M2): Short selling

Fig. 2. (continued)
Fig. 2. (continued)
short selling is allowed, since the investor's first order conditions hold in all states. Note that the patterns of the tilts in the risky-asset portfolio are similar with or without the availability of short selling. The right-hand side graphs indicate that the average allocation to portfolio M3 is much larger early in life than in the last period, while the converse is true for portfolios M2 and M1. Thus, the hedging demand induced by dividend yield as a predictive variable causes the investor to tilt her risky-asset portfolio away from small stocks early in life.

4.2.3. Book-to-market portfolios (3B set)

Fig. 3 presents allocation results when the investor has access to the 3 book-to-market portfolios, B3, B2, and B1. As with Fig. 2, Panel A of Fig. 3 reports the average allocation to the risky-asset portfolio, while Panel B reports the composition of the risky-asset portfolio. In particular, Panel B contains three sets of two graphs, each pair plotting the average allocation to a book-to-market portfolio scaled by the average allocation to the risky-asset portfolio.

Panel A of Fig. 3 shows that the average allocations to the risky-asset portfolio formed using assets B3, B2, and B1 are similar to the average allocations to VM in Fig. 1 when short selling is prohibited. In contrast, the average allocations to the risky-asset portfolio are lower using the 3B asset set rather than VM by 15–20% when short selling is allowed. However, the magnitude of the hedging demands are larger relative to those for VM, irrespective of whether short selling is allowed. In the absence of short selling, the average hedging demand is 19%, as compared to 15% for VM, while relaxing the short selling constraint results in a hedging demand of 24%, as compared to 16% for VM.

Turning to the graphs in Panel B of Fig. 3 that depict allocations in the absence of short selling, the composition of the risky-asset portfolio when the investor uses the i.i.d. distribution, \( U \), is 100% in B3, the high book-to-market portfolio. Thus, ignoring return predictability, the short selling restriction is binding, and the investor does not want to hold any of portfolios B2 or B1. When the investor is allowed to use dividend yield, the average allocation to portfolios B2 and B1 is still 0%, while the availability of term spread causes the average allocations to these portfolios to be less than 3%. Thus, the investor only wants to hold a positive amount of the high book-to-market portfolio, and return predictability does little to alter this conclusion.

Before examining the composition of the risky-asset portfolio when short selling is allowed, recall that the graphs in Panel B of Fig. 3 scale the average holding of each risky asset by the average total investment in the risky assets. Three of the graphs in Panel B plot the investor's allocations when short selling is allowed. Using the i.i.d. distribution \( U \), the average allocation to portfolio B3 is more than three times the average allocation to the risky-asset portfolio, while the average allocations to portfolios B2 and B1 are negative. Allowing
the investor to use the dividend yield to predict asset returns has little effect on average allocations to the three assets in the last period relative to the allocations using $U$. However, early in life, dividend yield information causes the investor to tilt her risky–asset portfolio away from the high book-to-market portfolio $B_3$ and towards portfolio $B_2$, and especially portfolio $B_1$. The average allocation to asset $B_3$ as a fraction of the average allocation to the risky-asset portfolio drops from more than 3.5 in the last period to less than 2.5 early in life. On the other hand, the average allocation to asset $B_2$ goes from negative to almost zero as the investor becomes younger. Finally, the average allocation to asset $B_1$, scaled by the average allocation to the risky-asset portfolio, goes from about $-2$ late in life to about $-1$ early in life. Thus, as the investor gets younger, hedging demand induced by dividend yield causes her to tilt her risky–asset portfolio away from $B_3$ and toward $B_2$ and $B_1$. While the magnitude of this tilt is large, it is troubling that the average allocation to the small book-to-market portfolio is still so negative, even early in life.

Finally, the hedging demands induced by the term spread variable when short selling is allowed are again negligible. This conclusion follows from both the average allocations to the risky-asset portfolio and the average composition of this portfolio.

### 4.3. A comparison of optimal portfolios to minimum-variance portfolios

Comparing the investor’s optimal portfolio to the minimum-variance portfolio with the same expected return yields interesting results. This exercise is done in the first five columns of Table 3 in which Panel A displays the results for the 3M set of assets, while Panel B displays the results for the 3B set of assets. The first three columns report the unconditional mean, the average

Fig. 3. Average portfolio allocations by an investor who has access to the riskless asset and 3 book-to-market portfolios of NYSE stocks: $B_3$, $B_2$, and $B_1$. The high book-to-market portfolio is denoted $B_3$ and the low book-to-market portfolio is denoted $B_1$. The investor has a relative risk aversion coefficient ($\gamma$) of 4. The investor’s allocation decision can be broken into two parts: the allocation to the risky-asset portfolio (which consists of the 3 book-to-market portfolios); and the composition of the risky-asset portfolio. Panel a reports the average allocation to the risky-asset portfolio while Panel b reports the composition of the risky-asset portfolio using three pairs of graphs. Each pair plots the average allocation to one of the book-to-market portfolios scaled by the average allocation to the risky-asset portfolio. For each pair, the top graph shows the allocation without short-selling while the bottom graph shows allocations with short-selling. Each graph shows the investor’s allocation as a function of the investor’s age $t$, where $t = 1$ is her first month and $t = 239$ is her last. Allocations for three sets of predictive variables are plotted in each graph. The unconditional ($U$) portfolio allocation is plotted together with the average allocation when the investor uses dividend yield ($D$) or term spread ($S$). Averaging is performed using the unconditional distribution for the predictive variable.
No short selling

Av. weight in risky-asset portfolio vs. Age in months (t)

- U
- D
- S

Short selling

Av. weight in risky-asset portfolio vs. Age in months (t)

- U
- D
- S

(a)
High book-to-market portfolio (B3): No short selling

High book-to-market portfolio (B3): Short selling

Fig. 3. (continued)
Medium book-to-market portfolio (B2): No short selling

Medium book-to-market portfolio (B2): Short selling

Fig. 3. (continued)
Low book-to-market portfolio (B1): No short selling

Fig. 3. (continued)

Low book-to-market portfolio (B1): Short selling

Fig. 3. (continued)
Table 3
Comparison of optimal and minimum-variance portfolios

Table 3 reports parameters for the optimal investor portfolio with short selling when returns are i.i.d. (U), or when the investor uses log dividend yield (D) or term spread (S) as a predictive variable (Z). The investor has a coefficient of relative risk aversion (\(g\)) of 4. The optimal portfolio for an investor of age (\(t\)) 1 month and 239 months is reported. The expected excess return (\(E[r_{W,t+1}]\)) on the portfolio is reported together with the average conditional volatility (\(\sigma_{t}[R_{W,t+1}]\)) and the average conditional covariance with the predictive variable (\(\sigma_{t}[R_{W,t+1}, Z_{t+1}]\)). These last two statistics are also reported for the minimum-variance portfolio with the same conditional mean as the investor’s portfolio in each state. When returns are i.i.d., only portfolio volatility is reported. Panel A reports results for an investor with access to three size portfolios plus the riskless asset while Panel B reports results for an investor with access to three book-to-market portfolios plus the riskless asset. In Panel A, the large and small firm portfolios are denoted M3 and M1 respectively, while in Panel B, the high and low book-to-market portfolios are denoted B3 and B1 respectively. The average intercept from a conditional regression of each asset’s excess return on the excess return of the investors portfolio is reported. When returns are i.i.d., this regression is unconditional. Returns are expressed per month and in percent.

<table>
<thead>
<tr>
<th>(Z_t)</th>
<th>Investor age ((t))</th>
<th>Optimal portfolio</th>
<th>Minimum-variance portfolio</th>
<th>Average intercept</th>
</tr>
</thead>
<tbody>
<tr>
<td>(E[r_{W,t+1}])</td>
<td>(\text{av. } \sigma_{t}[R_{W,t+1}])</td>
<td>(\text{av. } \sigma_{t}[R_{W,t+1}, Z_{t+1}])</td>
<td>(\text{av. } \sigma_{t}[R_{W,t+1}])</td>
<td>(\text{av. } \sigma_{t}[R_{W,t+1}, Z_{t+1}])</td>
</tr>
<tr>
<td>(U_t)</td>
<td>All</td>
<td>0.39</td>
<td>3.15</td>
<td>na</td>
</tr>
<tr>
<td></td>
<td>239</td>
<td>0.57</td>
<td>3.50</td>
<td>−0.77</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.72</td>
<td>4.61</td>
<td>−1.08</td>
</tr>
<tr>
<td>(S_t)</td>
<td>All</td>
<td>0.99</td>
<td>4.94</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>239</td>
<td>0.99</td>
<td>4.96</td>
<td>0.18</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.99</td>
<td>4.94</td>
<td>0.18</td>
</tr>
</tbody>
</table>
conditional volatility, and the average conditional covariance with the predictive variable of the investor’s optimal portfolio. Averaging is performed using the unconditional distribution for the predictive variable. The next two columns report the average conditional volatility and the average conditional covariance with the predictive variable for the conditional minimum variance portfolio with the same conditional mean as the optimal portfolio. The first row of each panel reports portfolio moments when returns are i.i.d., the $U$ distribution. In this case, the conditional distribution coincides with the unconditional distribution. The remaining 2 pairs of rows reports portfolio moments when returns are predictable using dividend yield, $D$, or term spread, $S$, based on the investor’s optimal portfolio in her first ($t = 1$) and last ($t = 239$) months of life.

The first row of each panel indicates that the investor’s optimal portfolio is indistinguishable from minimum variance when returns are i.i.d. The volatility of the optimal portfolio is equal to that of the minimum–variance portfolio with the same mean. So although returns are calibrated to be log-normal rather than normal, the investor still behaves like a mean–variance optimizer. When a predictive variable is available, the investor still behaves like a mean–variance optimizer in the last month of life ($t = 239$), but now uses the conditional rather than unconditional distribution. The average conditional volatility of the optimal portfolio is identical to that of the conditionally minimum-variance portfolio with the same conditional mean. This pattern is true for both sets of assets and either predictive variable.

Hedging demands have the greatest effect on the investor’s optimal portfolio early in life. When the investor use the dividend yield variable, $D$, the optimal portfolio has a higher average volatility in the first month of life ($t = 1$) than in the last month of life ($t = 239$). In particular, when allocating across the three size portfolios, as shown in Panel A of Table 3, average portfolio volatility increases from 3.50% to 4.61% as the investor gets younger. For the three book-to-market portfolios in Panel B, a similar increase from 4.70% to 5.95% is experienced as the investor goes from the last to the first month of life.

Comparing the optimal $t = 1$ portfolio using the dividend yield variable, $D$, to the minimum-variance portfolio with the same conditional mean, the optimal portfolio has a higher average conditional volatility but a more negative average covariance. When allocating across the three size portfolios, as shown in Panel A, the investor is prepared to accept an average volatility of 4.61% rather than the 4.59% from the minimum-variance portfolio to have an average covariance with $D$ of $-1.08$ rather than $-1.03$. Panel B shows that the investor makes a similar trade off when allocating across the three book-to-market portfolios.

Thus, when multiple risky assets are available, hedging demand induced by dividend yield has two effects. First, the investor is prepared to take on more volatility, given the volatility–mean tradeoff, because this greater volatility is
accompanied by a larger negative covariance with dividend yield. Second, for a given mean, the investor is prepared to accept higher volatility to obtain a larger negative covariance with dividend yield. The investor prefers the larger negative covariance with dividend yield because it allows her to hedge uncertainty about future opportunity sets. This behavior follows because $D$ covaries positively with future opportunity sets and the investor’s risk aversion is greater than 1. These results are consistent with previous portfolio choice papers like Barberis (2000), Campbell and Viceira (1999), and Lynch and Balduzzi (2000), which consider an investor with access to one risky asset calibrated to U.S. equity. These papers find that the negative covariance of dividend yield with U.S. equity induces a CRRA investor with risk aversion greater than 1 to hold more of the risky asset early in life than later in life.

It is interesting to examine whether this preference for negative portfolio covariance with $D$ translates into a tilt in the investor’s risky-asset portfolio toward assets with a large negative covariance with $D$. Tables 1 and 2 show that this covariance is most negative for the small firm portfolio M1 in the 3M asset set, and for the high book-to-market portfolio B3 in the 3B asset set. Despite a preference for negative portfolio covariance with $D$, the investor tilts her risky-asset portfolio away from assets M1 and B3 as she gets younger.

Thus, knowing just the covariances of the assets with the state variable is not sufficient to determine how hedging demand affects portfolio allocations across the assets. The reason is that, for a given portfolio covariance with $D$, the investor wants the portfolio with the lowest conditional volatility (see Fama (1996), for an excellent discussion of this point). The investor cares about how a change in portfolio composition affects both the portfolio’s covariance with $D$ and its volatility. In other words, the investor wants to obtain negative covariance with dividend yield at the cheapest cost in terms of increased portfolio variance. This intuition explains why the investor may want to hold relatively more of the risky assets with the less negative covariance with dividend yield. In fact, the entire covariance matrix for the assets and the state variable, and not just the vector of asset covariances with the state variable, affects hedging demands. This issue is explored in more detail in the next section.

Turning to the term spread variable, $S$, the investor’s portfolio in the last period of her life is indistinguishable from the minimum-variance portfolio with the same conditional mean. Moreover, the hedging demand induced by this variable has almost no impact on the average volatility of the investor’s portfolio, relative to the investor’s last-period portfolio.

---

3 Strictly speaking, the investor cares about more than just volatility, since asset returns are log-normally distributed rather than normally distributed. But, the investor’s holdings when returns are i.i.d. or when $t = 239$ indicate that, in the absence of hedging demands, the investor behaves like a mean–variance optimizer given the available assets.
4.4. Understanding the tilt in the risky-asset portfolio weights induced by hedging demands

This section provides theory and calibration results designed to explain the tilt in the risky-asset portfolio weights induced by hedging demand. The theory draws heavily on the work of Merton (1973) that considers dynamic portfolio allocation problems.

4.4.1. Theory

Fama (1996) and others have discretized Merton (1973) by treating returns and the predictive scalar \( Z \) as conditionally multivariate normal. If \( Z \) is the relevant state variable in this setting, a young multi-period investor cares about

\[
E_t[\frac{1}{2} \text{RW}_{t+1}]; \quad \text{cov}_t[\frac{1}{2} \text{RW}_{t+1}, \text{Z}_{t+1}];
\]

where the subscript \( t \) for the moments refers to conditioning on \( Z_t \):

Let \( V_t \) be the conditional covariance matrix for the risky return vector \( \text{R}_{t+1} \), \( \mu_t \) be the conditional mean vector for the risky excess return vector \( \text{r}_{t+1} = \text{R}_{t+1} - \text{R}^c \); and \( \delta_t \) be the covariance vector between \( \text{R}_{t+1} \) and \( \text{Z}_{t+1} \). The young investor’s optimal portfolio, \( \mathbf{a}_Y \), is a portfolio of the following 3 assets, whose weights in the portfolio depend on the parameters of the investor’s utility function, particularly risk aversion, and on the parameters of the return-generating process. The first asset is the riskless asset with return \( R_f \):

The second asset is the mean–variance optimal (MVO) portfolio, specified as

\[
\mathbf{a}_M = \frac{1}{\mu} V^{-1} \mu,
\]

in which \( \lambda_M = 1/(i_N' V^{-1} \mu) \) and so the risky-asset weights sum to 1. The third asset is the covariance–variance optimal (CVO) portfolio, specified as

\[
\mathbf{a}_C = \frac{1}{\delta} V^{-1} \delta,
\]

in which \( \lambda_C = 1/(i_N' V^{-1} \delta) \) and such that the risky asset weights sum to 1. The CVO portfolio is also a scalar multiple of the portfolio maximally correlated with the state variable, which is Merton’s (1973) hedging portfolio.

Moreover, using Merton (1973), the young investor’s portfolio can be approximated in the following way:

\[
\mathbf{a}_Y = \mathbf{a}_M + \omega_t \mathbf{a}_C,
\]

where the first term on the left hand side, denoted MVO, is the single-period investor’s portfolio, and \( \omega_t \) is the weight of the CVO portfolio in the young investor’s optimal portfolio. See Merton (1973) for an expression for \( \omega_t \) in terms of the cross and second partial derivatives of the young investor’s indirect utility function. Eq. (10), which holds exactly for an investor rebalancing continuously, emphasizes the dependence on risk aversion, \( \gamma \), of the investor’s allocation between the MVO portfolio, the CVO portfolio, and
the riskfree asset. Hedging demand in the Merton sense is given by the second term, \( \tilde{a}_t^C \), which does not depend on \( \mu_t \).

However, in terms of understanding cross-sectional variation in expected return, it is the tilt in the investor’s risky-asset portfolio away from the MVO portfolio that is relevant. The risky-asset portfolio, \( \tilde{a}_t^Y \), held by the young investor can be written as

\[
\tilde{a}_t^Y = \tilde{a}_t^M (1 - \tilde{a}) + \tilde{a}_t^C \tilde{a},
\]

(11)

For an old investor, \( \tilde{a} = 0 \) and 100% of the risky-asset portfolio is invested in the MVO portfolio. As the investor becomes young, the tilt in the risky-asset portfolio weights away from the MVO portfolio is given by: \( \tilde{a}[\tilde{a}_t^Y - \tilde{a}_t^M] \). This tilt depends on the fraction of the young investor’s risky-asset portfolio allocated to the CVO portfolio, \( \tilde{a} \), and the difference in weights between the CVO and the MVO portfolios. Notice that, since \( \tilde{a}_t^M \) depends on \( \mu_t \), the tilt does depend on \( \mu_t \).

The first term determining the tilt is the fraction of the young investor’s risky-asset portfolio allocated to the CVO portfolio, \( \tilde{a} \). An approach for determining the sign of \( \tilde{a} \) is outlined in the appendix. Intuitively, if a portfolio of \( R_t^f \) and the CVO portfolio has a more negative covariance with \( Z_{t+1} \) than a portfolio of \( R_t^f \) and the MVO portfolio, holding conditional expected return fixed, then the young investor’s preference for negative covariance with \( Z_{t+1} \), in general, translates into a positive weight for CVO in the young investor’s risky-asset portfolio. When the conditional excess return on CVO exactly equals that for MVO, Eq. (A.2) confirms that this intuition holds exactly (see the appendix).

The second term determining the tilt is the difference in weights between the CVO and the MVO portfolios. Examining the formulas in Eq. (8) and (9) for the weights in MVO and CVO, we see that the MVO weights are linear combinations of expected excess returns across the assets, while the CVO weights are the same linear combinations, up to a scaling factor, of the covariances with the state variable \( Z_{t+1} \) across the assets. Moreover, the linear combinations depend on the conditional covariance matrix of asset returns. It follows that the pattern of weights in MVO depend on the pattern of expected excess returns across the assets, while the pattern of weights in CVO depend on the pattern of covariances with the state variable \( Z_{t+1} \) across the assets. Thus, the direction of the tilt is going to depend on the pattern of covariances with the state variable \( Z_{t+1} \) compared to the pattern of expected excess returns across the risky assets. This intuition is applied to the 3M and 3B asset sets in the subsequent subsections.

4.4.2. Merton hedging demands for the 3M and 3B sets of assets using D

Before examining the tilt in the investor’s risky-asset portfolio, it is worth seeing how closely the investor’s allocation matches that for the case of
continuous rebalancing. Recall that Eq. (10) characterizes the investor’s allocation in that case. The investor’s average allocation to each risky asset for the two sets of assets 3M and 3B are graphed in Fig. 4: the three top graphs are for the 3M set, and the three bottom graphs are for the 3B set. The averaging is performed using the unconditional distribution for $D$. Also plotted is the single-period investor’s allocation based on the first term on the left-hand side of Eq. (10), and the investor’s allocation calculated from Eq. (10) with $o_t$ set equal to 1. Thus, the implicit average hedging demand for the investor, given by $a_C o_t$ in Eq. (10), can be determined by comparing the investor’s average allocation to her average single-period allocation.

First, note that the single-period investor’s allocation from Eq. (10) is very close to the investor’s actual allocation at $t = 239$. The implication of this observation is that Merton’s characterization of the single-period allocation in continuous time is a good approximation of the single-period allocation in the discrete time setting used here. Second, the investor’s hedging demands seem to be well approximated by Merton’s formula in Eq. (10). The graphs in Fig. 4 indicate that the average allocation to each asset can be expressed as the single-period allocation plus a fraction of the asset’s CVO weight, such that, for any given age ($t$), the fraction is approximately constant across all three assets. This result is exactly how Eq. (10) characterizes the investor’s allocation, with $o_t$ being the constant. Finally, the graphs show that the young investor’s desire to hold more stock dominates the hedging demand. For both sets of assets, the young investor allocates more of her portfolio to all three assets, on average, than the old investor.

4.4.3. The tilt in the risky-asset portfolio for the 3M set of assets using $D$

Table 4 reports the weight of the CVO portfolio in the investor’s risky-asset portfolio at an age ($t$) of 1 when the investor uses log dividend yield, $D$, as a predictive variable ($Z$), and is allowed to short sell. Panel A contains results for the 3M set of assets, and shows that the value of $a$ is positive in all 5 states. In fact, $a$ is positive in all 19 states. The reason for this outcome is that CVO has a
Large firm portfolio (M3)

High book-to-market portfolio (B3)
Fig. 4. (continued)
Fig. 4. (continued)
Table 4
Explaining the tilt in the young investor’s risky-asset portfolio

Table 4 reports the weight ($\tilde{a}$) of the CVO portfolio in the investor’s risky-asset portfolio at an age ($t$) of 1 month. The CVO portfolio is the portfolio maximally correlated with $D$ while the MVO portfolio is the minimum-variance portfolio. The investor has a coefficient of relative risk aversion ($\gamma$) of 4, uses log dividend yield ($D$) as a predictive variable, and is allowed to short sell. The table also reports the weights of the risky assets in the CVO ($\tilde{a}_{C;i;t}$) and MVO ($\tilde{a}_{M;i;t}$) portfolios, their conditional covariances with $D(\delta_{i;t})$, and their conditional expected excess returns ($\mu_{i;t}$). Panel A reports results for an investor with access to three size portfolios and the riskfree asset, while Panel B reports results for an investor with access to three book-to-market portfolios and the riskfree. In Panel A, the large and small firm portfolios are denoted M3 and M1 respectively, while in Panel B, the high and low book-to-market portfolios are denoted B3 and B1 respectively. The table reports results for five values of $D$. Returns are expressed per month and in percent.

<table>
<thead>
<tr>
<th>$D$ value</th>
<th>Weight of CVO ($\tilde{a}$)</th>
<th>CVO portfolio weights ($\tilde{a}_{C;i;t}$)</th>
<th>MVO portfolio weights ($\tilde{a}_{M;i;t}$)</th>
<th>Conditional covariance with $D(\delta_{i;t})$</th>
<th>Conditional expected excess return ($\mu_{i;t}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>M3</td>
<td>M2</td>
<td>M1</td>
<td>M3</td>
</tr>
<tr>
<td>Panel A: The three size portfolios</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−1.23</td>
<td>0.30</td>
<td>0.85</td>
<td>0.11</td>
<td>0.04</td>
<td>2.58</td>
</tr>
<tr>
<td>−0.59</td>
<td>0.34</td>
<td>0.85</td>
<td>0.11</td>
<td>0.04</td>
<td>1.20</td>
</tr>
<tr>
<td>0.00</td>
<td>0.35</td>
<td>0.85</td>
<td>0.11</td>
<td>0.04</td>
<td>0.24</td>
</tr>
<tr>
<td>0.59</td>
<td>0.34</td>
<td>0.85</td>
<td>0.11</td>
<td>0.04</td>
<td>−0.53</td>
</tr>
<tr>
<td>1.23</td>
<td>0.29</td>
<td>0.85</td>
<td>0.11</td>
<td>0.04</td>
<td>−1.19</td>
</tr>
<tr>
<td>Panel B: The three book-to-market portfolios</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>−1.23</td>
<td>0.59</td>
<td>0.18</td>
<td>0.61</td>
<td>0.22</td>
<td>2.63</td>
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<tr>
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<td>0.18</td>
<td>0.61</td>
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</tr>
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<td>0.61</td>
<td>0.22</td>
<td>3.69</td>
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<tr>
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<td>0.18</td>
<td>0.61</td>
<td>0.22</td>
<td>3.83</td>
</tr>
<tr>
<td>1.23</td>
<td>0.30</td>
<td>0.18</td>
<td>0.61</td>
<td>0.22</td>
<td>3.93</td>
</tr>
</tbody>
</table>
more negative covariance with \( D \) than MVO in all states, holding conditional expected return constant.\(^4\) As Table 3 demonstrates, the young investor likes negative covariance with dividend yield, and incorporates this reference by holding a risky-asset portfolio with a positive weight in CVO. Since \( \hat{a} \) is positive, the sign of the tilts is determined by the difference in weights between the CVO and the MVO portfolios.

Table 4 also reports, for the same five states, the weights of the risky assets in CVO and MVO, the conditional covariances with \( D \), and the conditional expected excess returns for the risky assets. Again, Panel A contains results for the 3M set of assets. Since the VAR is homoskedastic, the weights of the risky assets in CVO are identical across the five states. For Table 4, I use the conditional covariance matrix for each state in the calculations, and not the covariance matrix for the residuals from the VAR reported in Table 1. The three CVO weights are all positive and increase monotonically, going from the small firm portfolio M1, to the large firm portfolio M3. Comparing the risky-asset weights in the CVO and MVO portfolios reveals that the weight of portfolio M1 in CVO is lower than its weight in MVO for all five states. Since portfolio CVO has a positive weight in the young investor’s risky-asset portfolio, it follows that the investor tilts her risky-asset portfolio, in all five states, away from M1 as she gets younger. This pattern is confirmed in Fig. 5, which plots both the allocation to the risky-asset portfolio and the composition of the risky-asset portfolio for the same five states. The graph for portfolio M1 shows that its allocation in the risky-asset portfolio is always lower at \( t = 1 \) than at \( t = 239 \). Turning to the M3 portfolio, the CVO allocation is lower than the MVO allocation for the dividend states in which \( D \) equals \(-1.23\) and \(-0.59\), but is higher for the states in which \( D \) is equal to 0, 0.59, and 1.23. Consistent with this observation, the graph for portfolio M3 in Fig. 5 shows that its allocation in the risky-asset portfolio is slightly lower at \( t = 1 \) than at \( t = 239 \) for the dividend states in which \( D \) equals \(-1.23\) and \(-0.59\), but is higher for the other three states.

Turning to the conditional covariances with \( D \) and the conditional expected excess returns for the risky assets, it is clear that the expected excess return pattern over the three risky assets is changing across the five states. Expected excess return always increases monotonically going from portfolio M3 to portfolio M1, but the pattern becomes increasingly steeper going from the low \( D \) to the high \( D \) states. At the same time, the pattern of conditional covariances with \( D \) is roughly the same for all the states, with the covariance becoming more negative going from portfolio M3 to portfolio M1. Interestingly, the covariance pattern is steeper than the expected excess return pattern for the low dividend states, but the converse holds for the high states. This observation

\(^4\)These results are not reported, and are available from the author upon request.
helps explain the changes in the tilts going from the low to the high dividend yield states.

4.4.4. The tilt in the risky-asset portfolio for the 3B set of assets using $D$

Panel B of Table 4 and Fig. 6 presents results for the 3B set of assets that are analogous to the results in Panel A of Table 4 and Fig. 5 for the 3M set. Again, Table 4 shows that the CVO portfolio has a positive weight, $\tilde{a}$, in the young investor’s risky-asset portfolio. As with the 3M set of assets, the positive values for $\tilde{a}$ are due to the young investor having a preference for negative covariance with $D$, and CVO, rather than MVO, having a more negative covariance with $D$, holding conditional expected return constant. These covariances are not reported.

Turning to the composition of portfolio CVO, all three risky-asset weights in CVO are positive, though now the B2 asset has the largest weight. Comparing the risky-asset weights in CVO and MVO reveals that the weight of B3 in CVO is always lower than its weight in MVO, and that the weight of B1 is always higher in CVO than MVO. Since CVO has a positive weight in the young investor’s risky-asset portfolio, it follows that the investor tilts her risky-asset portfolio away from B3 and toward B1 as she gets younger in all five states. This pattern is confirmed in Fig. 6. The graph for the B3 asset shows that its allocation in the risky-asset portfolio is always lower at $t = 1$ than at $t = 239$, while the graph for the B1 asset shows its allocation is always higher. With respect to asset B2, its CVO allocation is lower than its MVO allocation for the dividend states in which $D$ is $-1.23$ and $-0.59$, but is higher for those in which $D$ is $0$, $0.59$ and $1.23$. Consistent with this pattern, the graph for portfolio B2 in Fig. 6 shows that its allocation in the risky-asset portfolio is slightly lower at $t = 1$ than at $t = 239$ for the dividend states in which $D$ is equal to $-1.23$ and $-0.59$, but is higher for the other three states.

For all five states, expected excess return increases monotonically, going from portfolio B1 to B3, while the conditional covariance with $D$ becomes monotonically more negative. However, the pattern of conditional covariances is much less steep than the pattern of expected excess returns in all five states.

Fig. 5. State-by-state portfolio allocations by an investor who uses dividend yield ($D$) and has access to the riskless asset and 3 size portfolios of NYSE stocks: M3, M2, and M1. The large firm portfolio is denoted M3 and the small firm portfolio is denoted M1. The investor is allowed to short sell and has a relative risk aversion coefficient ($\gamma$) of 4. The plotted allocations are for five values of $D$. The investor’s allocation decision can be broken into two parts: the allocation to the risky-asset portfolio; and the composition of the risky-asset portfolio. The first graph reports the investor’s allocation to the risky-asset portfolio, while the last three graphs plot the weights of the three size portfolios in the risky-asset portfolio for each $D$ state. Each graph shows the investor’s allocation as a function of the investor’s age $t$, where $t = 1$ is her first month and $t = 239$ is her last.
Risky-asset portfolio: Three size portfolios

Age in months (t)

Weight in risky-asset portfolio

D = -1.23
D = -0.59
D = 0.00
D = 0.59
D = 1.23

Large firm portfolio (M3)

Age in months (t)

Weight in M3/Weight in risky portfolio

D = -1.23
D = -0.59
D = 0.00
D = 0.59
D = 1.23
Fig. 5. (continued)
So portfolio B3 has a more negative covariance with $D$ than portfolio B1, but the ratio is much smaller than the ratio of the expected excess returns for B3 and B1. This observation helps explain why the young investor likes negative covariance with $D$, but tilts away from the asset having the largest negative covariance with dividend yield, portfolio B3. The reason is that the old investor overweights this asset in her portfolio because of its high expected excess return relative to the other two assets.

4.5. Impact of hedging demands on abnormal returns

The previous subsections describe how the hedging demands induced by dividend yield are large and cause investors to tilt their risky-asset portfolios away from high book-to-market stocks and small stocks. Thus, hedging demand may help explain the size and book-to-market effects in the literature. However, these effects are typically measured using abnormal return, which is obtained from a market model regression of excess return on the excess return of a market proxy. The small firm effect means that this abnormal return is larger for a small firm portfolio rather than a large firm portfolio, while the book-to-market effect means that this abnormal return is larger for a high rather than a low book-to-market portfolio. On the other hand, hedging demands have been measured, above, in terms of their impact on portfolio holdings, which is the traditional practice in the portfolio choice literature.

Thus, it would be useful to assess the impact of hedging demand on abnormal returns rather than on portfolio holdings. One way to perform this assessment is to run conditional regressions of asset excess returns on the excess return from the investor’s optimal portfolio, and examine the magnitude and cross-sectional variation in the intercepts. If the investor chooses a conditionally minimum-variance portfolio, then the intercepts will be zero. If the investor chooses a different portfolio, the intercepts will be non-zero. These intercepts can be interpreted as the CAPM abnormal returns for an economy whose representative agent is the investor. The reason is as follows. If the young investor is the representative agent, then the intercepts are those that would be obtained from regressions of asset excess returns on the market excess return. However, I do not want to lean too heavily on this interpretation, since the young investor’s optimal risky asset holdings are often negative. Moreover, while the partial equilibrium analysis in this paper invites speculation about the role of hedging demands for the cross-section of expected returns, definitive conclusions require a general equilibrium model. Such a model is outside the scope of the current paper.

By producing an optimal portfolio that lies inside the conditional mean–variance frontier, as documented in Section 4.3, the hedging demand induced by dividend yield can be expected to generate non-zero intercepts. However, as Kandel and Stambaugh (1995) demonstrate, the closeness of an inefficient
portfolio to the minimum-variance frontier says nothing about the relation between expected return and the beta coefficient with respect to that portfolio, unless the observations are weighted appropriately. Consequently, both the size and direction of any cross-sectional variation in these intercepts are open questions. Considerable cross-sectional variation in the intercepts would indicate that hedging demands can materially affect abnormal return calculations. Further, if the variation is in the direction implied by the size or book-to-market effects, the implication is that hedging demands can help explain the particular effect.

Abnormal returns are reported in the last three columns of Table 3. Panel A reports abnormal returns for each size portfolio in the 3M asset set, and Panel B reports abnormal returns for each book-to-market portfolio in the 3B asset set. The first row of each panel reports abnormal returns when returns are i.i.d., the $U$ investor. The remaining 2 pairs of rows report average abnormal returns when returns are predictable, using either dividend yield, $D$, or term spread, $S$. For each predictive variable, abnormal returns are reported using the investor’s allocation in her first ($t = 1$) and last ($t = 239$) months of life.

The abnormal returns obtained using the $U$ investor’s optimal portfolio are virtually zero, for either set of assets, consistent with her optimal portfolio being minimum-variance in each case. The same result holds true in the last month ($t = 239$) of the investor’s life using either $D$ or $S$. While the approximation generates log-normally distributed asset returns in the limit, the investor appears to behave like a mean–variance optimizer in the absence of any hedging demand.

However, the $D$ row with $t = 1$ in each panel shows that the hedging demand induced by knowing dividend yield, $D$, causes abnormal returns to deviate from zero. Across the three size portfolios in Panel A, abnormal returns range from $-0.029\%$, for the large firm portfolio, up to $-0.010\%$, for the small firm portfolio. This cross-sectional dispersion in abnormal return of $0.018\%$ is approximately one-seventh of the dispersion of $0.131\%$ reported in Table 1, using the VW portfolio as the market proxy. Further, abnormal return is decreasing in firm size in each case. Turning to the book-to-market portfolios

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*Fig. 6. State-by-state portfolio allocations by an investor who uses dividend yield ($D$) and has access to the riskless asset and 3 book-to-market portfolios of NYSE stocks: B3, B2, and B1. The high book-to-market portfolio is denoted B3 and the low book-to-market portfolio is denoted B1. The investor is allowed to short sell and has a relative risk aversion coefficient ($\gamma$) of 4. The plotted allocations are for five values of $D$. The investor’s allocation decision can be broken into two parts: the allocation to the risky-asset portfolio; and the composition of the risky-asset portfolio. The first graph reports the investor’s allocation to the risky-asset portfolio, while the last three graphs plot the weights of the three book-to-market portfolios in the risky-asset portfolio for each $D$ state. Each graph shows the investor’s allocation as a function of the investor’s age $t$, where $t = 1$ is her first month and $t = 239$ is her last.*
Risky-asset portfolio: Three book-to-market portfolios

High book-to-market portfolio (B3)
Fig. 6. (continued)
in Panel B, abnormal return is higher for the high than for the low book-to-market portfolio by 0.055%, though again both abnormal returns are negative. Table 1 shows that abnormal return relative to VM is also increasing in book-to-market ratio, although the dispersion of 0.351% is about 7 times larger than that obtained for the optimal $t = 1$ portfolio using $D$. To summarize, the patterns of abnormal returns for size and book-to-market portfolios obtained relative to the investor’s optimal $t = 1$ portfolio are consistent with those obtained relative to the market portfolio, but are not as extreme. Thus, hedging demands induced by information about $D$ may be a partial explanation for the small firm and book-to-market effects.

Turning to the hedging demands induced by term spread, $S$, the abnormal returns are close to zero for both sets of assets, 3M and 3B, and exhibit almost no cross-sectional variation. Thus, the hedging demand induced by $S$ has a negligible effect on asset abnormal returns, as well as portfolio holdings and portfolio characteristics.

Finally, related work by Roll and Ross (1994) examines how the closeness of a portfolio to the minimum-variance frontier affects the cross-sectional relation between expected asset return and asset beta with respect to the portfolio. Roll and Ross hold the set of assets fixed, and are concerned with identifying the set of portfolios that imply a given OLS slope coefficient between expected return and beta with respect to the portfolio. However, they do not consider the magnitude or pattern of deviations from the OLS linear relation between beta coefficient and expected return. Here, the focus is on the magnitude and pattern of expected return deviations from a linear relation with the beta coefficient. Note that by running market model regressions to obtain abnormal returns, I am implicitly restricting the linear relation between expected return and beta coefficient to have a slop coefficient equal to the excess return on the portfolio, rather than equal to the OLS slope coefficient.

4.6. Volatility of portfolio allocations across states

Table 5 reports the standard deviations of the investor’s portfolio allocations when using dividend yield, $D$, or term spread, $S$. A number of results are worth noting. The weights of the risky assets in the investor’s portfolio are quite volatile, with standard deviations as high as 0.85. The volatility of the total allocation to stocks is typically much lower, though still substantial. For example, when the 3B set of assets is available, and $D$ has predictive ability that is being exploited, the volatilities of the allocations to the individual assets range from 0.85 for asset B2 down to 0.27 for asset B1, while the volatility of the total allocation to stocks is only 0.17. Since the total allocation to stocks equals the sum of the individual allocations, the low volatility for the total allocation indicates that the individual allocations must move in opposite directions going across the states.
Table 5
Volatility of portfolio allocations

Table 5 reports standard deviations for the weights of the risky assets and the risky-asset portfolio in the investor’s optimal portfolio. “Risky” refers to the investor’s risky-asset portfolio. The investor rebalances monthly and has a relative risk aversion coefficient ($\gamma$) of 4, intermediate consumption, and a 20-year horizon. Standard deviations are reported for the investor’s portfolio at ages ($t$) of 1 month and 239 months, both with and without short selling. The table reports results for an investor with access to each of three sets of assets in addition to the riskfree: the value-weighted market portfolio (VM); three size portfolios (3M); and three book-to-market portfolios (3B). The large and small firm portfolios are denoted M3 and M1 respectively, while the high and low book-to-market portfolios are denoted B3 and B1 respectively. For each set of assets, allocations using log dividend yield ($D$) and term spread ($S$) as the predictive variable ($Z$) are considered, with standard deviations being calculated using the unconditional distribution for the predictive variable. Return distributions are obtained using quadrature approximations that are calibrated to VARs estimated for U.S. data from July 1927 to November 1996.

<table>
<thead>
<tr>
<th>Predictive variable ($Z$)</th>
<th>Investor age ($t$)</th>
<th>Value-weighted market (VM)</th>
<th>3M asset set</th>
<th>3B asset set</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Risky</td>
<td>M3</td>
</tr>
<tr>
<td><strong>Panel A: No short selling</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td>239</td>
<td>0.25</td>
<td>0.19</td>
<td>0.14</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.25</td>
<td>0.20</td>
<td>0.21</td>
</tr>
<tr>
<td>$S$</td>
<td>239</td>
<td>0.30</td>
<td>0.25</td>
<td>0.09</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.30</td>
<td>0.25</td>
<td>0.10</td>
</tr>
<tr>
<td><strong>Panel B: Short selling</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D$</td>
<td>239</td>
<td>0.25</td>
<td>0.09</td>
<td>0.63</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.26</td>
<td>0.13</td>
<td>0.61</td>
</tr>
<tr>
<td>$S$</td>
<td>239</td>
<td>0.33</td>
<td>0.22</td>
<td>0.23</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>0.33</td>
<td>0.22</td>
<td>0.22</td>
</tr>
</tbody>
</table>
Intuition suggests that volatility is lower when short selling is prohibited. While this prediction always holds for individual asset allocations, it is not always true of the total allocation when more than one risky asset is available, such as in the 3M and 3B asset sets. Although using S as a predictive variable generates small hedging demands, Table 5 shows that the resulting portfolio allocations exhibit substantial volatility. In fact, the total allocation to stocks is almost always more volatile when investors have access to information about S rather than D. Thus, S allows the investor to time the market (so the old investor’s total allocation to stocks is volatile), even though the induced hedging demand is small (so the young investor’s allocation is similar to the old investor’s allocation). This result explains why the utility cost of not using S is found to be quite high in Table 6, which is discussed in the next subsection. Finally, there appears to be only small volatility differences between allocations made by young and old investors.

4.7. Utility cost calculations

This section calculates the utility costs of either using a value-weighted equity index or ignoring predictability. Because the investor in this paper cannot be regarded as the representative agent, these utility cost calculations should not be regarded as welfare cost calculations. Rather, they represent the costs to an individual investor with CRRA preferences of ignoring the various aspects of the return generating process considered below.

Table 6 contains the utility cost calculations. Panel A reports the cost of using the value-weighted index rather than one of the asset sets formed using equity characteristics. The left-hand side presents utility costs assuming no short selling, while the right-hand side shows how the cost calculations change when short selling is allowed. Each row refers to the set of predictive variables available to the investor, with U referring to the i.i.d. return case. Each column heading refers to the particular set of assets that the investor is comparing to VM.

A number of results in Panel A are worth mentioning. First, being able to allocate across book-to-market portfolios is much more valuable than being able to allocate across size portfolios. With dividend yield available and no short selling, the investor is prepared to give up 22.1% of her wealth to get the 3 book-to-market portfolios, but only 9.2% to get the 3 size portfolios. Second, the benefits from using book-to-market portfolios are much larger when short selling is allowed. If using dividend yield information, access to the 3 book-to-

\[ \text{Row } U \text{ can be obtained using the unconditional distribution for } R \text{ implied by the conditional distribution obtained for any of the sets of predictive variables. However, in unreported results, the utility costs are similar for } U, \text{ irrespective of the set of predictive variables used, which is further evidence that the quadrature approximation is mimicking essential features of the data.} \]
Table 6
Utility cost calculations

Table 6 reports the fraction of wealth (in percent) that an investor would be prepared to give up to be given access to a better information set or a better set of assets. The investor rebalances monthly and has a relative risk aversion coefficient (γ) of 4, intermediate consumption, and a 20-year horizon. The left-hand side of each panel presents utility costs assuming no short selling, while the right-hand side shows how the cost calculations change when short selling is allowed. Panel A reports the cost of having access to the value-weighted market portfolio (VM) and the riskfree rather than one of the other asset sets and the riskfree. The three asset sets are three size portfolios (the 3M set), three book-to-market portfolios (the 3B set), and six portfolios formed on the basis of size and book-to-market (the 3B2M set). Each row refers to the set of predictive variables available to the investor, with U referring to the i.i.d. return case. The three sets of predictive variables (Z) are the log dividend yield (D), term spread (S) and these two variables together (F). Each column heading refers to the particular set of assets that the investor is comparing to VM. Panel B reports the utility cost associated with ignoring return predictability. Each row refers to the set of predictive variables whose predictive ability is being ignored. Each column refers to the set of assets available to the investor. Return distributions are obtained using quadrature approximations that are calibrated to VARs estimated for U.S. data from July 1927 to November 1996.

<table>
<thead>
<tr>
<th>Predictive variables (Z)</th>
<th>No short selling</th>
<th>Short selling</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3M</td>
<td>3B</td>
</tr>
<tr>
<td>U</td>
<td>2.3</td>
<td>12.5</td>
</tr>
<tr>
<td>D</td>
<td>9.2</td>
<td>22.1</td>
</tr>
<tr>
<td>S</td>
<td>4.1</td>
<td>11.1</td>
</tr>
<tr>
<td>F = [D, S]</td>
<td>10.7</td>
<td>20.8</td>
</tr>
</tbody>
</table>

Panel A: Cost of having access to the value-weighted market portfolio (VM) in addition to the riskfree

<table>
<thead>
<tr>
<th>VM</th>
<th>3M</th>
<th>3B</th>
<th>3B2M</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>7.0</td>
<td>12.2</td>
<td>13.1</td>
</tr>
<tr>
<td>S</td>
<td>6.8</td>
<td>9.1</td>
<td>6.1</td>
</tr>
<tr>
<td>F  = [D, S]</td>
<td>11.3</td>
<td>17.9</td>
<td>16.0</td>
</tr>
</tbody>
</table>

Panel B: Cost of using the unconditional distribution (U)

<table>
<thead>
<tr>
<th></th>
<th>VM</th>
<th>3M</th>
<th>3B</th>
<th>3B2M</th>
</tr>
</thead>
<tbody>
<tr>
<td>D</td>
<td>7.1</td>
<td>13.6</td>
<td>14.4</td>
<td>14.6</td>
</tr>
<tr>
<td>S</td>
<td>7.0</td>
<td>9.5</td>
<td>9.9</td>
<td>11.4</td>
</tr>
<tr>
<td>F = [D, S]</td>
<td>13.0</td>
<td>20.1</td>
<td>21.4</td>
<td></td>
</tr>
</tbody>
</table>
market portfolios is worth 22.1% in the absence of short selling, but is worth 29.3% when short selling is allowed. Finally, being able to use both book-to-market and size portfolios is only valuable relative to using book-to-market portfolios, if short selling is allowed. For example, with no short selling, the investor is prepared to give up the same fraction of her wealth, approximately 22%, to get access to the three book-to-market portfolios (the 3B set) or the six book-to-market and size-based portfolios (the 3B2M set). In contrast, when short selling is allowed, the investor is willing to give up a much larger fraction of her wealth (36.6%), to use the six book-to-market and size-based portfolios set than to use the three book-to-market portfolios (29.3%).

Panel B reports the utility cost associated with ignoring return predictability. Each row refers to the set of predictive variables whose predictive ability is being ignored. Each column refers to the set of assets available to the investor. The left-hand side presents costs with no short selling, while the right-hand side reports costs when short selling is allowed. The cost of ignoring dividend yield information is typically higher than the cost of ignoring term spread information. For example, if using the 3 size portfolios, and short selling is not allowed, the investor is prepared to give up 9.1% of her wealth to exploit the term spread variable, but an even larger 11.9% to exploit dividend yield information. Even so, as the example illustrates, the cost of ignoring term spread is still substantial. Thus, while term spread fails to generate large hedging demands, it allows the investor to time the market in a way that is quite valuable.

Moreover, it appears that the term spread variable has some incremental benefit over and above the dividend yield variable. For example, the investor who uses the 3 size portfolios and is not allowed to short sell, is prepared to give up 17.9% of her wealth to simultaneously use dividend yield and term spread information. This 17.9% can be compared to the 12.2% that this investor would give up to use the dividend yield variable alone. This result is consistent with the view that dividend yield and term spread information predict distinct components of returns (see Fama and French, 1989). Finally, the cost of ignoring predictability is always higher when short selling is allowed.

4.8. Consumption and the set of assets available

Fig. 7 presents average consumption when the investor (γ = 4) is allowed to sell short and has access to either the value-weighted market (VM), the three size portfolios (3M), or the 3 book-to-market portfolios (3B) in addition to the riskless asset. The consumption number plotted is the percentage of wealth consumed multiplied by the number of periods until T plus 1, referred to as scaled consumption. The first graph, U, shows scaled consumption when the investor uses the i.i.d. unconditional distribution, while the last two graphs
show average scaled consumption when the investor uses information about dividend yield, $D$, or term spread, $S$. Each graph shows the investor’s scaled consumption as a function of the investor’s age $t$, where $t = 1$ is her first month and $t = T = 240$ is her last.

Fig. 7 indicates that average scaled consumption increases as the investor gets younger irrespective of the set of assets or the predictive variable being used by the investor. At $T$, the consumption fraction is 1. As the investor gets younger, average scaled consumption becomes increasingly larger than 1. In other words, Fig. 7 shows that average scaled consumption is always equal to 1 at age $T$ and is always a negative function of age. For any of the three distributional assumptions considered, the slope is always steepest for the 3B set of assets and the flattest for the VM asset. This result implies that, for any given investor age $t$, average scaled consumption, and also average consumption, is largest for the 3B set and smallest for the VM asset.

Since $\gamma$ is greater than 1, the income effect of an improved opportunity set dominates the substitution effect for this investor. More precisely, it can be shown (see, for example, Ingersoll, 1987) that the investor’s optimal consumption fraction, $\kappa_t$, and value function constant, $a(Z_t, t)$, are related in the following way:

$$\kappa_t = a(Z_t, t)^{-1/\gamma}. \quad (12)$$

When $\gamma > 1$, the investor’s Bellman equation specified in Eq. (4), shows that utility is decreasing in $a(Z_t, t)$. Thus, for a given investor age $t$, a better opportunity set translates into a smaller $a(Z_t, t)$, which implies, using Eq. (12), a larger $\kappa_t$. For this reason, the steeper slopes for average scaled consumption, going from VM to 3M to 3B, are consistent with the finding in Table 6 discussed above. The investor with access to VM would give up a positive fraction of her wealth to have access to the 3M asset set, and an even larger positive fraction to have access to the 3B asset set.

The finding of negative slopes as a function of age in Fig. 7 is also not surprising given the dominant income effect. With $\beta = 1/R^f$ and $R^f > 1$, it can be shown that, whenever $R^f$ is the only asset available, scaled consumption by
Investor uses the unconditional distribution (U)

Investor uses dividend yield (D)
this investor is a negative function of age.\textsuperscript{6} The availability of risky assets improves the opportunity set relative to the riskless asset alone, and so the dominant income effect means that their availability makes the scaled consumption in Fig. 7 an even more negatively sloped function of age.

Finally, the slope for any given asset set is steepest when the investor uses dividend yield as the predictor. Again, this result is consistent with the finding in Table 6 that the utility cost of ignoring predictability using dividend yield is positive and larger than that of ignoring predictability using term spread.

4.9. Robustness checks and sensitivity analysis

This section performs sensitivity analysis and robustness checks. I first vary the parameters of the return generating process that determine the nature and extent of the return predictability available to the investor. Then I vary other parameters like risk aversion, rate of time preference, and rebalancing period.

\textsuperscript{6}The proof uses Eq. (12), the first order condition for $k_t$ from Eq. (4), and some algebra. Details are available from the author.
4.9.1. Predictability parameters

It would be useful to know how the predictability parameters affect the magnitude of the hedging demand. Table 7 explores the relative importance of the persistence of the predictive variable, $b_Z$, specified in Eq. (6), and the vector of conditional correlations between log-returns and the predictive variables, denoted $\rho[r, Z]$. With the exception of the first line, each line in the table reports allocation results for the young investor, at time $t = 1$, given a different set of assumptions about the generating processes (GP) for returns and the predictive variable. All generating processes for all asset sets match the unconditional log-return covariance matrix to that for the data and the slopes of the predictive regressions, specified as $b_r$ in Eq. (5), to those for the term spread variable, $S$, in the data. All predictive variables are calibrated to be zero mean and unit variance. Thus, the conditional single-period return distribution is the same for all 6 generating processes, and the portfolio allocations for the single-period investor ($t = T - 1$) in the first row of Table 7 apply to all 6 generating processes.\footnote{While the return generating processes are calibrated to have the same single-period conditional distribution, it is worth checking whether the time $T - 1$ allocations are the same across the discretized processes. In unreported results, the $T - 1$ allocations for discretizations I through V are virtually identical to those in the reported in the first row of Table 7.} Across generating processes I through III, return correlations with the state variable are fixed at the correlations with term spread, $S$, in the data, which are small, while the persistence parameter for the predictive variable, $b_Z$, is allowed to vary. Across generating processes IV to VI, return correlations with the state variable are fixed at those for log dividend yield, $D$, in the data, which are large negative values, and, again, the persistence parameter for the predictive variable, $b_Z$, is allowed to vary. As above, the investor’s allocation decision can be broken into two parts: the allocation to the risky-asset portfolio, and the composition of the risky-asset portfolio. Short selling is allowed.

Theoretically, if the correlations between returns and the state variable are exactly zero, then the hedging demand is also zero. Generating processes I through III allow an examination of the effect of varying the persistence of the state variable when return correlations with the state variable are small, but still non-zero. The average allocation to the risky-asset portfolio by the young investor is similar under generating processes I through III to that of the myopic investor reported in the first row. Even when the persistence is matched to that of dividend yield in the data, as in generating process III, the average allocation to the risky-asset portfolio differs from the myopic allocation by less than 1.5% across all three asset sets. However, for the asset sets with multiple risky assets, the sets 3M and 3B, the composition to the risky-asset portfolio is affected by increasing the persistence parameter even though return correlations with the state variable are small. In contrast, when the correlations
between returns and the state variable are large, as in processes IV–VI, varying the predictive parameter from 0 to 0.85 to 0.96 has a large impact on both the average allocation to the risky-asset portfolio and its composition. For example, when the predictive parameter increases from 0.85 to 0.96 going from process V to process VI, the average allocation to the risky-asset portfolio increases by at least 6%, irrespective of the asset set used.

The correlation vector clearly has a large main effect on both the average allocation to the risky-asset portfolio and its composition. Holding the persistence parameter fixed by comparing processes I with IV, II with V, or III with VI, Table 7 shows large increases in the average allocation to the risky-asset portfolio and greatly increased tilts in composition, relative to the myopic case, when the correlation vector goes from that for $S$ to that for $D$. Moreover, when the persistence parameter is larger, comparing, for example, III vs. VI rather than I vs. IV, switching from the small $S$ correlation to the large $D$ correlation vector has a larger effect on portfolio allocation. For the 3B set, going from III to VI causes the average risky-asset allocation to increase from 34.8% to 48.1%, while going from I to IV only causes an increase in this allocation of 1.7%, from 34.6% to 36.3%. The results are similar for the other two asset sets.

Thus, both larger persistence and larger correlation with returns are driving the larger hedging demands associated with using predictive variable $D$ instead of $S$. At the same time, these two predictability parameters interact in such a way that a larger magnitude for one means that the portfolio choice made by a young investor is more sensitive to varying the magnitude of the other.

4.9.2. Other parameters

The first parameter to be varied is the risk aversion coefficient, $\gamma$, which is increased from 4 to 10. The results, which are not reported here, are qualitatively similar to those described above for $\gamma = 4$. While the investor holds less equity, intertemporal hedging demands induced by dividend yield still cause the investor to tilt her risky-asset portfolio away from the small firm portfolio, M1, and away from the high book-to-market portfolio, B3. Moreover, abnormal returns relative to the investor’s optimal early life portfolio using $D$ exhibit greater cross-sectional dispersion than when $\gamma$ is 4. These results are consistent with the following intuition from Merton (1973). The young investor, irrespective of $\gamma$, holds combinations of $R^f$, MVO and CVO, and an increase in $\gamma$ reduces her holding of MVO, while increasing her holding of CVO.

For all the calibrations reported above, the parameter for the rate of time preference, $\beta$, is set equal to the reciprocal of the riskfree rate. There is a concern that this value for $\beta$ is too high, since the expected return on the investor’s optimal portfolio is likely to be much higher than $R^f$. To assess the sensitivity of the results to the choice of $\beta$, the investor’s problem was also
Table 7
The young investor’s portfolio allocations as a function of the predictability parameters

Table 7 reports allocation results for an investor with relative risk aversion (γ) of 4, intermediate consumption, a 20-year horizon (T = 240) and monthly rebalancing. The investor has access to each of three sets of assets plus the riskfree: the value-weighted market portfolio (VM); three size portfolios (3M); and three book-to-market portfolios (3B). For the 3M set, the large and small firm portfolios are denoted M3 and M1 respectively, while for the 3B set, the high and low book-to-market portfolios are denoted B3 and B1 respectively. The investor always uses a predictive variable (Z) with zero mean and unit variance. The first row of the table shows the portfolio allocations for a single-period investor (t = T – 1). The remaining rows report allocation results for a young investor (time t = 1) facing six different generating processes for returns and the predictive variable, all with the same conditional single-period return distribution as the one faced by the single-period investor. In particular, all generating processes for all asset sets match the unconditional log-return covariance matrix to that for the data and the slopes of the predictive regressions (β) to those for the term spread variable (S) in the data. So for all six generating processes, the β’s are small and the regression R² are less than 1%. Across generating processes I–III, return correlations with the state variable (ρ(r, Z)) are fixed at those for term spread (S) in the data (which are close to zero) and the persistence parameter for the predictive variable (βZ) is allowed to vary. Across generating processes IV–VI, return correlations with the state variable (ρ(r, Z)) are fixed at those for log dividend yield (D) in the data (which are large negative values) and again the persistence parameter for the predictive variable (βZ) is allowed to vary. The investor’s allocation decision can be broken into two parts: the allocation to the risky-asset portfolio; and the composition of the risky-asset portfolio. “Risky” denotes the average allocation to the risky-asset portfolio. For the asset sets with more than one risky asset, “Composition” refers to the average allocation to each portfolio scaled by the average allocation to the risky-asset portfolio. Short selling is allowed.

<table>
<thead>
<tr>
<th>Generating process</th>
<th>Persistence of Z (βZ)</th>
<th>Correlation of Z with r(ρ(r, Z))</th>
<th>VM</th>
<th>3M</th>
<th>3B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Risky</td>
<td>Risky</td>
<td>Composition</td>
</tr>
<tr>
<td>All</td>
<td>Myopic allocation</td>
<td>0.536</td>
<td>0.452</td>
<td>0.225</td>
<td>0.480</td>
</tr>
<tr>
<td>I</td>
<td>0</td>
<td>Close to zero</td>
<td>0.536</td>
<td>0.454</td>
<td>0.241</td>
</tr>
<tr>
<td>II</td>
<td>0.85</td>
<td>Close to zero</td>
<td>0.536</td>
<td>0.459</td>
<td>0.280</td>
</tr>
<tr>
<td>III</td>
<td>0.96</td>
<td>Close to zero</td>
<td>0.537</td>
<td>0.464</td>
<td>0.319</td>
</tr>
<tr>
<td>IV</td>
<td>0</td>
<td>Large negative</td>
<td>0.566</td>
<td>0.488</td>
<td>0.272</td>
</tr>
<tr>
<td>V</td>
<td>0.85</td>
<td>Large negative</td>
<td>0.653</td>
<td>0.592</td>
<td>0.374</td>
</tr>
<tr>
<td>VI</td>
<td>0.96</td>
<td>Large negative</td>
<td>0.762</td>
<td>0.722</td>
<td>0.458</td>
</tr>
</tbody>
</table>
solved setting the rate of time preference equal to the reciprocal of the average return on the value-weighted market proxy, VM, in the data. Since the average risky-asset holding is typically less than 1, this strategy should produce a $\beta$ value that, together with $\frac{1}{R^f}$, brackets the range of likely $\beta$ values investors use. The portfolio allocation results using this low $\beta$, which are not reported, are virtually indistinguishable from those reported above.

Another interesting question is how varying the rebalancing period affects portfolio allocations for a 20-year investor. One potential complication is whether consumption frequency changes as the rebalancing frequency changes. To finesse this problem, I consider, in unreported calculations, the portfolio allocation problem for a 20-year investor with utility over terminal wealth only. Due to computational limitations, rebalancing periods up to only three months are considered. For each set of assets, reducing rebalancing frequency from monthly to quarterly, when returns are log-normal and i.i.d., has virtually no impact on the investor’s average allocation to risky assets. Barberis (2000) finds the same insensitivity in the case of one risky asset. Turning to the results when returns are predictable, reducing rebalancing frequency from monthly to quarterly causes both the old ($t = T$) and the young ($t = 1$) investor, using dividend yield, to increase her average allocation to risky assets slightly, irrespective of the asset set.

5. Conclusion

This paper examines portfolio allocation across stock portfolios formed on the basis of equity characteristics like firm size and book-to-market ratio. The investor lives for many periods, and return predictability is calibrated to U.S. data. Comparing the investor’s allocation in her last investment period to her allocation early in life, return predictability with dividend yield causes the investor to tilt her risky-asset portfolio away from high book-to-market stocks and away from small stocks early in life. Abnormal returns relative to the investor’s optimal early-life portfolio are also calculated. These abnormal returns are found to exhibit the same cross-sectional patterns as abnormal returns calculated relative to the market portfolio: higher abnormal returns for small than for large firms, and higher abnormal returns for high, as opposed to low, book-to-market firms. Thus, hedging demand may be a partial explanation for the high expected returns documented empirically for small firms and high book-to-market firms. However, even with this hedging demand, the investor wants to short-sell the low book-to-market portfolio to hold the high book-to-market portfolio.

The paper suggests that investor hedging demands may help explain cross-sectional variation in expected returns. However, the analysis is a partial-equilibrium one, and market-clearing conditions are not being imposed. In
fact, the results for the book-to-market portfolios are unable to explain why U.S. investors hold positive amounts of low book-to-market stocks. For this reason, care must be taken not to overstate the implications of the results for the cross-section of expected returns. At the same time, the hedging demands induced by dividend yield as a predictor cause the investor’s risky-asset portfolios to tilt in directions that invite speculation about the ability of hedging demands to at least partially explain the size and book-to-market effects. More analysis is needed to better understand how hedging demands induced by return predictability affect equilibrium asset prices.

A number of extensions to this study are possible. In particular, the current study treats sample moments and regression coefficients as population moments and population coefficients. It would be interesting to assess the effects of parameter uncertainty (see, for example, Kandel and Stambaugh, 1996; Barberis, 2000). Another direction is to incorporate transaction costs into the analysis, as in Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000). Finally, given evidence in Jagannathan and Wang (1996) and Ferson and Harvey (1999), that conditional risk loadings are time-varying, it would be instructive to allow the conditional covariance matrix for the assets and state variables to be state-dependent.

Appendix. Determining the weight of the CVO portfolio in the young investor’s risky-asset portfolio

One way to ascertain $\hat{a}$ is to compare the covariance of the young investor’s portfolio with $Z_{t+1}$ to that for two portfolios, both with the same conditional expected excess return as the young investor’s portfolio $\mu^Y_t$. One is a portfolio of $R^f$ and the MVO (denoted by MVO,Y), and the other is a portfolio of $R^f$ and the CVO (denoted by CVO,Y). The young investor’s portfolio can be written as a portfolio of MVO,Y and CVO,Y. The weight of CVO,Y in this portfolio, denoted by $a$, can be obtained by exploiting the fact that covariance of portfolio $j$ with $Z_{t+1}$ ($\delta^j_{t+1}$) is a linear operator,

$$\delta^Y_t = \delta^M_{t} Y_t (1 - a) + \delta^C_{t} Y_t a,$$

where $\delta^M_{t} Y_t$ and $\delta^C_{t} Y_t$ are the conditional covariances of $Z_{t+1}$ with portfolios MVO,Y and CVO,Y, respectively. The weight of CVO,Y in this portfolio $a$ can then be used to obtain $\hat{a}$ by exploiting the following result:

$$\hat{a} = \frac{a(\mu^M_t / \mu^C_t)}{1 - a + a(\mu^M_t / \mu^C_t)},$$

where $\mu^M_t = \mu^M_t Z_t^M$ is the conditional excess return on MVO, and $\mu^C_t = \mu^C_t Z_t^C$ is the conditional excess return on CVO.
The proof of Eq. (A.2) is as follows. Recall that \( \mu_Y \) is the expected excess return on the young investor's optimal portfolio in state \( Z_t \). It is possible to express the MVO,Y portfolio as a multiple of the MVO portfolio:

\[
\alpha_t^{M,Y} = \hat{\alpha}_t^M \frac{\mu_Y}{\mu_t^M}.
\]  
(A.3)

Similarly, it is possible to express the CVO,Y portfolio as a multiple of the CVO portfolio:

\[
\alpha_t^{C,Y} = \hat{\alpha}_t^C \frac{\mu_Y}{\mu_t^C}.
\]  
(A.4)

Thus, the young investor's optimal portfolio can be rewritten as

\[
\alpha_t^Y = \hat{\alpha}_t^M \frac{\mu_Y}{\mu_t^M} (1 - a) + \hat{\alpha}_t^C \frac{\mu_Y}{\mu_t^C} a.
\]  
(A.5)

which means that the young investor has \([(\mu_Y / \mu_t^M)(1 - a) + (\mu_Y / \mu_t^C)a]\) invested in her risky asset portfolio. Thus, we can obtain the following expression for the young investor's risky-asset portfolio:

\[
\hat{\alpha}_t^Y = \hat{\alpha}_t^M \frac{(1 - a) \mu_Y / \mu_t^M + a \mu_Y / \mu_t^C}{(1 - a) \mu_t^M + a \mu_t^C} + \hat{\alpha}_t^C \frac{a \mu_Y / \mu_t^C}{(1 - a) \mu_t^M + a \mu_t^C}.
\]  
(A.6)

Using both the definition of \( \hat{\alpha} \) in Eq. (11) and some algebra gives Eq. (A.2).

References


