In this note, I discuss some handy facts about steady states and local approximations to them in the context of the Solow (1956) growth model.

**Setup**

Consider the Solow growth model in its most stripped down form. This consists of a resource constraint, a production function, a law of motion for capital accumulation, and the behavioral assumption that a constant fraction $0 < s < 1$ of output is saved and invested.

In standard notation

\[
\begin{align*}
    c_t + i_t & = y_t = f(k_t) \\
    k_{t+1} & = (1 - \delta)k_t + i_t \\
    c_t & = (1 - s)y_t
\end{align*}
\]

for some constant depreciation rate $0 < \delta < 1$ and given initial condition $k_0$.

The following assumptions are made about the production function: It is strictly increasing, strictly concave, needs some capital to produce anything, and satisfies the Inada conditions

\[
\begin{align*}
    f(k) & \geq 0, \quad f(0) = 0 \\
    f'(k) & > 0, \quad f''(k) < 0 \\
    f'(0) & = \infty, \quad f'(\infty) = 0
\end{align*}
\]

**Scalar difference equation**

Putting this together gives a first order non-linear difference equation

\[
k_{t+1} = H(k_t) \equiv sf(k_t) + (1 - \delta)k_t
\]

Given the assumptions about $f$, the function $H$ is also strictly increasing, strictly concave, etc

\[
\begin{align*}
    H(0) & = sf(0) + (1 - \delta)0 = 0 \\
    H'(k) & = sf'(k) + (1 - \delta) > 0 \\
    H''(k) & = sf''(k) < 0 \\
    H'(0) & = sf'(0) + (1 - \delta) = \infty \\
    H'(\infty) & = sf'(\infty) + (1 - \delta) = (1 - \delta) < 1
\end{align*}
\]

This model has a single state variable, the capital stock $k_t$. Given the capital stock, it’s trivial to find output $y_t = f(k_t)$, consumption $c_t = (1 - s)f(k_t)$, and investment $i_t = sf(k_t)$.

A solution to a difference equation is a function that expresses the capital stock as a function of time, $t$ an initial condition, $k_0$ and other parameters of the model. Linear difference equations are trivial to solve. For example, suppose

\[
x_{t+1} = Ax_t
\]
Then the solution is found by recursive substitution

\[ x_1 = Ax_0 \]
\[ x_2 = Ax_1 = A^2x_0 \]
\[ \vdots \]
\[ x_t = A^tx_0 \]

Unfortunately, non-linear difference equations are not so easy to solve. It’s traditional to analyze their qualitative behavior in terms of steady states and phase diagrams.

**Steady states**

A steady state is a **fixed point** of the mapping \( k_{t+1} = H(k_t) \). Put differently, it is a situation where the state variable is unchanging so that \( k_{t+1} = k_t = k \). The Solow model has two fixed points. A steady state has to satisfy

\[ k = H(k) = sf(k) + (1 - \delta)k \]

One solution is \( k = 0 \)

\[ 0 = H(0) = sf(0) + (1 - \delta)0 \]

another is the unique positive solution to

\[ \delta k = sf(k) \]

The left hand side is a straight line through the origin with positive slope \( 0 < \delta < 1 \). This is the steady state level of investment, the amount of investment required to keep the capital stock constant. The right hand side is a strictly increasing strictly concave function that is exactly a scalar multiple of the production function \( f(k) \). Given the Inada conditions, there is a unique positive \( \bar{k} > 0 \) determined by the intersection of the two curves \( \delta \bar{k} = sf(\bar{k}) \).

**Local stability**

Consider the first order Taylor series approximation to the function \( H \) at some point \( \bar{k} \)

\[ H(k_t) \approx H(\bar{k}) + H'(\bar{k})(k_t - \bar{k}) \]

A natural point around which to approximate \( H \) is a fixed point. In this case,

\[
H(k_t) \approx H(\bar{k}) + H'(\bar{k})(k_t - \bar{k}) \\
= \bar{k} + H'(\bar{k})(k_t - \bar{k})
\]

where the second line follows because of the definition of \( \bar{k} \) as a point that satisfies \( \bar{k} = H(\bar{k}) \). If we treat this approximation as exact, we have the linear difference equation

\[ k_{t+1} = \bar{k} + H'(\bar{k})(k_t - \bar{k}) \]

An intuitive understanding of whether a fixed point is locally stable can be obtained by considering what would have to be true in order for sequences of \( k_t \) that satisfy this linear
difference equation to converge to \( \bar{k} \). Writing \( x_{t+1} = k_{t+1} - \bar{k} \) so that \( x_{t+1} = H'(\bar{k})x_t \), it is clear that this linear difference equation has the solution

\[
x_t = H'(\bar{k})^t x_0
\]
or

\[
k_t = \bar{k} + H'(\bar{k})^t (k_0 - \bar{k})
\]

Clearly, \( k_t \to \bar{k} \) if and only if \( H'(\bar{k})^t \to 0 \) and this in turn requires that \( |H'(\bar{k})| < 1 \). Thus in order for a fixed point to be locally stable, we need the absolute value of the slope of \( H \) at the fixed point to be less than one. If \( 0 < H'(\bar{k}) < 1 \), the convergence to the fixed point is monotone. If \(-1 < H'(\bar{k}) < 0 \), the convergence to the fixed point takes the form of dampened oscillations.

Consider the Solow model. The trivial fixed point \( \bar{k} = 0 \) is locally unstable — it repels capital sequences — because the derivative at this point is \( H'(0) = \infty \). On the other hand, the interior fixed point is locally stable — it attracts capital sequences — because the derivative at this point is positive but less than one.

Of course, just because a fixed point is locally stable, doesn’t mean that it is globally stable (that would require \( k_t \to \bar{k} \) for all initial conditions \( k_0 \) in the domain of the original non-linear map \( H \)).

## Log-linearizations

Log-linearizing a model has the convenience of providing coefficients that are readily interpretable as elasticities. The log deviation of a variable \( x_t \) from a steady state level \( \bar{x} \) is just \( \hat{x}_t \equiv \log(x_t/\bar{x}) \). Multiplied by 100, this is approximately the percentage deviation of \( x_t \) from \( \bar{x} \). Mechanically, the log linearization of a model proceeds by replacing \( x_t \) with \( \bar{x} \exp(\hat{x}_t) \) and then linearizing the equations of the model with respect to \( \hat{x}_t \) in a neighborhood of zero.

To illustrate, consider the fundamental non-linear difference equation of the Solow model

\[
k_{t+1} = H(k_t) \equiv sf(k_t) + (1 - \delta)k_t
\]

Replace \( k_t \) by \( \bar{k} \exp(\hat{k}_t) \) to get

\[
\bar{k} \exp(\hat{k}_{t+1}) = H[\bar{k} \exp(\hat{k}_t)] \equiv sf[\bar{k} \exp(\hat{k}_t)] + (1 - \delta)\bar{k} \exp(\hat{k}_t)
\]

A first order Taylor series approximation around zero of the term on the far left gives

\[
\bar{k} \exp(\hat{k}_{t+1}) \approx \bar{k} \exp(0) + \bar{k} \exp(0)\hat{k}_{t+1} = \bar{k} + \bar{k} \hat{k}_{t+1}
\]

Similarly

\[
H[\bar{k} \exp(\hat{k}_t)] \approx H[\bar{k} \exp(0)] + H'[\bar{k} \exp(0)]\bar{k} \exp(0)\hat{k}_t
\]

\[
= H(\bar{k}) + H'(\bar{k})\bar{k} \hat{k}_t
\]

\[
= \bar{k} + [sf'(\bar{k}) + (1 - \delta)]\bar{k} \hat{k}_t
\]

Equating (1) and (2) and simplifying gives

\[
\hat{k}_{t+1} \approx [sf'(\bar{k}) + (1 - \delta)]\hat{k}_t
\]

Here are some rules that make taking log-linearization easier (you should derive these results yourself to test your understanding).
1. MULTIPLICATION

\[ z_t = x_t y_t \]

\[ \Rightarrow \hat{z}_t = \hat{x}_t + \hat{y}_t \]

**Application:** A consumer’s first order condition is often

\[ 1 = \beta \frac{U''(c_{t+1})}{U'(c_t)} R_{t+1} \]

This implies the log linearization

\[ \hat{1} = \hat{\beta} + U'(\hat{c}_{t+1}) - U'(\hat{c}_t) + \hat{R}_{t+1} \]

But since the log deviations of constants are zero, this is just

\[ 0 = U'(\hat{c}_{t+1}) - U'(\hat{c}_t) + \hat{R}_{t+1} \]

2. DIVISION

\[ z_t = \frac{x_t}{y_t} \]

\[ \Rightarrow \hat{z}_t = \hat{x}_t - \hat{y}_t \]

**Application:** The law of motion for money supply is often written

\[ M_{t+1} = \mu_t M_t \]

This implies the law of motion for *real balances*

\[ m_{t+1} = \frac{M_{t+1}}{P_t} = \mu_t \frac{M_t}{P_{t-1}} \frac{1}{\pi_t} \]

(where \(\pi_t = P_t/P_{t-1}\)). So we have the log linearization

\[ \hat{m}_{t+1} = \hat{\mu}_t + \hat{m}_t - \hat{\pi}_t \]

3. ADDITION/SUBTRACTION

\[ z_t = x_t + y_t \]

\[ \Rightarrow \bar{z}_t = \bar{x}_t + \bar{y}_t \]

**Applications:** The resource constraint is often

\[ c_t + i_t = y_t \]

This implies the log linearization

\[ \bar{c}_t + \bar{i}_t = \bar{y}_t \]
Another example is that gross returns are often written as one plus net returns

\[ R_{t+1} = 1 + r_{t+1} \]

This implies

\[ \hat{R} \hat{R}_{t+1} = \hat{r} \hat{r}_{t+1} \]

or

\[ \hat{R}_{t+1} = \frac{\hat{r}}{1 + \hat{r}} \hat{r}_{t+1} \]

4. SMOOTH FUNCTIONS

\[ z_t = f(x_t) \]

\[ \implies \hat{z}_t = f'(\bar{x})\hat{x}_t \]

**Applications:** The marginal utility of consumption may be \( U'(c) \). This implies the log linearization

\[ U'(\bar{c})U'(c_t) = U''(\bar{c})\bar{c} c_t \]

As a leading example, if we have constant relative risk aversion preferences with coefficient \( \sigma > 0 \), this implies

\[ U'(c_t) = \frac{U''(\bar{c})C}{U'(\bar{c})} \hat{c}_t = -\sigma \hat{c}_t \]

As a further example, if we have the rental rate of capital equal to its marginal product

\[ r_t = F'(k_t) \]

this implies

\[ \hat{r}_t = \frac{F''(k)\hat{k}_t}{F'(k)} \hat{k}_t \]

5. MULTIVARIATE SMOOTH FUNCTIONS

\[ z_t = f(x_t, y_t) \]

\[ \implies \hat{z}_t = f_x(\bar{x}, \bar{y})\hat{x}_t + f_y(\bar{x}, \bar{y})\hat{y}_t \]

**Applications:** The production function might be \( y = F(k, n) \). This implies the log linearization

\[ \hat{y}_t = F_k(\bar{k}, \bar{n})\hat{k}_t + F_n(\bar{k}, \bar{n})\hat{n}_t \]
or
\[ \hat{y}_t = \frac{F_k(\bar{k}, \bar{n})}{F(k, \bar{n})} \beta_k + \frac{F_n(\bar{k}, \bar{n})}{F(k, \bar{n})} \beta_n \]

As a leading example, if we have a Cobb-Douglas production function \( F(k, n) = k^\alpha n^{1-\alpha} \) for \( 0 < \alpha < 1 \), this is
\[ \hat{y}_t = \alpha \beta_k + (1 - \alpha) \beta_n \]

As a further example, if we have the marginal utility of consumption \( U_c(c, \ell) \), the associated log linearization is
\[ U_c(c_t, \ell_t) = \frac{U_{cc}(\bar{c}, \bar{\ell})}{U_c(\bar{c}, \bar{\ell})} \beta_{c_t} + \frac{U_{ct}(\bar{c}, \bar{\ell})}{U_c(\bar{c}, \bar{\ell})} \beta_{\ell_t} \]

And if we use the approximation implied by Rule 3 to write \( \bar{n} \beta_n + \bar{\ell} \beta_\ell = 0 \) with \( \bar{\ell} = 1 - \bar{n} \), we can also write
\[ U_c(c_t, \ell_t) = \frac{U_{cc}(\bar{c}, 1 - \bar{n})}{U_c(\bar{c}, 1 - \bar{n})} \beta_{c_t} - \frac{U_{ct}(\bar{c}, 1 - \bar{n})}{U_c(\bar{c}, 1 - \bar{n})} \beta_{\ell_t} \]

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