Let me briefly recap on the dynamics of current accounts in small open economies. Consider the problem of a representative consumer in a country that is perfectly integrated with world capital markets and that takes as given a constant world real interest rate \( r > 0 \). The consumer is born at date \( t = 0 \) and lives until \( t = T \) with preferences \( U(c) \) over the consumption vector

\[
c = (c_0, c_1, \cdots, c_T)
\]

For simplicity, I assume that the consumer discounts the future at a geometric rate and has time-separable preferences of the form

\[
U(c) = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots + \beta^T u(c_T)
\]

\[
= \sum_{t=0}^{T} \beta^t u(c_t)
\]

This consumer faces a sequence of flow budget constraints, each of the form

\[
B_{t+1} - B_t = rB_t + y_t - c_t - i_t - g_t
\]

The change in net foreign assets \( B_{t+1} - B_t \) is the country’s current account balance. If \( B_{t+1} > B_t \), the country runs a current account surplus in date \( t \) while if \( B_{t+1} < B_t \), the country runs a current account deficit. Government expenditure \( \{g_t\} \) is a known exogenous sequence. The sum \( rB_t + y_t \) is Gross National Product (GNP) with Gross Domestic Product (GDP) denoted by \( y_t \). GDP is determined by the physical capital stock (labor is not a factor of production) according to a production function

\[
y_t = F(k_t)
\]

Investment is the change in the capital stock net of depreciation, \( i_t = k_{t+1} - k_t - \delta k_t \) where \( \delta \) denotes the depreciation rate. I will assume that \( \delta = 0 \) so that physical capital never depreciates. This implies that

\[
i_t = k_{t+1} - k_t
\]

The initial capital stock \( k_0 > 0 \) is a given parameter of the model. Choosing an investment plan is equivalent to choosing a sequence of capital installations \( \{k_{t+1}\} \).

### Intertemporal budget constraint

The sequence of flow budget constraints can be integrated to give a single intertemporal (or present value) budget constraint. This is done by recursive substitution. The basic idea is to continuously eliminate the future asset terms, \( B_{t+1} \), from the constraints. Mechanically,

\[
B_1 = (1 + r)B_0 + y_0 - c_0 - i_0 - g_0
\]

\[
B_2 = (1 + r)B_1 + y_1 - c_1 - i_1 - g_1
\]
Substituting $B_1$ into the second equation gives
\[ B_2 = (1 + r)[(1 + r)B_0 + y_0 - c_0 - i_0 - g_0] + y_1 - c_1 - i_1 - g_1 \]
Now write out an expression for $B_3$
\[ B_3 = (1 + r)B_2 + y_2 - c_2 - i_2 - g_2 \]
\[ = (1 + r)[(1 + r)B_0 + y_0 - c_0 - i_0 - g_0] + y_1 - c_1 - i_1 - g_1 + y_2 - c_2 - i_2 - g_2 \]
More generally, for any $t$
\[ B_{t+1} = (1 + r)^{t+1}B_0 + \sum_{s=0}^{t} (1 + r)^{t-s}(y_s - c_s - i_s - g_s) \]
Dividing throughout by the common factor $(1 + r)^t$, evaluating at $t = T$ and rearranging gives the intertemporal budget constraint
\[ \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t c_t + \left( \frac{1}{1 + r} \right)^T B_{T+1} = (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t) \]

**Intertemporal optimization**

The consumer’s problem is to choose a consumption vector $c$ and an investment plan to maximize her utility function subject to the budget constraint, the production function, and the definition of investment. The Lagrangian for this problem is
\[ L = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda \left[ (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t - c_t) - \left( \frac{1}{1 + r} \right)^T B_{T+1} \right] \]
\[ = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda \left[ (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (F(k_t) - (k_{t+1} - k_t) - g_t) - \left( \frac{1}{1 + r} \right)^T B_{T+1} \right] \]
where $\lambda$ denotes a Lagrange multiplier. The first order conditions that characterize this problem include
\[ \frac{\partial L}{\partial c_t} = 0 \quad \Rightarrow \quad \beta^t u'(c_t) - \lambda \left( \frac{1}{1 + r} \right)^t = 0 \quad \text{each } t \]
\[ \frac{\partial L}{\partial k_{t+1}} = 0 \quad \Rightarrow \quad -\lambda \left( \frac{1}{1 + r} \right)^t + \lambda \left( \frac{1}{1 + r} \right)^{t+1} F'(k_{t+1}) = 0 \quad \text{each } t \]
(We can also derive the obvious conclusion that $B_{T+1} = 0$ by noting that there is a cost to acquiring assets in the last period but no offsetting benefit). The optimality conditions can be rearranged to give the familiar consumption-smoothing condition and the requirement that investment take place up to the point where the marginal product of capital equals the given world real interest rate. In this notation,
\[ u'(c_t) = \beta(1 + r)u'(c_{t+1}) \]
\[ r = F'(k_{t+1}) \]
We can invert the last condition to solve for the capital stock in terms of $r$. When $r$ is constant, $k_{t+1}$ is constant at some $k = (F')^{-1}(r)$ too. With a constant exogenous world real interest rate, capital accumulation is not determined simultaneously with consumption.
The consumption function

To solve for consumption, we have to combine the first order condition

\[ u'(c_t) = \beta(1 + r)u'(c_{t+1}) \]

with the budget constraint

\[ \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t c_t = (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t) \]

(I have used the fact that \( B_{T+1} = 0 \)).

**Example 1.**

Suppose that \( \beta(1 + r) = 1 \) so that the discount rate \( \rho \equiv \beta - 1 \) is equal to the world real interest rate \( r \). Then

\[ u'(c_t) = u'(c_{t+1}) \]

implies that

\[ c_t = c_{t+1} = \bar{c} \quad \text{each } t \]

We still need to solve for this level \( \bar{c} \) of consumption. Substituting into the budget constraint

\[ \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t \bar{c} = (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t) \]

Since \( \bar{c} \) is the same for all \( t \) we can pull it outside of the sum

\[ \bar{c} \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t = (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t) \]

Now evaluating the sum on the left hand side gives (from a standard formula for geometric series, \( \sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x} \) for \( 0 < x < 1 \)),

\[ \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t = \frac{1 - \left( \frac{1}{1 + r} \right)^{T+1}}{1 - \left( \frac{1}{1 + r} \right)} = \left[ 1 - (1 + r)^{-(T+1)} \right] \left( \frac{1 + r}{r} \right) \]

So our consumption function is

\[ \bar{c} = \frac{1}{1 - (1 + r)^{-(T+1)}} \frac{r}{1 + r} \left[ (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t) \right] \]

(Recall that \( k = (F')^{-1}(r) \) so that everything on the right hand side can be written in terms of exogenous variables). This is a version of the permanent income hypothesis. The
main determinant of consumption is intertemporal wealth (or "permanent income"). The marginal propensity to consume out of wealth depends on $T$. As $T$ become large,

$$\bar{c} = \lim_{T \to \infty} \left\{ \frac{1}{1 - (1 + r)^{-(T+1)}} \frac{r}{1 + r} \left[ (1 + r)B_0 + \sum_{t=0}^{T} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t) \right] \right\}$$

$$= \frac{r}{1 + r} \left( (1 + r)B_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t) \right)$$

Thus in the long-horizon limit, consumption is simply proportional to intertemporal wealth.

### The infinite-horizon model

In this case, the consumer’s preferences are ordered by

$$U(c) = u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots$$

$$= \sum_{t=0}^{\infty} \beta^t u(c_t)$$

which is well defined if $0 < \beta < 1$ and the period utility function is either i) bounded, or ii) such that consumption does not grow too fast. The natural infinite-horizon budget constraint is

$$\sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t c_t + \lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^T B_{T+1} = (1 + r)B_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t)$$

In order to make this well defined, it is standard practice to impose a "no-Ponzi-game" constraint of the form

$$\lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^T B_{T+1} \geq 0$$

to ensure that the consumer cannot roll-over debt continuously. This leads to the requirement that the present value of consumption satisfy

$$\sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t c_t \leq (1 + r)B_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1 + r} \right)^t (y_t - i_t - g_t)$$

(Of course, if $u(c_t)$ is strictly increasing in $c_t$ this will always hold with equality). The same first order conditions can be obtained, namely,

$$u'(c_t) = \beta (1 + r) u'(c_{t+1})$$

$$r = F'(k_{t+1})$$
**Example 2.**

Now suppose that period utility has the isoelastic form

\[ u(c) = \frac{c^{1-\frac{1}{\sigma}} - 1}{1 - \frac{1}{\sigma}} \]

where \( \sigma > 0 \) denotes the constant intertemporal elasticity of substitution of the consumer. Then the marginal utility of consumption at date \( t \) is \( u'(c_t) = c_t^{-\frac{1}{\sigma}} \) so that the consumption smoothing condition can be written

\[ c_t^{-\frac{1}{\sigma}} = \beta(1+r)c_{t+1}^{-\frac{1}{\sigma}} \]

or

\[ c_{t+1} = \beta^\sigma (1+r)^\sigma c_t \]

If \( \beta(1+r) = 1 \) we again have that \( c_{t+1} = c_t \). More generally, we have

\[ c_t = [\beta^\sigma (1+r)^\sigma]^t c_0 \]

so that consumption at any date is a scaled up or down version of consumption at date zero. As before, if the consumer is relatively patient — so that she discounts less than the world interest rate — she has a growing consumption path, while if the consumer is relatively impatient she has a shrinking consumption path.

Now combine the formula \( c_t = [\beta^\sigma (1+r)^\sigma]^t c_0 \) with the intertemporal budget constraint

\[ \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t c_t = (1+r)B_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t (y_t - i_t - g_t) \]

to determine the initial consumption \( c_0 \). Obviously,

\[ c_0 \sum_{t=0}^{\infty} [\beta^\sigma (1+r)^{\sigma-1}]^t = (1+r)B_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t (y_t - i_t - g_t) \]

But

\[ \sum_{t=0}^{\infty} [\beta^\sigma (1+r)^{\sigma-1}]^t = \frac{1}{1 - \beta^\sigma (1+r)^{\sigma-1}} = \frac{1+r}{1 + r - \beta^\sigma (1+r)^\sigma} \]

(Assuming that \( 0 < \beta^\sigma (1+r)^{\sigma-1} < 1 \)). Hence

\[ c_0 = \frac{r + v}{1+r} \left( (1+r)B_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^t (y_t - i_t - g_t) \right) \]

where the number \( v \) is

\[ v \equiv 1 - \beta^\sigma (1+r)^\sigma \]

\( v \) summarizes the influence of \( \sigma \) and of \( \beta(1+r) \neq 1 \). If \( \beta(1+r) = 1 \), we have the same consumption function as in Example 1 with \( v = 0 \).
Dynamics of the current account

Let me introduce some notation which is helpful for discussing present value budget constraints. For any variable $x$, let $\tilde{x}_t$ denote the permanent value of $x$ at date $t$. This is the solution to

$$\sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} \tilde{x}_t = \sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} x_s$$

For a given world real interest rate $r > 0$, this is a mapping from the sequence $\{x_s\}$ to the single number $\tilde{x}_t$. Specifically,

$$\tilde{x}_t = \frac{r}{1+r} \sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} x_s$$

(Using $\sum_{i=0}^{\infty} z^i = (1 - z)^{-1}$ for $0 < z < 1$ and rearranging). Hence the permanent value is a measure of the central tendency of the sequence $\{x_s\}$ weighted by the discount factors.

Now suppose that $\beta(1+r) = 1$ as in Example 1. Then as in that example, the consumption function is

$$c_0 = \frac{r}{1+r} \left[ (1+r)B_0 + \sum_{t=0}^{\infty} \left( \frac{1}{1+r} \right)^{t} (y_t - i_t - g_t) \right]$$

Or at any initial date $t$,

$$c_t = \frac{r}{1+r} \left[ (1+r)B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} (y_s - i_s - g_s) \right]$$

(This follows using the changes of variable $0 \leftrightarrow t$ and $t \leftrightarrow s - t$). In terms of permanent values, this is just

$$c_t = rB_t + \tilde{y}_t - \tilde{i}_t - \tilde{g}_t$$

Now recall the flow budget constraint

$$B_{t+1} - B_t = rB_t + y_t - c_t - i_t - g_t$$

and eliminate consumption using $c_t = rB_t + \tilde{y}_t - \tilde{i}_t - \tilde{g}_t$. This gives

$$B_{t+1} - B_t = (y_t - \tilde{y}_t) - (i_t - \tilde{i}_t) - (g_t - \tilde{g}_t)$$

In this example, the current account $B_{t+1} - B_t$ is the sum of three terms, each the difference between a variable and its permanent value. If $y_t$ is relatively high, so that $y_t > \tilde{y}_t$, there will (ceteris paribus) be a current account surplus, $B_{t+1} > B_t$. Similarly, if $i_t$ or $g_t$ is relatively high, there will be a current account deficit. Over time, of course, the present value of current accounts must be zero.

Chris Edmond, 11 August 2003