Consider the problem of a representative consumer in a country that is perfectly integrated with world capital markets and that has access to a riskless world real interest rate \( r > 0 \). The consumer is born at date \( t = 0 \) and lives forever and has preferences \( U(c) \) over random consumption vectors

\[
c = (c_0, c_1, \ldots)
\]

This consumer faces a random endowment process \( \{y_t\} \) and has the flow budget constraints

\[
B_{t+1} - B_t = rB_t + y_t - c_t
\]

The consumer’s preferences are given by an expected utility function

\[
U(c) = E_0 \left( \sum_{t=0}^{\infty} \beta^t u(c_t) \right) \quad 0 < \beta < 1
\]

where \( E_t\{w\} \) denotes the expectation of the random variable \( w \) conditional on the information set at time \( t \). In the notation I will use, a variable subscripted \( t \) is known at date \( t \); thus at date \( t \), \( u'(c_t) \) is known but \( u'(c_{t+1}) \) is not.

The Lagrangian for this problem can be written

\[
L = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t [(1 + r)B_t + y_t - c_t - B_{t+1}] \right\}
\]

where the Lagrange multipliers \( \{\lambda_t\} \) on the flow constraints are generally stochastic too (because they reflect the marginal utility of wealth which depends on the stochastic process \( \{y_t\} \)).

The first order conditions for this problem include

\[
\frac{\partial L}{\partial c_t} = 0 \quad \iff \quad E_0 E_t\{\beta^t u'(c_t)\} = E_0 E_t\{\lambda_t\}
\]

\[
\frac{\partial L}{\partial B_{t+1}} = 0 \quad \iff \quad E_0 E_t\{\lambda_t\} = E_0 E_t\{(1 + r)\lambda_{t+1}\}
\]

which can be simplified using the Law of Iterated Expectations to give

\[
\beta^t u'(c_t) = \lambda_t
\]

\[
\lambda_t = (1 + r)E_t\{\lambda_{t+1}\}
\]

These are to be interpreted as holding for every date and state of nature (i.e., for every date and possible realization of \( y_t \)). They can be combined to give the stochastic Euler equation

\[
u'(c_t) = \beta(1 + r)E_t\{u'(c_{t+1})\}
\]

which characterizes optimal consumption-smoothing behavior for this problem.
Investment

Now let’s extend this problem by including production and capital accumulation. Let the production function be

\[ y_t = A_t F(k_t) \]

where \( A_t \) denotes a random technology shock and \( k_t \) denotes physical capital. Capital is accumulated by making investments and does not depreciate

\[ k_{t+1} = k_t + i_t \]

And the flow budget constraint of the consumer is now

\[ (B_{t+1} - B_t) + (k_{t+1} - k_t) = r B_t + A_t F(k_t) - c_t \]

The consumer’s Lagrangian is

\[ L = E_0 \left\{ \sum_{t=0}^{\infty} \beta^t u(c_t) + \lambda_t [(1 + r) B_t + A_t F(k_t) + k_t - c_t - B_{t+1} - k_{t+1}] \right\} \]

And her decision problem is characterized by three sets of first order conditions, one each for consumption, bonds, and capital accumulation

\[ \frac{\partial L}{\partial c_t} = 0 \quad \Leftrightarrow \quad \beta^t u'(c_t) = \lambda_t \]
\[ \frac{\partial L}{\partial B_{t+1}} = 0 \quad \Leftrightarrow \quad \lambda_t = E_t \{(1 + r) \lambda_{t+1}\} \]
\[ \frac{\partial L}{\partial k_{t+1}} = 0 \quad \Leftrightarrow \quad \lambda_t = E_t \{[1 + A_{t+1} F'(k_{t+1})] \lambda_{t+1}\} \]

The last two of these conditions are asset pricing equations. In each case, the consumer foregoes some consumption and pays a cost \( \lambda_t \). At an optimum, this price must be equalized with a benefit. Indeed, the benefits from acquiring bonds or holding capital must be the same. In each case, part of the benefit comes from \( \lambda_{t+1} \). In the case of bonds, there is a sure return \( 1 + r \) per \( \lambda_{t+1} \). In the case of capital, there is a risky return \( 1 + A_{t+1} F'(k_{t+1}) \) that depends on the marginal product of capital at \( t + 1 \) which is unknown as of date \( t \).

Rewriting the first order conditions gives

\[ u'(c_t) = \beta E_t \{(1 + r) u'(c_{t+1})\} \]
\[ u'(c_t) = \beta E_t \{[1 + A_{t+1} F'(k_{t+1})] u'(c_{t+1})\} \]

which imply

\[ \frac{1}{1+r} = E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} \]

and

\[ 1 = E_t \left\{ [1 + A_{t+1} F'(k_{t+1})] \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} = E_t \left\{ 1 + A_{t+1} F'(k_{t+1}) \right\} E_t \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} + \text{Cov}_t \left\{ A_{t+1} F'(k_{t+1}), \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} \right\} \]
(Using the definition of a covariance, \( \text{Cov}(X, Y) = E(XY) - E(X)E(Y) \) and that \( \text{Cov}(z + X, Y) = \text{Cov}(X, Y) \) for any constant \( z \)). Now using the price of a bond, \((1 + r)^{-1}\) to simplify this last expression

\[
1 = E_t \left\{ 1 + A_{t+1}F'(k_{t+1}) \right\} \frac{1}{1 + r} + \text{Cov}_t \left[ \left\{ A_{t+1}F'(k_{t+1}) \right\}, \left\{ \beta \frac{u'(c_{t+1})}{u'(c_t)} \right\} \right]
\]

or

\[
E_t \left\{ A_{t+1}F'(k_{t+1}) \right\} = r - \beta(1 + r)\text{Cov}_t \left[ \left\{ A_{t+1}F'(k_{t+1}) \right\}, \left\{ \frac{u'(c_{t+1})}{u'(c_t)} \right\} \right]
\]

(which uses \( \text{Cov}(zX, Y) = z\text{Cov}(X, Y) \) for any constant \( z \)).

Thus capital accumulation takes place until the expected marginal product of capital is set equal to the riskless rate minus a conditional covariance term. Since a positive technology shock will increase consumption and lower the marginal utility of consumption, this covariance term is negative. The expected marginal product of capital is the riskless rate plus a risk premium

\[
E_t \left\{ r_{t+1}^{\text{risky}} \right\} = r_{t+1}^{\text{riskless}} + \text{risk premium}_t
\]

A high risk premium is demanded of an asset that is poor from an insurance perspective, i.e., an asset that has a high return only when the marginal utility of consumption is low (and consumption itself is relatively high). In general, this risk premium is time-varying.

Chris Edmond, 18 August 2003