316-406 ADVANCED MACROECONOMIC TECHNIQUES **Midterm**

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This exam lasts **90 minutes** and has three questions, each of equal marks. Within each question there are a number of parts and the weight given to each part is also indicated. You have **10 minutes perusal** before you can start writing answers.

Question 1. *Stochastic Solow Growth Model* (30 marks): Let time be discrete, $t = 0, 1, ...$ Let the national resource constraint be

$$c_t + i_t = y_t = z_t f(k_t)$$

where c_t denotes consumption, i_t denotes investment, y_t denotes output, and k_t denotes capital. The production function has the properties

$$\begin{aligned} f(0) &= 0 \\ f'(k) &> 0, \qquad f''(k) < 0 \\ \lim_{k \to 0} f'(k) &= \infty, \qquad \lim_{k \to \infty} f'(k) = 0 \end{aligned}$$

The production function is buffeted by random IID technology shocks $\{z_t\}$ with properties to be described below and with a given initial condition z_0. Let capital accumulation be given by

$$k_{t+1} = i_t, \qquad k_0 \text{ given}$$

(i.e., there is "full depreciation"). Finally, let consumption be a fixed fraction of national output

$$c_t = (1 - s) y_t, \qquad 0 < s < 1$$

where s denotes the national saving rate.

(a) (6 marks): Show that this model can be reduced to a single non-linear stochastic difference equation in k_t and z_t, i.e., that you can write the model as

$$k_{t+1} = \psi(k_t, z_t), \qquad k_0 \text{ and } z_0 \text{ given}$$

Provide an explicit formula for the function ψ.

(b) (6 marks): Let $z_t = \bar{z} = 1$ always and let $\bar{k}$ denote a solution to

$$\bar{k} = \psi(\bar{k}, 1)$$

How many such $\bar{k}$ are there? Linearize the function $\psi(k_t, 1)$ around each of these points and determine the local stability or instability of each such point.

(c) (6 marks): Suppose that the production function is Cobb-Douglas with capital share α,

$$f(k) \equiv k^{\alpha}, \qquad 0 < \alpha < 1$$

Provide an explicit solution for each $\bar{k}$ (continuing to hold $z_t = \bar{z} = 1$). Explain how the fixed points depend on the parameters α and s. Given economic interpretations.

(d) (12 marks): Suppose that $\log(z_t)$ are IID Gaussian with mean 0 and variance σ^2. Let

$$\hat{x}_t = \log\left(\frac{x_t}{\bar{x}}\right)$$

be the log deviation of some variable from its "non-stochastic steady state". Log-linearize $\psi(k_t, z_t)$ to derive an approximate linear stochastic difference equation in the state $\hat{k}_t$ and the shocks $\hat{z}_t$ (assume as above that $f(k)$ is Cobb-Douglas). Solve for the stationary distribution of $\hat{k}_t$ and explain how its mean and variance depend on the parameters α, s and σ. Give economic interpretations.

Question 2. *Stochastic Labor Demand* (30 marks): Suppose that a firm faces a stochastic real wage rate each period which follows an m-state Markov chain (w, P, μ_0) where w is an m-vector, P is a transition matrix and μ_0 is an initial distribution. Suppose that each period, the firm solves the static profit maximization problem over employment

$$\pi(w) = \max_n \{f(n) - wn\}$$

where w is this period's random wage realization and n is the firm's labor force. Suppose that $f(n)$ is strictly increasing and strictly concave in n.

(a) (5 marks): Explain how a labor demand schedule of the form

$$n = \varphi(w)$$

can be derived from the optimization problem. Explain how you characterize the function φ. What is the sign of $\varphi'(w)$? Why?

(b) (5 marks): Explain the stochastic dynamics that n_t exhibits. Carefully explain how you could simulate the optimal labor demand choices.

(c) (10 marks): Suppose that the production function is

$$f(n) \equiv n^\gamma, \qquad 0 < \gamma < 1$$

Provide an explicit solution for $n = \varphi(w)$ and for profits $\pi(w)$. What pattern of labor demand would one observe given the fluctuations in w? How does your answer depend on the labor share γ? What is the elasticity of labor demand?

(d) (5 marks). Let the production function be as in part (c). Suppose that the Markov chain has $m = 2$ states with

$$w = \begin{pmatrix} w_L & w_H \end{pmatrix}$$

and transition matrix

$$P = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}, \qquad 0 < p, q < 1$$

Finally, the initial distribution is

$$\mu_0 = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

Solve for the stationary distribution of wages. Explain how the mean and variance of the stationary distribution of wages depends on the transition probabilities p and q.

(e) (5 marks). Suppose further that we have $p = q$ and $w_L = \omega - 1$, $w_H = \omega + 1$ for some $\omega > 1$. Compute the mean and variance of the implied stationary distribution of labor demand. Explain how your answers depend on the parameters ω and γ. Give economic intuition.

Question 3. *Cake Eating* (30 marks): Consider a consumer with utility function

$$\sum_{t=0}^{\infty} \beta^t U(c_t), \qquad 0 < \beta < 1$$

The consumer is endowed with a cake of size x_0 at time $t = 0$. Each period, she has cake x_t and can either consume some, c_t, or hold some cake over to next period, x_{t+1}.

(a) (10 marks): Provide a dynamic programming representation of this problem. In your answer, let $V(x)$ denote the utility value of a cake of size x.

(b) (15 marks): Let the period utility function be

$$U(c) \equiv \frac{c^{1-\sigma}}{1-\sigma}, \qquad \sigma > 0$$

Guess that the value function $V(x)$ and policy function $g(x)$ for your dynamic programming problem have the forms

$$\begin{aligned} V(x) &= \alpha \frac{x^{1-\sigma}}{1-\sigma} \\ g(x) &= \theta x \end{aligned}$$

for some unknown coefficients $\alpha > 0$ and $0 < \theta < 1$. Solve for the unknown coefficients.

(c) (5 marks): Verify that over the infinite time horizon, the consumer eats all the cake, namely

$$\sum_{t=0}^{\infty} c_t = x_0$$

At what rate does the cake diminish? Provide a formula that calculates how long it takes for there to only be $\varepsilon > 0$ crumbs of cake left. How does this rate of consumption depend on the parameters β and σ? Give economic intuition.