Aside on quantity and price indices with CES utility

Consider a static 2-good utility maximization problem of the form: maximize utility

\[ U(c_1, c_2) = V(C(c_1, c_2)) \]

where

\[ C(c_1, c_2) \equiv \left[ \frac{\theta^{-1}}{\alpha c_1^{\theta} + (1 - \alpha) c_2^{\theta}} \right]^{\frac{\theta}{\theta - 1}}, \quad 0 < \alpha < 1, \text{ and } \theta > 0 \]

subject to the budget constraint

\[ p_1 c_1 + p_2 c_2 \leq y \]

The function \( C \) is a constant elasticity of substitution **aggregator** and overall utility \( U \) is some monotonic increasing transformation \( V \) of \( C \). The parameter \( \theta > 0 \) is the **elasticity of substitution** between \( c_1 \) and \( c_2 \). When \( \theta \to \infty \) the two goods are **perfect substitutes**, when \( \theta \to 0 \) the two goods are **perfect complements**, and when \( \theta \to 1 \) the utility function is Cobb-Douglas. (You need to use l’Hôpital’s rule to show this last claim). The parameter \( \alpha \) will turn out to measure an expenditure share.

Let’s solve this problem. The first order conditions are

\[ V'(C(c_1, c_2)) \frac{\partial C(c_1, c_2)}{\partial c_1} = \lambda p_1 \]
\[ V'(C(c_1, c_2)) \frac{\partial C(c_1, c_2)}{\partial c_2} = \lambda p_2 \]

for some unknown Lagrange multiplier \( \lambda \). Computing the marginal utilities on the left hand side

\[ \frac{\partial C(c_1, c_2)}{\partial c_1} = \frac{\theta}{\theta - 1} \left[ \frac{\theta^{-1}}{\alpha c_1^{\theta} + (1 - \alpha) c_2^{\theta}} \right]^{\frac{\theta - 1}{\theta - 1}} \frac{\theta - 1}{\theta} \frac{\theta^{-1}}{\alpha c_1^{\theta} - 1} \]
\[ \frac{\partial C(c_1, c_2)}{\partial c_2} = \frac{\theta}{\theta - 1} \left[ \frac{\theta^{-1}}{\alpha c_1^{\theta} + (1 - \alpha) c_2^{\theta}} \right]^{\frac{\theta - 1}{\theta - 1}} \frac{\theta - 1}{\theta} (1 - \alpha) \frac{\theta^{-1}}{\alpha c_2^{\theta - 1}} \]

The marginal rate of substitution at an optimum is therefore

\[ \frac{\alpha}{1 - \alpha} \frac{c_1^{\frac{\theta - 1}{\theta}}}{c_2^{\frac{\theta - 1}{\theta}} - 1} = \frac{\alpha}{1 - \alpha} \frac{c_1^{\frac{\theta - 1}{\theta}}}{c_2^{\frac{\theta - 1}{\theta}}} = \frac{\alpha}{1 - \alpha} \left( \frac{c_1}{c_2} \right)^{\frac{1}{\theta}} = \left( \frac{p_1}{p_2} \right) \]
or
\[
\left( \frac{c_1}{c_2} \right) = \left( \frac{1 - \alpha}{\alpha} \right)^\theta \left( \frac{p_1}{p_2} \right)^{-\theta}
\]

Notice that this implies
\[
\frac{d \log(c_1)}{d \log(p_1)} = \frac{d \log(c_2)}{d \log(p_2)} = -\theta
\]
which justifies the name given to the aggregator.

Now we can solve for the demand functions by combining this tangency condition with the budget constraint. That is,
\[
y = p_1 c_1 + p_2 c_2 = \left[ p_1 \left( \frac{1 - \alpha}{\alpha} \right)^\theta \left( \frac{p_1}{p_2} \right)^{-\theta} + p_2 \right] c_2
\]
so
\[
\hat{c}_2 = \frac{1}{1 + \left( \frac{1 - \alpha}{\alpha} \right)^\theta \left( \frac{p_1}{p_2} \right)^{-\theta}} \left( \frac{y}{p_2} \right) = \frac{\alpha^\theta p_2^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \left( \frac{y}{p_2} \right)
\]
\[
\hat{c}_1 = \frac{\left( \frac{1 - \alpha}{\alpha} \right)^\theta \left( \frac{p_1}{p_2} \right)^{-\theta}}{1 + \left( \frac{1 - \alpha}{\alpha} \right)^\theta \left( \frac{p_1}{p_2} \right)^{-\theta}} \left( \frac{y}{p_2} \right) = \frac{(1 - \alpha)^\theta p_1^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \left( \frac{y}{p_1} \right)
\]

**Computing the price index**

We now want to find functions $C(\hat{c}_1, \hat{c}_2)$ and $P(p_1, p_2)$ such that
\[
p_1 \hat{c}_1 + p_2 \hat{c}_2 = P(p_1, p_2)C(\hat{c}_1, \hat{c}_2)
\]
and
\[
U(\hat{c}_1, \hat{c}_2) = V(C(\hat{c}_1, \hat{c}_2))
\]
at the utility maximizing demands $\hat{c}_1, \hat{c}_2$. Mechanically, we do this by minimizing expenditure $PC$ subject to the constraint that $C = 1$.

Now $C = 1$ if and only if
\[
U(\hat{c}_1, \hat{c}_2) = \left[ \alpha \hat{c}_1^{\frac{\theta - 1}{\theta}} + (1 - \alpha) \hat{c}_2^{\frac{\theta - 1}{\theta}} \right]^{\frac{\theta}{\theta - 1}} = 1
\]
Plugging in the demand functions with \( y = PC = P \) gives

\[
1 = \left\{ \alpha \left[ \frac{\alpha^\theta p_2^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \left( \frac{P}{p_2} \right) \right]^{\frac{\theta - 1}{\theta}} + (1 - \alpha) \left[ \frac{(1 - \alpha)^\theta p_1^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \left( \frac{P}{p_1} \right) \right]^{\frac{\theta - 1}{\theta}} \right\}^{\frac{\theta}{\theta - 1}}
\]

We need to solve this expression for \( P \) as a function of \( p_1 \) and \( p_2 \). Write

\[
1 = \left\{ \alpha^\theta \left[ \frac{p_2^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \left( \frac{P}{p_2} \right) \right]^{\frac{\theta - 1}{\theta}} + (1 - \alpha)^\theta \left[ \frac{p_1^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \left( \frac{P}{p_1} \right) \right]^{\frac{\theta - 1}{\theta}} \right\}^{\frac{\theta}{\theta - 1}}
\]

\[
= \left\{ \alpha^\theta \left[ \frac{p_2^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \right]^{\frac{\theta - 1}{\theta}} + (1 - \alpha)^\theta \left[ \frac{p_1^{1-\theta}}{\alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta}} \right]^{\frac{\theta - 1}{\theta}} \right\}^{\frac{\theta}{\theta - 1}}
\]

Hence

\[
\left[ \alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta} \right]^{\frac{\theta - 1}{\theta}} = \left[ \alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta} \right]^{-1}
\]

or

\[
P = \left[ \alpha^\theta p_2^{1-\theta} + (1 - \alpha)^\theta p_1^{1-\theta} \right]^{\frac{1}{\theta - 1}}
\]

After all that algebra, we see that the price index is itself a CES aggregate of the individual prices \( p_1 \) and \( p_2 \). We will use this result in our model of complete markets with non-traded goods (see Note 4a).

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