Question 1. The key first order condition for a consumer’s problem is

$$\beta U'[c_i(s)]\pi(s) = U'(c_0^i)q_1(s)$$

a) If period utility is $$U(c) = \log(c)$$, then the first order condition can be re-written

$$c_0^i\beta\pi(s) = q_1(s)c_1^i(s)$$

Summing over the $$i$$ countries and rearranging gives solutions for the state prices

$$q_1(s) = \beta\pi(s)\frac{Y_0}{Y_1(s)}$$

Now plugging this expression for $$q_1(s)$$ back into the first order conditions gives

$$\frac{c_0^i}{c_1^i(s)} = \frac{Y_0}{Y_1(s)} = \mu^i$$

where $$\mu^i$$ is a time and state independent constant. We solve for $$\mu^i$$ from the intertemporal budget constraint, namely

$$\mu^i \left[ Y_0 + \sum_s q_1(s)Y_1(s) \right] = y_0^i + \sum_s q_1(s)y_1^i(s)$$

or

$$\mu^i = \frac{y_0^i + \sum_s q_1(s)y_1^i(s)}{Y_0 + \sum_s q_1(s)Y_1(s)} = \frac{y_0^i + \sum_s \beta\pi(s)\frac{Y_0}{Y_1(s)}y_1^i(s)}{Y_0 + \sum_s \beta\pi(s)\frac{Y_0}{Y_1(s)}Y_1(s)}$$

$$= \frac{1}{1 + \beta Y_0} + \frac{\beta}{1 + \beta} \sum_s \pi(s)\frac{y_1^i(s)}{Y_1(s)}$$

This is a weighted average of country $$i$$’s income share in period $$t = 0$$ and its expected income share in period $$t = 1$$. The weights correspond to the relative importance of each period, as measured by $$\beta$$. Finally, we can solve for the world interest rate via the relationship

$$\frac{1}{1 + r} = \sum_s q_1(s) = \beta \sum_s \pi(s)\frac{Y_0}{Y_1(s)} = \beta E_0 \left\{ \frac{Y_0}{Y_1} \right\}$$

so that the real interest rate is higher when the expected growth rate of the world endowment is higher.

b) Similarly, if $$U(c) = -\gamma \exp(-\gamma c)$$, then the key first order condition can be written

$$\gamma\beta \exp[-\gamma c_1^i(s)]\pi(s) = \gamma \exp(-\gamma c_0^i)q_1(s)$$

Taking logs of both sides and simplifying

$$\log[\beta\pi(s)] - \gamma c_1^i(s) = -\gamma c_0^i + \log[q_1(s)]$$
Summing over the $I$ countries and using the market clearing conditions

$$I \log[\beta \pi(s)] - \gamma Y_1(s) = -\gamma Y_0 + I \log[q_1(s)]$$

Hence

$$q_1(s) = \beta \pi(s) \exp \left\{ \frac{\gamma [Y_0 - Y_1(s)]}{I} \right\}$$

The world real interest rate is then given by

$$\frac{1}{1 + r} = \sum_s q_1(s) = \beta \sum_s \pi(s) \exp \left\{ \frac{\gamma [Y_0 - Y_1(s)]}{I} \right\} = \beta E_0 \left\{ \frac{\gamma (Y_0 - Y_1)}{I} \right\}$$

Using the expressions for $q_1(s)$, consumption allocations are now given by

$$c_i^0 - c_i^1(s) = \frac{Y_0 - Y_1(s)}{I}$$

which is independent of $\gamma$. Guess that

$$c_i^0 = \frac{Y_0}{I} - \mu^i$$
$$c_i^1(s) = \frac{Y_1(s)}{I} - \mu^i$$

for some time and state independent constants $\mu^i$. Again, we can solve for this constant using the intertemporal budget constraint

$$Y_0 - I \mu^i + \sum_s q_1(s)(Y_1(s) - I \mu^i) = I y_0^i + \sum_s q_1(s)I y_1^i(s)$$

Rearranging

$$\mu^i = \frac{(Y_0/I) - y_0^i + \sum_s q_1(s)(Y_1(s)/I) - y_1^i(s)}{1 + \sum_s q_1(s)}$$

where the state prices are given as above.

c) Now we have $I = 2$ and first order conditions

$$\gamma_1 \beta \exp[-\gamma_1 c_1^1(s)] \pi(s) = \gamma_1 \exp(-\gamma_1 c_0^1) q_1(s)$$
$$\gamma_2 \beta \exp[-\gamma_2 c_1^2(s)] \pi(s) = \gamma_2 \exp(-\gamma_2 c_0^2) q_1(s)$$

Taking logs of both sides and simplifying

$$\log[\beta \pi(s)] - \gamma_1 c_1^1(s) = -\gamma_1 c_0^1 + \log[q_1(s)]$$
$$\log[\beta \pi(s)] - \gamma_2 c_1^2(s) = -\gamma_2 c_0^2 + \log[q_1(s)]$$

or

$$c_0^1 - c_1^1(s) = \frac{1}{\gamma_1} \{ \log[q_1(s)] - \log[\beta \pi(s)] \}$$
$$c_0^2 - c_1^2(s) = \frac{1}{\gamma_2} \{ \log[q_1(s)] - \log[\beta \pi(s)] \}$$
Now summing over countries
\[ Y_0 - Y_1(s) = \frac{\gamma_1 + \gamma_2}{\gamma_1 \gamma_2} \{ \log[q_1(s)] - \log[\beta \pi(s)] \} \]
and solving for the state prices
\[ q_1(s) = \beta \pi(s) \exp \left\{ \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} [Y_0 - Y_1(s)] \right\} \]
Notice that if \( \gamma_1 = \gamma_2 = \gamma \), the term \( \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2} = \frac{\gamma}{2} \) which is what we would have from part b), but with \( I = 2 \). Now using this expression for the state prices to solve for relative consumption
\[ c_0^1 - c_1^1(s) = \frac{\gamma_2}{\gamma_1 + \gamma_2} [Y_0 - Y_1(s)] \]
\[ c_0^2 - c_1^2(s) = \frac{\gamma_1}{\gamma_1 + \gamma_2} [Y_0 - Y_1(s)] \]
Hence it seems that consumption allocations are given by
\[ c_0^1 = \frac{\gamma_2}{\gamma_1 + \gamma_2} Y_0 - \mu^1 \]
\[ c_1^1(s) = \frac{\gamma_2}{\gamma_1 + \gamma_2} Y_1(s) - \mu^1 \]
and similarly for country 2. Again, we could solve for the time and state independent constants \( \mu^i \) from each country’s intertemporal budget constraint. Notice that if country 1 is more risk averse, \( \gamma_1 > \gamma_2 \), then country 2 gets a larger share of world output.

**Question 2.** The equivalent planning problem is to choose allocations \( a_i^t(z^t), b_i^t(z^t) \) for \( i = 1, 2 \) to maximize the social welfare function
\[ \sum_i \sum_{t=0}^{\infty} \sum_{z^t} \omega_i \beta^t \left[ \frac{a_i^t(z^t)^{1-\alpha} + b_i^t(z^t)^{1-\alpha}}{1-\alpha} \right] \pi_t(z^t) \]
subject to the resource constraints
\[ \sum_i a_i^t(z^t) \leq x_t(z^t) \]
\[ \sum_i b_i^t(z^t) \leq y_t(z^t) \]
Let \( Q^t_f(z^t) \equiv \beta^t \pi_t(z^t)q^t_f(z^t) \) and \( Q^t_p(z^t) \equiv \beta^t \pi_t(z^t)q^t_p(z^t) \) denote the planner’s Lagrange multipliers. Then the problem breaks down into a sequence of static maximization problems, one for each date and state.
a) The key first order conditions are
\[ \omega_i a_i^t(z^t)^{-\alpha} = q_x^t(z^t), \quad i = 1, 2 \]
\[ \omega_i b_i^t(z^t)^{-\alpha} = q_y^t(z^t), \quad i = 1, 2 \]
along with the resource constraints.

b) Rewrite the first order conditions as
\[ a_i^t(z^t) = \left( \frac{\omega_i}{q_x^t(z^t)} \right)^{1/\alpha}, \quad i = 1, 2 \]
\[ b_i^t(z^t) = \left( \frac{\omega_i}{q_y^t(z^t)} \right)^{1/\alpha}, \quad i = 1, 2 \]
and now sum over \( i = 1, 2 \) to get
\[ x_t(z^t)q_x^t(z^t)^{1/\alpha} = (\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) \]
\[ y_t(z^t)q_y^t(z^t)^{1/\alpha} = (\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) \]

Hence the Lagrange multipliers are
\[ q_x^t(z^t) = (\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) x_t(z^t)^{-\alpha} \]
\[ q_y^t(z^t) = (\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) y_t(z^t)^{-\alpha} \]
which vary inversely with the supply of each good with elasticity \(-\alpha\). Now using these to solve for the consumption allocations
\[ a_i^t(z^t) = \frac{\omega_i^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} x_t(z^t) \]
\[ b_i^t(z^t) = \frac{\omega_i^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} y_t(z^t) \]

Finally, the terms of trade are given by
\[ \text{tot}_t(z^t) = \frac{q_y^t(z^t)}{q_x^t(z^t)} = \left( \frac{x_t(z^t)}{y_t(z^t)} \right)^\alpha \]
So when the world supply of \( x \) rises, its (shadow) price declines and country 1’s terms of trade worsen (i.e., \( \text{tot}_t(z^t) \) declines).

c) The implied welfare weights are found by considering a market economy. Let \( \lambda_i \) denote the Lagrange multiplier on a country’s intertemporal budget constraint. Then if \( \lambda_i = 1/\omega_i \), the market allocations will be the same as the planner’s allocations. Now the intertemporal budget constraint for country \( i = 1 \) is
\[ \sum_{t=0}^{\infty} \sum_{z^t} \left[ Q_x^t(z^t) a_1^t(z^t) a_1^t(z^t) + Q_y^t(z^t) b_1^t(z^t) \right] = \sum_{t=0}^{\infty} \sum_{z^t} Q_x^t(z^t) x_t(z^t) \]
Using the planners Lagrange multipliers

\[ \omega_1^{1/\alpha} \sum_{t=0}^{\infty} \sum_{z^t} \{ \beta^t \pi_t(z^t)[x_t(z^t)^{1-\alpha} + y_t(z^t)^{1-\alpha}] \} = (\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi_t(z^t)x_t(z^t)^{1-\alpha} \]

Imposing the normalization \((\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) = 1\), which is implicitly a normalization of the price system, we have

\[ \omega_1^{1/\alpha} = \frac{\sum_{t=0}^{\infty} \sum_{z^t} \beta^t \pi_t(z^t)x_t(z^t)^{1-\alpha}}{\sum_{t=0}^{\infty} \sum_{z^t} \{ \beta^t \pi_t(z^t)[x_t(z^t)^{1-\alpha} + y_t(z^t)^{1-\alpha}] \}} \]

So the implicit weights correspond to relative shares of intertemporal wealth.

d) The trade balance for country 1 is

\[ t_b(z^t) = x_t(z^t) - \frac{\omega_1^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} x_t(z^t) - \left( \frac{x_t(z^t)}{y_t(z^t)} \right)^\alpha \frac{\omega_1^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} y_t(z^t) \]

Plugging in our previous solutions

\[ t_b(z^t) = x_t(z^t) - \frac{\omega_1^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} x_t(z^t) - \left( \frac{x_t(z^t)}{y_t(z^t)} \right)^\alpha \frac{\omega_1^{1/\alpha}}{\omega_1^{1/\alpha} + \omega_2^{1/\alpha}} y_t(z^t) \]

Again using the normalization \((\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) = 1\), this simplifies to

\[ t_b(z^t) = (1 - \omega_1^{1/\alpha}) x_t(z^t) - \omega_1^{1/\alpha} x_t(z^t)^\alpha y_t(z^t)^{1-\alpha} \]

With the normalization \((\omega_1^{1/\alpha} + \omega_2^{1/\alpha}) = 1\), it must be the case that \(0 < \omega_1^{1/\alpha} < 1\).

So one effect of an increase in the supply of the domestic \(x\) good is to improve the trade balance. But there is also an offsetting terms of trade effect where the increase in the supply of \(x\) reduces the world price for that commodity and so worsens the domestic terms of trade. An increase in the supply of the foreign good \(y\) worsens the trade balance if \(1 - \alpha > 0\) and improves the trade balance otherwise. Put differently, if \(\alpha < 1\), then the increase in the supply of \(y\) is sufficient to improve the domestic terms of trade so much that the domestic trade balance improves. So if \(\alpha < 1\), the terms of trade and the trade balance tend to covary in a positive manner.

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