

The Aggregate Production Function

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Economic systems transform inputs — labor, capital, raw materials — into products. We use a theoretical construct called a *production function* to summarize the connection between inputs and outputs. Doing this for an entire economy is something of a leap of faith, but it's an extremely useful device for thinking about economic performance.

The production function

Economic organizations transform inputs (factories, office buildings, machines, labor with a variety of skills, intermediate inputs, and so on) into outputs or products. Boeing, for example, owns factories, hires workers, buys electricity and avionics, and uses them to produce aircraft. American Express's credit card business uses computers, buildings, labor, and small amounts of plastic to produce payment services. Pfizer hires scientists, MBAs, and others to develop, produce, and market drugs. McKinsey takes labor and information technology to produce consulting services.

For an economy as a whole, we might think of all the labor and capital used in the economy as producing GDP, the total value of goods and services. A production function is a mathematical relation between inputs and output that makes this idea concrete:

$$Y = AF(K, L),$$

where Y is output (real GDP), K is the quantity of physical capital (plant and equipment) used in production, L is the quantity of labor, and A is a measure of the productivity of the economy. More on each of these shortly.

The production function tells us how different amounts of capital and labor may be combined to produce output. The critical ingredient here is the function F . Among its properties are

- More input leads to more output. In economic terms, the marginal products of capital and labor are positive. In mathematical terms, the function F increases in both K and L :

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial F}{\partial L} > 0.$$

Consult the '[Mathematics Review](#)' if this seems overly mysterious to you.

- Diminishing marginal products of capital and labor. Increases in capital and labor lead to increases in output, but they do so at a decreasing rate: the more labor we add, the less additional output we get. You can see this in Figure 1: for a given capital stock \bar{K} , increasing labor by Δ starting from L_1 has a larger effect on output than increasing labor by the same amount starting from L_2 . That is: $AF(\bar{K}, L_1 + \Delta) - AF(\bar{K}, L_1) > AF(\bar{K}, L_2 + \Delta) - AF(\bar{K}, L_2)$. This condition translates into properties of the second derivatives of the production function:

$$\frac{\partial^2 F}{\partial K^2} < 0, \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

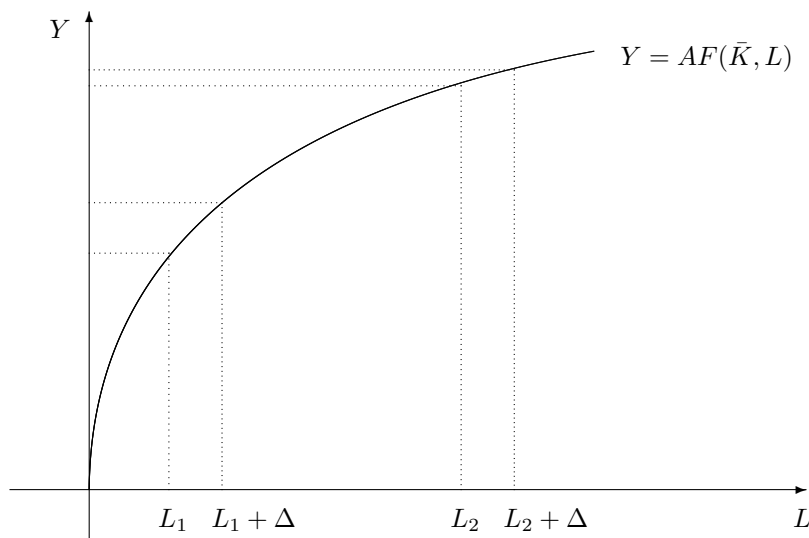


Figure 1: The Production Function.

- Constant returns to scale. This property says that if we (say) double all the inputs, the output doubles, too. More formally, if we multiply both inputs by the same number $\lambda > 0$, then we multiply output by the same amount:

$$AF(\lambda K, \lambda L) = \lambda AF(K, L).$$

Thus there is no inherent advantage or disadvantage of size.

These properties are more than we need for most purposes, but we mention them because they play a (sometimes hidden) role in the applications that follow.

Our favorite example of a production function is

$$Y = AK^\alpha L^{1-\alpha}$$

for a number (‘parameter’) α between zero and one. Circle this equation so you remember it! It’s referred to as the *Cobb-Douglas* version of the production function to commemorate two of the

earliest people to use it. (Charles Cobb was a mathematician. Paul Douglas was an economist and later a US senator). Let's verify that it satisfies the properties we suggested. First, the marginal products of capital and labor are

$$\begin{aligned}\partial Y/\partial K &= \alpha AK^{\alpha-1}L^{1-\alpha} = \alpha Y/K \\ \partial Y/\partial L &= (1-\alpha)AK^{\alpha}L^{-\alpha} = (1-\alpha)Y/L.\end{aligned}$$

Note that both are positive. Second, the marginal products are both decreasing. We show this by differentiating the first derivatives to get second derivatives:

$$\begin{aligned}\partial^2 Y/\partial K^2 &= \alpha(\alpha-1)AK^{\alpha-2}L^{1-\alpha} \\ \partial^2 Y/\partial L^2 &= -\alpha(1-\alpha)AK^{\alpha}L^{-\alpha-1}.\end{aligned}$$

Note that both are negative. Finally, the function exhibits constant returns to scale. If we multiply both inputs by $\lambda > 0$, the result is

$$A(\lambda K)^{\alpha}(\lambda L)^{1-\alpha} = A\lambda^{\alpha}K^{\alpha}\lambda^{1-\alpha}L^{1-\alpha} = \lambda AK^{\alpha}L^{1-\alpha},$$

as needed.

Productivity

You may see the word productivity used to mean several different things. The most common measure of productivity is the ratio of output to labor input, which we'll call the *average product of labor*. This is typically what government agencies mean when they report productivity data. It differs from the *marginal product of labor* for the same reason that average cost differs from marginal cost. *Total factor productivity*, or TFP, is the letter A in the production function. It measures the overall efficiency of the economy in transforming inputs into outputs. Mathematically, the three definitions are

$$\begin{aligned}\text{Average Product of Labor} &= Y/L \\ \text{Marginal Product of Labor} &= \partial Y/\partial L \\ \text{Total Factor Productivity} &= Y/F(K, L).\end{aligned}$$

For the Cobb-Douglas production function they are

$$\begin{aligned}\text{Average Product of Labor} &= A(K/L)^{\alpha} \\ \text{Marginal Product of Labor} &= (1-\alpha)A(K/L)^{\alpha} \\ \text{Total Factor Productivity} &= A.\end{aligned}$$

Holding A constant, the first two increase when we increase the ratio of capital to labor. The idea? You can be more productive if you have (say) more equipment to work with. TFP is an attempt to measure productivity independently of the amount of capital each worker has.

Capital input

The capital input (or capital stock) K is the total quantity of plant and equipment used in production. We value different kinds of capital (machines, office buildings, computers) at their base-year prices, just as we do with real GDP in the National Income and Product Accounts. It's somewhat heroic to combine so many different kinds of capital into one number, but that's the kind of people we are.

Fine points:

- How does capital change over time? Typically capital increases with investment (purchased of new plant and equipment) and decreases with depreciation. Mathematically, we might write

$$K_{t+1} - K_t = I_t - D_t,$$

where I_t is investment and D_t is depreciation. We often make the simplifying assumption that depreciation is proportional to capital, $D_t = \delta K_t$ with δ a number like 0.10. This is because, on average, the capital stock depreciates about 10% a year. Still, you should be aware that just as there are really different kinds of capital, these different kinds of capital depreciate at different rates (buildings have longer useful lives than computers).

Wars and disasters can also have an impact. We estimate that the German and Japanese capital stocks declined by about 50% between the start and end of World War II.

- Does land count? The short answer: no. In principle maybe it should, but in modern economies land is far less important than plant and equipment. To the extent it matters, its impact shows up in A .

Labor input

The quantity of labor is measured in most countries by the number of people employed, and in most cases that will be our initial measure of L . Working people also differ in the number of hours they work, and we will take this into account when we can.

Labor also differs in quality. Most of us don't have the skill to be a star shortstop for the Yankees. American workers earn more than Mexican workers, in large part because their skills are better. There are many skills we might want to measure, but the most important for a country is the level of education of the workforce. In 2000, the average Korean worker had 10.46 years of schooling, and the average Mexican worker had 6.73 years. We know that individuals with more education

have higher salaries, on average, so we might guess that Koreans have higher average skills than Mexicans. We take this into account by modifying our production function to include education:

$$Y = AF(K, HL),$$

where H is the average years of education (human capital). We refer to this relation as the *augmented production function*. If we do not adjust L for skill, the effect of H shows up implicitly in A . With a Cobb-Douglas production function,

$$Y = AK^\alpha(HL)^{1-\alpha} = (AH^{1-\alpha})K^\alpha L^{1-\alpha},$$

so that $(AH^{1-\alpha})$ is our overall measure of productivity. Of course part of it here is the result of increases in the skill of the workforce.

Executive summary

1. A production function links output to inputs.
2. Inputs include physical capital (primarily plant and equipment) and labor.
3. Total Factor Productivity (TFP) is a measure of productive efficiency.
4. Labor varies in quantity (number of people working, numbers of hours) and quality (skill, education).

Review questions

1. Suppose an economy has the production function

$$Y = AK^{1/4}L^{3/4}.$$

If $Y = 10$, $K = 15$, and $L = 5$, what is total factor productivity A ?

Answer. $A = Y/(K^{1/4}L^{3/4}) = 1.520$.

2. Suppose the production function is

$$Y = 2K^{1/4}L^{3/4}$$

and $K = L = 1$. How much output is produced? If we reduced L by 10%, how much would K need to be increased to produce the same output?

Answer. With $K = L = 1$, $Y = 2$. If L falls to 0.9, $K = 1/0.9^3 = 1.372$ (a 37% increase in K). The reason for the difference between the magnitudes in the changes in K and L is the difference in their exponents in the production function.

3. Worker 1 has 10 years of education, worker 2 has 15. How much more would you expect worker 2 to earn? Why?

Answer. If H = years of education, then one hour of worker 2's time is equivalent to 1.5 (= 15/10) hours of worker 1's time, so we'd expect her to be paid 50% more. A more complex answer is that skill may increase in a more complicated way with years of education, and that types of education may differ in their impact on earning power (an MBA may be worth more in this sense than a PhD in cultural anthropology, however interesting the latter may be).

4. Consider the augmented production function

$$Y = K^{1/3}(HL)^{2/3}.$$

If $K = 10$, $H = 10$, and $L = 5$, what is the average product of labor? How much does the average product increase if H rises to 12?

Answer. Output is $Y = 29.24$ so $Y/L = 5.85$. If H rises to 12, $Y/L = 6.60$.