Sluggish responses of prices and inflation to monetary shocks in an inventory model of money demand*

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ABSTRACT

We examine the responses of prices and inflation to monetary shocks in an inventory-theoretic model of money demand. We show that the price level responds sluggishly to an exogenous increase in the money stock because the dynamics of households’ money inventories leads to a partially offsetting endogenous reduction in velocity. We also show that inflation responds sluggishly to an exogenous increase in the nominal interest rate. To contrast with the sticky price and rational inattention literatures on nominal sluggishness, we show our results under the stark assumption that all prices are always set optimally and with full information. In a quantitative example, we show that this nominal sluggishness is substantial and persistent if inventories in the model are calibrated to match households’ holdings of M2.

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1. Introduction

In this paper, we examine the dynamics of money, velocity, prices, interest rates, and inflation in an inventory-theoretic model of the demand for money. We show that our inventory-theoretic model offers new answers to two important questions: why do prices respond sluggishly to changes in money? and why does inflation respond sluggishly to changes in the short-term nominal interest rate? We first show analytically how prices and inflation are both sluggish in our model, even though price setting is fully flexible. We then show through a quantitative example that this sluggishness is substantial and persistent when our inventory theoretic model is interpreted as applying to a broad monetary aggregate such as M2.

Our model is inspired by the analyses of money demand developed by Baumol (1952) and Tobin (1956). In their models, households carry money (despite the fact that money is dominated in rate of return by interest bearing assets) because they face a fixed cost of trading money and these other assets. Our model is a simplified version of their framework. We study a cash-in-advance model with physically separated asset and goods markets. Households have two financial accounts: a brokerage account in the asset market in which they hold a portfolio of interest bearing assets and a bank account in the goods market in which they hold money to pay for consumption. We assume that households do not have the opportunity to exchange funds between their brokerage and bank accounts every period. Instead, we assume they have the opportunity to transfer funds between accounts only once every \( N \geq 1 \) periods. Hence, households maintain an inventory of money in their bank account large enough to pay for consumption expenditures for several periods. They replenish this inventory with a transfer from their brokerage account once every \( N \) periods. As households optimally manage this inventory, their money holdings follow a sawtooth pattern — rising rapidly with each periodic transfer from their brokerage account and then falling slowly as these funds are spent smoothly over time — similar to the sawtooth pattern of money holdings originally derived by Baumol (1952) and Tobin (1956), and more recently by Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007). Here, we focus on the implications of our model for the response of prices to a change in money growth and the response of inflation to a change in interest.

\(^1\)Traditionally, the literature on inventory-theoretic models of money demand has focused on the steady-state implications of these models for money demand (see, for example, Barro 1976, Jovanovic 1982, Romer 1986, Chatterjee and Corbae 1992). Here we examine the implications of an inventory-theoretic model of the demand for money for the dynamics of prices and inflation following a shock to money or to interest rates.
rates. To highlight the specific mechanisms at work, we make the stark assumptions that price setting is fully flexible and that output in the model is exogenous so that our results can easily be compared to those from a flexible-price, constant-velocity, benchmark model.

Our first result is that prices respond sluggishly to a change in money in our model. Prices respond sluggishly in our model because an exogenous increase in the stock of money leads endogenously, through the dynamics of households' inventories of money, to a partially offsetting decrease in the velocity of money. As a result of this endogenous fall in velocity, prices respond on impact less than one-for-one to the change in money. Prices respond fully only in the long-run when households' inventories of money, and hence aggregate velocity, settle back down to their steady-state values. The sluggish response of prices to a change in money in our model can then be understood not as a consequence of a sticky-price setting policy of firms but as a simple consequence of the sluggish response of nominal expenditure to a change in money inherent in an inventory-theoretic approach to money demand.

We are motivated to highlight this implication of our inventory-theoretic model of money demand by the fact that a strong negative correlation between fluctuations in money and velocity can be seen clearly in U.S. data. In figure 1, we illustrate this short-run behavior of money and velocity. We plot the ratio of M2 to consumption and the consumption velocity of M2 as deviations from a trend extracted using an HP-filter. As is readily apparent, these two series are strongly negatively correlated. After presenting our analytical results, we examine the extent to which our model can reproduce this comovement of money and velocity in a quantitative example.

The mechanism through which our model produces a negative correlation between fluctuations in money and velocity and hence sluggish prices can be understood in two steps. First, consider how aggregate velocity is determined in this inventory-theoretic model of money demand. Households at different points in the cycle of depleting and replenishing their inventories of money in their bank accounts have different propensities to spend the money that they have on hand, or, equivalently, different individual velocities of money. Those households that have recently transferred funds from their brokerage account to their bank account have a large stock of money in their bank account and tend to spend this stock of

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2 In applying the HP-filter, we used a parameter of $3^2 \times 1600 = 14400$ for monthly data. As discussed in the Appendix, similar results are obtained using alternative measures of the short-run fluctuations in money and velocity.
money slowly to spread their spending smoothly over the interval of time that remains before they next have the opportunity to replenish their bank account. Hence, these households have a relatively low individual velocity of money. In contrast, those households that have not transferred funds from their brokerage account in the recent past and anticipate having the opportunity to make such a transfer soon tend to spend the money that they have in the bank at a relatively rapid rate, and thus have a relatively high individual velocity of money. Aggregate velocity is determined by the weighted average of these individual velocities of money of all of the households in the economy, with weights determined by the distribution of money across households.

Now consider the effects on aggregate velocity of an increase in the money supply brought about by an open market operation. In this open market operation, the government trades newly created money for interest bearing securities, and households, on the opposite side of the transaction, trade interest bearing securities held in their brokerage accounts for newly created money. If the nominal interest rate is positive, this new money is purchased only by those households that currently have the opportunity to transfer funds from their brokerage account to their bank account since these are the only households that currently have the opportunity to begin spending this money. All other households choose not to participate in the open market operation since these households would have to leave this money sitting idle in their brokerage accounts where it would be dominated in rate of return by interest bearing securities. Hence, as a result of this open market operation, the fraction of the money stock held by those households currently able to transfer resources from their brokerage account to their bank account rises. Since these households have a lower-than-average propensity to spend this money, aggregate velocity falls. In this way, an exogenous increase in the supply of money leads to an endogenous reduction in the aggregate velocity of money and hence, a diminished, or sluggish, response of the price level.

It is common to model changes in monetary policy not as exogenously specified changes in money but as exogenously specified changes in the short-term nominal interest rate. When we model monetary policy in this way, we find our second result, that expected inflation responds sluggishly to a change in the short-term nominal interest rate.

To gain some intuition for this result that expected inflation responds sluggishly to a change in the short-term nominal interest rate, it is useful to consider the Fisher equation to decompose any change in the nominal interest rate into its two components — a change in
the real interest rate and a change in expected inflation. For example, in a standard flexible price constant endowment cash-in-advance model, the real interest rate is always constant, so that, given the Fisher equation, any change in the nominal interest rate must always be accompanied by a matching change in expected inflation. In this sense, in this model, expected inflation must respond immediately to a change in the nominal interest rate. More generally, if a model is to generate a sluggish response of expected inflation to a change in the nominal interest rate caused by a change in monetary policy implemented through open market operations, it must do so because those open market operations generate a change in the real interest rate that is roughly as large as the change in the nominal interest rate. In our inventory theoretic model of money demand, money injections implemented through open market operations have an effect on the real interest rate because the asset market is segmented, and it is this effect of open market operations on the real interest rate that is the source of the inflation sluggishness in our model.

Asset markets are segmented in our model in the sense that only those agents who currently have the opportunity to transfer money between their brokerage and bank accounts are at the margin in participating in open market operations and in determining asset prices. This asset market segmentation arises naturally in an inventory theoretic model of the demand for money because those agents who do not have the opportunity to transfer money between the asset market and the goods market have no desire to purchase new money being injected into the asset market through an open market operation because these agents have no ability to spend that money in the current period and they find that interest bearing bonds dominate money as a store of value in the asset market.3 Because only the subset of agents who currently have the opportunity to transfer money from the asset market to the goods market are at the margin in trading money and bonds with the monetary authority, money injections implemented through open market operations have a disproportionate impact on the marginal utility of a dollar for these marginal investors that is manifest as a movement in real interest rates. In our model, as \(N\) gets large, the asset market becomes highly segmented and a given change in expected money growth has a large effect on the real interest rate, Hence, only a small change in money injections and a correspondingly small change in expected inflation is

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3These agents choose not to participate in the open market operation as long as the short-term nominal interest rate remains positive. Note that financial intermediaries also choose not to hold money injected through open market operations as long as the short-term nominal interest rate remains positive.
required to implement a given change in the nominal interest rate, because this small change in money injections has a large impact on the real interest rate. It is in this sense that expected inflation shows a sluggish response to a change in the nominal interest rate.

In our analytical results, we show that the parameter $N$ governing the frequency with which households replenish their bank accounts determines our model’s implications for the magnitude of the responses of prices and interest rates to changes in monetary policy. This parameter also determines our model’s implications for steady-state aggregate velocity — since $N$ determines the size of the inventory of money households must hold to purchase consumption. Thus, the empirical implications of our model for the sluggishness of prices and inflation are largely determined by how we define money (since that definition determines the measure of velocity and hence $N$). In our model, defining money comes down to answering the question: What assets correspond to those that households hold in their bank accounts and what assets do households hold and trade less frequently in their brokerage accounts?

We examine the implications of our model in a quantitative example using a broad measure of money: U.S. households’ holdings of currency, demand deposits, savings deposits, and time deposits. In the data, U.S. households hold a large stock of such accounts, roughly 1/2 to 2/3 of annual personal consumption expenditure. We argue for the interpretation of this broad collection of accounts in the data as corresponding to bank accounts in our model because we find in the data that U.S. households pay a large opportunity cost in terms of forgone interest to hold such accounts — on the order of 150-200 basis points. This opportunity cost is not substantially different from the opportunity cost U.S. households pay to hold a narrower definition of money such as M1. To parameterize our model to match the observed ratio of U.S. households’ holdings of this broad money relative to personal consumption expenditure, we assume households transfer funds between their brokerage and bank accounts very infrequently — on the order of one every one and a half to three years. We argue that this assumption is not inconsistent with evidence summarized by Vissing-Jorgensen (2002) regarding the frequency with which U.S. households trade in high-yield assets. Our interpretation of a bank account used for transactions replenished by transfers from a high-yield managed portfolio of risky and riskless assets is the same as used in the models of Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007).

We conduct two quantitative exercises with our model. In the first, we feed into the model the shocks to the stock of M2 and aggregate consumption observed in the U.S. economy
in monthly data over the past 40 years and examine the model’s predictions for velocity in
the short-run. The model produces fluctuations in velocity that have a surprisingly high
correlation of 0.60 with the fluctuations in velocity observed in the data. This result stands
in sharp contrast to the implications of a standard cash-in-advance model (this model with
\( N = 1 \)). In such a model, aggregate velocity is constant regardless of the pattern of money
growth. We also find that the short-run fluctuations in velocity in our model are only 40% as
large as those in the data. From the finding that the short run fluctuations in velocity in our
model are highly correlated with those observed in the data, we conclude that a substantial
portion of the unconditional negative correlation of the ratio of money to consumption and
velocity might reasonably be attributed to the response of velocity to exogenous movements
in money. From the finding that the short run fluctuations in velocity in our model are not
as large as those in the data, however, we conclude that there may be other shocks to the
demand for money which we have not modeled here.

In our second quantitative exercise, we consider the response of money, prices and
velocity to an exogenous shock to monetary policy, modeled as an exogenous, persistent shock
to the short-term nominal interest rate similar to that estimated in the literature which uses
vector autoregressions (VARs) to draw inferences about the effects of monetary policy. The
consensus in that literature is that the impulse response of inflation to a monetary policy
shock is sluggish.\(^4\) In our model we find that the impulse response of inflation is also quite
sluggish, as are the responses of money and the price level. All three of these responses
from our model are quite similar to the estimated responses of these variables in this VAR
literature. While our model is incomplete in that we have assumed for simplicity that output
is exogenous, these findings suggest that our model can account for a substantial portion of
the sluggish responses of nominal variables to a change in the nominal interest rate simply
through asset market segmentation.

Our model is related to a growing literature on segmented asset markets. Grossman
and Weiss (1983) and Rotemberg (1984) were the first to point out that open market op-
ervations could have effects on real interest rates and a delayed impact on the price level in
inventory-theoretic models of money demand. The models they present are similar to this

\(^4\)See Cochrane (1994), Leeper, Sims and Zha (1996), for early estimates, Christiano, Eichenbaum and
estimates.
model when the parameter $N = 2$. Those authors examine the impact of a surprise money injection in the context of otherwise deterministic models. Here we study a fully stochastic model as in Alvarez and Atkeson (1997). That model is similar to the one presented here in that agents have separate financial accounts in asset and goods markets and cannot transfer funds between these accounts in every period. In that earlier paper however, in equilibrium, the individual velocity of money is the same for all households and is constant over time so that aggregate velocity is also constant. This result follows from the assumptions in that paper that households have logarithmic utility and a constant probability of being able to transfer money between the asset market and the goods market. The asset pricing implications of our model are closely related to those obtained by Grossman and Weiss (1983), Rotemberg (1984), and Alvarez and Atkeson (1997). In particular, our model has predictions for the effects of money injections on real interest rates arising from the segmentation of the asset market related to the predictions in those papers and those in Alvarez, Atkeson, and Kehoe (2002, 2007) and Alvarez, Lucas, and Weber (2001). Alvarez, Atkeson, and Kehoe (2002, 2007) study the implications of models with segmented asset markets in which households pay a fixed cost to transfer money between bank and brokerage accounts. In that paper, they focus on equilibria in which all households spend all of the money in their bank account every period so that, again, velocity is constant.

Khan and Thomas (2007) explore numerically the dynamics of money and velocity in a version of our model in which households both store money in their bank account and can pay a fixed cost to transfer money between their brokerage and bank accounts. They find that consideration of these costs can substantially reinforce the sluggishness of prices and the persistence of liquidity effects relative to that seen in our model.

2. An inventory-theoretic model of money demand

Consider a cash-in-advance economy in which the asset market and the goods market are physically separated. There is a unit mass of households each comprised of a worker and a shopper. Each household has access to two financial intermediaries: one that manages its portfolio of assets and another that manages its money held in a transactions account in the goods market. We refer to the household’s account with the financial intermediary in the asset market as its brokerage account and its account with the financial intermediary in the goods market as its bank account. There is a government that injects money into the
asset market in this economy via open market operations. Households that participate in the open market operation purchase this money with assets held in their brokerage accounts. These households must transfer this money to their bank account before they can spend it on consumption.

Time is discrete and denoted \( t = 0, 1, 2, \ldots \). The exogenous shocks in this economy are shocks to the money growth rate \( \mu_t \) and shocks to the endowment of each household \( y_t \). Since all households receive the same endowment, \( y_t \) is also the aggregate endowment of goods in the economy. Let \( h_t = (\mu_t, y_t) \) denote the realized shocks in the current period. The history of shocks is denoted \( h^t = (h_0, h_1, \ldots, h_t) \). From the perspective of time zero, the probability distribution over histories \( h^t \) has density \( f_t(h^t) \).

As in a standard cash-in-advance model, each period is divided into two sub-periods. In the first sub-period, each household trades assets held in its brokerage account in the asset market. In the second sub-period, the shopper purchases consumption in the goods market using money held in the household’s bank account while the worker sells the endowment in the goods market for money \( P_t y_t \) where \( P_t \) denotes the price level in the current period. In the next period, a fraction \( \gamma \in [0, 1] \) of the worker’s earnings is deposited in the bank account in the goods market while the remaining \( 1 - \gamma \) of these earnings are deposited in brokerage account in the asset market. We interpret \( \gamma \) as the fraction of total income households receive regularly deposited into their transactions accounts or as currency. We refer to \( \gamma \) as the \emph{paycheck parameter} and to \( \gamma P_{t-1} y_{t-1} \) as the household’s \emph{paycheck}. We interpret \( 1 - \gamma \) as the fraction of total income households receive in the form of interest and dividends paid on assets held in their brokerage accounts.

Unlike a standard cash-in-advance model, households cannot transfer money between the asset market and the goods market every period. Instead, each household has the opportunity to transfer money between its brokerage account and its bank account only once every \( N \) periods. In other periods, a household can trade assets in its brokerage account and use money in its bank account to purchase goods, it simply cannot move money between these two accounts. We refer to households that currently have the opportunity to transfer money between their accounts as \emph{active households}.

Each period a fraction \( 1/N \) of the households are active. We index each household by the number of periods since it was last active, here denoted by \( s = 0, 1, \ldots, N - 1 \). A household of type \( s < N - 1 \) in the current period will be type \( s + 1 \) in the next period. A household
of type \( s = N - 1 \) in the current period will be type \( s = 0 \) in the next period. Hence a household of type \( s = 0 \) is active in this period, a household of type \( s = 1 \) was active last period, and a household \( s = N - 1 \) will be active next period. In period 0, each household has an initial type \( s_0 \), with fraction \( 1/N \) of the households of each type \( s_0 = 0, 1, ..., N - 1 \). Let \( S(t, s_0) \) denote the type in period \( t \) of a household that was initially of type \( s_0 \).

The quantity of money a household \( s \) has on hand in its bank account at the beginning of goods market trade is \( M_t(s) \). The shopper in this household spends some of this money on goods, \( P_t c_t(s) \), and the household carries the unspent balance in its bank account into next period, \( Z_t(s) \). For an inactive household of type \( s > 0 \), the balance at the beginning of the period is equal to the quantity of money that it held over in its bank account last period \( Z_{t-1}(s-1) \) plus its paycheck \( \gamma P_{t-1} y_{t-1} \). Thus, the evolution of money holdings and consumption for inactive households is:

\[
M_t(s, h^t) = Z_{t-1}(s-1, h^{t-1}) + \gamma P_{t-1}(h^{t-1})y_{t-1}(h^{t-1}),
\]

\[
M_t(s, h^t) \geq P_t(h^t)c_t(s, h^t) + Z_t(s, h^t).
\]

When a household is of type \( s = 0 \) and hence active it also chooses a transfer of money \( P_t x_t \) from its brokerage account in the asset market into its bank account in the goods market. Hence, the money holdings and consumption of active households satisfy:

\[
M_t(0, h^t) = Z_{t-1}(N-1, h^{t-1}) + \gamma P_{t-1}(h^{t-1})y_{t-1}(h^{t-1}) + P_t(h^t)x_t(h^t),
\]

\[
M_t(0, h^t) \geq P_t(h^t)c_t(0, h^t) + Z_t(0, h^t).
\]

In addition to the bank account constraints, equations (1)-(4) above, the household also faces a sequence of brokerage account constraints. In each period the household can trade in a complete set of one-period state contingent bonds which pay one dollar into the household’s brokerage account next period if the relevant contingency is realized. Let \( B_{t-1}(s-1, h^t) \) denote the stock of bonds held by households of type \( s \) at the beginning of period \( t \) following history \( h^t \) and let \( B_t(s, h^t, h') \) denote bonds purchased at price \( q_t(h^t, h') \) that will pay off next period if \( h' \) is realized. Let \( A_t(s, h^t) \) denote money held by the household in its brokerage account at the end of the period. Since an inactive household of type \( s > 0 \) cannot transfer money between its brokerage account and its bank account, this household’s bond and money
holdings in its brokerage account must satisfy:

\[ B_{t-1}(s - 1, h^t) + A_{t-1}(s - 1, h^{t-1}) + (1 - \gamma)P_{t-1}(h^{t-1})y_{t-1}(h^{t-1}) - P_t(h^t)\tau_t(h^t) \geq \int q_t(h^t, h')B_t(s, h^t, h')dh' + A_t(s, h^t), \]

where \( P_t(h^t)\tau_t(h^t) \) are nominal lump-sum taxes. Each household’s real bondholdings must remain within arbitrarily large bounds. The analogous constraint for active households is:

\[ B_{t-1}(N - 1, h^t) + A_{t-1}(N - 1, h^{t-1}) + (1 - \gamma)P_{t-1}(h^{t-1})y_{t-1}(h^{t-1}) - P_t(h^t)\tau_t(h^t) \geq \int q_t(h^t, h')B_t(0, h^t, h')dh' + P_t(h^t)x_t(h^t) + A_t(0, h^t), \]

where \( P_t(h^t)x_t(h^t) \) is the active household’s transfer of money from brokerage to bank account.

At the beginning of period 0 initially inactive households begin with balances \( \bar{M}_0(s_0) \) in their bank accounts in the goods market. This quantity is the balance on the left side of (2) in period 0. For initially active households, the initial balance \( \bar{M}_0(0) \) in (4) is composed of an initial given balance \( \bar{Z}_0 \) and a transfer \( P_0x_0 \) of their choosing. Each household also begins with initial balance \( \bar{B}_0(s_0) \) in its brokerage account on the left side of constraints (5) and (6). The households initially have no money corresponding to \( A_{-1} \) in their brokerage accounts.

For each date and state and taking as given the prices and aggregate variables, each household of initial type \( s_0 \) chooses complete contingent plans for transfers, consumption, bond, and money holdings to maximize expected utility:

\[ \sum_{t=0}^{\infty} \beta^t \int u[c_t(s, h^t)]f_t(h^t)dh^t, \quad s = S(t, s_0) \]

subject to the constraints (1), (2), and (5) in those periods \( t \) in which \( S(t, s_0) > 0 \), and constraints (3), (4), and (6) in those periods \( t \) in which \( S(t, s_0) = 0 \).

Let \( B_t(h^t) \) be the total stock of government bonds. The government faces a sequence of budget constraints:

\[ B_{t-1}(h^t) = M_t(h^t) - M_{t-1}(h^{t-1}) + P_t(h^t)\tau_t(h^t) + \int q_t(h^t, h')B_t(h^t, h')dh'. \]
together with arbitrarily large bounds on the government’s real bond issuance. We denote the government’s policy for money injections as \( \mu_t(h^t) = M_t(h^t)/M_{t-1}(h^{t-1}) \). In period 0, the initial stock of government debt is \( \bar{B}_0 \) and \( M_0 - M_{-1} \) is the initial monetary injection. This budget constraint implies that the government pays off its initial debt with a combination of lump-sum taxes and money injections achieved through open market operations.

An *equilibrium* of this economy is a collection of prices, complete contingent plans for households, and government policy such that (i) taking as given prices and the government policy, the complete contingent plans solve each household’s problem, and (ii) the goods, money and bond markets all clear at each date and state:

\[
\frac{1}{N} \sum_{s=0}^{N-1} c_t(s, h^t) = y_t(h^t),
\]

\[
\frac{1}{N} \sum_{s=0}^{N-1} \left[ M_t(s, h^t) + A_t(s, h^t) \right] = M_t(h^t),
\]

\[
\frac{1}{N} \sum_{s=0}^{N-1} B_t(s, h^t, h') = B_t(h^t, h').
\]

To understand equilibrium money demand and asset prices, we examine the household’s first order conditions. Let \( \eta_t(s, h^t) \) denote Lagrange multipliers on the bank account constraints (2) and (4) of household \( s \) and let \( \lambda_t(s, h^t) \) denote Lagrange multipliers on the brokerage account constraints (5) and (6). Active households choose transfers \( x_t(h^t) \) to equate the multipliers on the bank and brokerage accounts:

\[
\eta_t(0, h^t) = \lambda_t(0, h^t). \tag{7}
\]

For households of type \( s \) the marginal utility of a dollar satisfies:

\[
\eta_t(s, h^t) = \beta_t u'[c_t(s, h^t)] f_t(h^t). \tag{8}
\]

The multipliers on the bank accounts satisfy the inequalities:

\[
\eta_t(s, h^t) \geq \int \eta_{t+1}(s + 1, h^t, h') dh'. \tag{9}
\]
which hold with equality if $Z_t(s, h^t) > 0$. Combining (8)-(9) we have the consumption Euler equations that determine a household’s money demand:

$$1 \geq \int \beta \frac{u'[c_{t+1}(s + 1, h^t, h')]}{u'[c_t(s, h^t)]} \frac{P_t(h^t)}{P_{t+1}(h^t, h')} \frac{f_{t+1}(h^t, h')}{f_t(h^t)} dh',$$

(10)

again, which holds with equality if $Z_t(s, h^t) > 0$. The evolution of the marginal utility of a dollar in the brokerage account is the same for all households and is determined by state contingent bond prices:

$$q_t(h^t, h') = \frac{\lambda_{t+1}(s + 1, h^t, h')}{\lambda_t(s, h^t)}.$$  (11)

Let $Q_t(h^t)$ denote the price in period 0 of one dollar delivered in the asset market in period $t$ following $h^t$. These prices satisfy the recursion $Q_t(h^t) = Q_{t-1}(h^{t-1}) q_t(h^{t-1}, h_t)$ for $t \geq 1$. From (11) we then have that for all households:

$$Q_t(h^t) = \lambda_t(s, h^t).$$  (12)

where we assume that initial conditions are such that the initial Lagrange multipliers on the brokerage account $\lambda_0(s_0)$ are the same for all households.\(^5\) Therefore from (7)-(8), we have that asset prices are determined by the marginal utility for active households:

$$Q_t(h^t) = \beta u'[c_t(0, h^t)] \frac{P_t(h^t)}{P_t(h^t)} f_t(h^t),$$

(13)

and state contingent bond prices are given by:

$$q_t(h^t, h') = \beta \frac{u'[c_{t+1}(0, h^t, h')]}{u'[c_t(0, h^t)]} \frac{P_t(h^t)}{P_{t+1}(h^t, h')} \frac{f_{t+1}(h^t, h')}{f_t(h^t)}.$$  (14)

The key asset price for our analysis is the price of an uncontingent bond paying interest $i_t(h^t)$ in nominal terms:

$$\frac{1}{1 + i_t(h^t)} = \int q_t(h^t, h') dh' = \int \beta \frac{u'[c_{t+1}(0, h^t, h')]}{u'[c_t(0, h^t)]} \frac{P_t(h^t)}{P_{t+1}(h^t, h')} \frac{f_{t+1}(h^t, h')}{f_t(h^t)} dh'.$$

(15)

\(^5\)This can be ensured by an appropriate choice of initial bond holdings $B_0(s_0)$. 

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To this point, we have made explicit reference to uncertainty in the notation so as to give a clear characterization of state contingent asset prices. For the remainder of the paper we suppress reference to histories $h^t$ to conserve notation. The inequalities governing money demand can therefore be written:

$$1 \geq E_t \left\{ \beta u'[c_{t+1}(s+1)] \frac{P_t}{P_{t+1}} \right\},$$  \hspace{1cm} (16)$$

with strict equality if $Z_t(s) > 0$, while the price for bonds can be written:

$$\frac{1}{1 + i_t} = E_t \left\{ \beta u'[c_{t+1}(0)] \frac{P_t}{P_{t+1}} \right\}. \hspace{1cm} (17)$$

In the brokerage accounts (5)-(6) we have allowed households the option of holding (noninterest bearing) money $A_t(s) \geq 0$ in the asset market. If the equilibrium nominal interest rate $i_t$ is always positive, no household would ever wish to do so since interest bearing bonds would dominate money. We focus on equilibria in which the nominal interest rate $i_t$ is always positive. After solving the model under this assumptions, one can always use (14) and (15) to check that the implied interest rates are positive.

### 3. How the model works

In this section, we solve our model for a special case in which agents have utility $u(c) = \log(c)$ and the paycheck parameter is $\gamma = 0$. In this specification, households of type $s$ spend a constant fraction $v(s)$ of their current money holdings and carry the remaining fraction $1 - v(s)$ into the next period, irrespective of the future path of money and prices. As a result of the fact that agents choose this simple pattern of expenditure we can, in this special case, solve analytically for the dynamic, stochastic equilibrium of our model. In this section we use this analytical example to first show how the price level responds sluggishly to an exogenous change in money growth and then show how inflation responds sluggishly to an exogenous change in the nominal interest rate. In the next section, we explore the quantitative implications of our model for illustrative examples in which household expenditure does vary with the future path of money and prices.
A. Money and velocity

In our model, households periodically withdraw money from the asset market and then spend that money slowly in the goods market to ensure it lasts until they have another opportunity to withdraw money from the asset market. As a result, households’ equilibrium paths for money holdings have the familiar *saw-toothed* shape characteristic of inventory-theoretic models of money demand. Here we discuss how this saw-toothed pattern of money holdings plays a key role in shaping our model’s implications for the dynamics of money, velocity, and prices.

Given our assumption that households have utility $u(c) = \log(c)$ and the paycheck parameter is $\gamma = 0$, households’ cash flows and spending are given by:

$$Z_t(s) = (1 - v(s))M_t(s) \quad \text{and} \quad P_t c_t(s) = v(s)M_t(s),$$

with

$$v(s) \equiv \frac{1 - \beta}{1 - \beta^{N-s}}.$$  

(19)

We refer to the fraction $v(s)$ as the *individual velocity of money*. Note that, in this special case of our model, these individual velocities of money are constant over time regardless of expectations of the future path of money and prices. Observe that these individual velocities $v(s)$ converge to $1/(N-s)$ as $\beta$ approaches one and thus, in this limit, approach the individual velocities obtained in the original Baumol-Tobin framework in which one assumes directly that households maintain constant nominal expenditure while inactive.

Given that individual velocities $v(s)$ are constant in this specification of our model, aggregate velocity at any date or state is simply a function of the distribution of money across these households with different individual velocities. If the nominal interest rate is positive, so that households do not hold any money in the asset market, money market clearing implies:

$$M_t = \frac{1}{N} \sum_{s=0}^{N-1} M_t(s).$$

(20)

Accordingly, we interpret $\{M_t(s)/M_t\}_{s=0}^{N-1}$ as the distribution of money holdings across households. Goods market clearing then implies the aggregate velocity of money is a weighted
average of the individual velocities of money where the weights are given by the distribution of money holdings across households:

\[ v_t \equiv \frac{P_t y_t}{M_t} = \frac{1}{N} \sum_{s=0}^{N-1} \frac{P_tC_t(s)}{M_t} = \frac{1}{N} \sum_{s=0}^{N-1} v(s) \left( \frac{M_t(s)}{M_t} \right). \]

(21)

In a steady-state with constant money growth, the distribution of money holdings across households of different types is constant. Hence aggregate velocity is also constant and the steady-state inflation rate is equal to the money growth rate. Therefore our model predicts that in the long-run, along a steady-state growth path, the price level and the money supply grow together while the aggregate velocity of money stays constant.

Out of steady-state, however, as a result of the fact that the individual velocities of money \( v(s) \) vary across households with difference values of \( s \), fluctuations in aggregate money growth cause fluctuations in the distribution of money across households and this in turn causes fluctuations in aggregate velocity. More specifically, the dynamics of prices, velocity, and money are determined by two factors: first, the differences in individual velocities \( v(s) \) across households of different types and second, the effect of a money injection on the distribution of money holdings across households. How these factors affect fluctuations in aggregate velocity can be understood intuitively as follows.

First, consider the differences in individual velocities \( v(s) \). These measures of individual velocity equal the flow of consumption obtained by that household relative to its money holdings at the beginning of the period. From (19), we immediately see that \( v(s) \) is increasing in \( s \). A household of type \( s \) close to zero holds a large stock of money relative to his consumption while a household of type \( s \) close to \( N - 1 \) holds only a small stock of money relative to his consumption.

Next consider how a money injection affects the distribution of money across households. From (18), the evolution of the distribution of money for households of type \( s = 1, \ldots, N - 1 \) is given by:

\[ \frac{M_t(s)}{M_t} = (1 - v(s - 1)) \frac{M_{t-1}(s - 1)}{M_{t-1}} \frac{1}{\mu_t}. \]

(22)

Since the distribution of money must sum to one, the money holdings of active households
Given an initial distribution of money holdings across households and a process for money growth \( \mu_t \), equations (22) and (23) completely characterize the equilibrium dynamics of the distribution of money holdings across households and hence the equilibrium dynamics of aggregate velocity and the price level.

This law of motion for the distribution of money has two key implications. First, in response to an increase in the money supply, aggregate velocity falls and thus the price level responds less than one-for-one with the money supply. Hence, prices in this model are sluggish in that they move less than would be predicted by the simplest quantity theory. Specifically, with large \( N \), the response of prices on impact is roughly half as large as the change in the supply of money. Second, there is a persistently sluggish response of prices to changes in the quantity of money, and the extent of persistence is increasing in \( N \).

To see these implications, consider first the impact effect of a money injection on velocity. By redistributing money towards the active households, an increase in the supply of money tilts the distribution of money holdings towards agents with low individual velocities and away from agents with high individual velocities, lowering aggregate velocity. To derive this result analytically, from (21), (22), and (23) observe:

\[
\frac{\partial}{\partial \mu_t} \left( v_t \mu_t \right) = v(0).
\]

Hence the elasticity of velocity with respect to money growth is given by:

\[
\frac{\partial \log(v_t)}{\partial \log(\mu_t)} = \left[ \frac{\partial (v_t \mu_t)}{\partial \mu_t} - v_t \right] \frac{1}{v_t} = \frac{v(0)}{v_t} - 1.
\]

Since the individual velocity of active households is less than aggregate velocity \( (v(0) < v_t) \), aggregate velocity declines when money growth increases. Given the exchange equation \( M_t v_t = P_t y_t \), we see that the price level does not respond on impact one-for-one with an increase in the money supply since that increase in the money supply leads to an endogenous decrease in aggregate velocity.
To see how this elasticity of velocity with respect to money growth depends on $N$, consider the equilibrium of this model as $\beta/\bar{\mu} \to 1$ (where $\bar{\mu}$ is the steady-state rate of money growth). In this limiting case, the nominal expenditure of each household is constant over time as in the original Baumol-Tobin framework. In this limit, $v(0) = 1/N$ and steady-state aggregate velocity is $\bar{v} = 2/(N + 1)$ so that, under these assumptions, the elasticity of aggregate velocity with respect to a change in money growth is:

$$\frac{\partial \log(v)}{\partial \log(\mu)} = -\frac{1}{2} \frac{N - 1}{N}$$
and

$$\frac{\partial \log(\pi)}{\partial \log(\mu)} = \frac{1}{2} \frac{N + 1}{N},$$

(24)

where these derivatives are evaluated at steady-state and where $\pi$ denotes the inflation rate.

We can see here that if $N = 1$, as in a standard cash-in-advance model, inflation responds one-for-one with the shock to money growth and velocity is constant. In contrast, for large $N$, inflation respond only about $1/2$ as much as money growth. This result follows from the geometry of money holdings implied by an inventory-theoretic model — a household that has just replenished its bank account will hold roughly twice as much money as an average household and hence have roughly half the velocity of the average household.

Figure 2 illustrates the dynamics of money, velocity and prices following a one-time shock to the money growth rate when households have log utility and paycheck parameter $\gamma = 0$. In response to this money injection, aggregate velocity falls and the price level responds less than one-for-one with the change in the money supply. Over time, aggregate velocity and prices rise, even overshooting their steady-state levels, and then gradually converge to steady-state with dampened oscillations.

These dynamics can be understood as follows. Since the money growth rate is high for only one period, from (22) we see that the households who were active at the time of the money injection carry an abnormally large stock of money until they next have the opportunity to transfer funds from their brokerage account. As shown in (19), their individual velocities rise each period until this next visit occurs. Thus, aggregate velocity remains below its steady-state level for a time initially as these agents have a low individual velocity and then rises past its steady-state level as the individual velocity for these agents rises. After $N$ periods these agents have spent all of their money and they visit the asset market again. If this were the only effect, we would expect aggregate velocity to return to its steady-state value in $N/2$ periods. However, we show in the Appendix that aggregate velocity remains below its steady-state
value for approximately \( N \log(2) \) periods, well over \( N/2 \) periods [\( \log(2) \approx 0.69 \), see equation (A8)]. In this sense, there is persistence in the sluggish response of prices to changes in the quantity of money and this persistence is increasing in \( N \). The periodic structure of the model introduces a sequence of dampened oscillations in velocity as the changes in the distribution of money holdings work their way through the system. After the first \( N \) periods, however, these effects of a money growth shock on velocity are quite small.

**B. Interest rates and inflation**

Until now, we have taken as given the path of money growth and examined our model’s implications for the responses of velocity and the price level to a shock to money growth. An alternative approach is to discuss monetary policy in terms of interest rates and solve endogenously for the responses of money growth, velocity, and inflation consistent with a shock to nominal interest rates. We turn now to such an analysis. Here we show our main result that, when \( N \) is large, inflation responds sluggishly to a shock to interest rates.

We demonstrate analytically that the response of inflation to a change in the nominal interest rate is sluggish in our model when \( N \) is large, again under the assumptions that \( u(c) = \log(c) \) and \( \gamma = 0 \) so that individual velocities \( v(s) \) are time-invariant. We solve for the responses of money growth, velocity, and inflation to a change in the nominal interest rate in a deterministic setting. Specifically, we assume that the nominal interest rate, inflation, money growth, and the distribution of money holdings across households (and hence velocity) are all initially at steady-state values corresponding to a constant interest rate \( \bar{i} \). We fix at \( t = 0 \) an increase in the nominal rate above steady-state, \( i_0 > \bar{i} \). We solve for the response of inflation, money growth, and velocity consistent with this change in the nominal interest rate at \( t = 0 \).

To solve for these responses, we use the pricing formula for nominal bonds (17). In a deterministic setting, this formula can be rewritten as a Fisher equation relating nominal interest rates, real interest rates and inflation between the current period and the next:

\[
\hat{r}_t = \hat{r}_t + \hat{\pi}_{t+1},
\]

where a “hat” denotes log-deviation from steady-state and where we repeatedly use approximations of the form \( \log(1 + \hat{i}_t) \approx \hat{i}_t \).
We use this Fisher equation to find a path for money growth such that the implied paths for inflation and the real interest rate are consistent with the exogenously specified path for the nominal interest rate. Recall that in our model changes in the path of money growth have an impact on velocity, inflation, and real interest rates, with the magnitude of these changes depending on $N$.

As a benchmark, consider first the responses of money growth, velocity, and inflation when $N = 1$ (so that our model is a standard constant-velocity cash-in-advance model). With $N = 1$, all households are active, velocity is constant, and the consumption of active households is also constant at $c_t(0) = y$. As a result, in this case, inflation is equal to money growth ($\hat{\pi}_{t+1} = \hat{\mu}_{t+1}$) and the real interest rate is constant ($\hat{r}_t = 0$). With these results, we see that any path of money growth that is consistent with our exogenously specified path of nominal interest rates must have money growth $\hat{\mu}_1$ and inflation $\hat{\pi}_1$ responding one-for-one to the change in the nominal interest rate in period 0. That is, $\hat{\mu}_1 = \hat{i}_0$. Clearly, in this case, the response of inflation from period $t = 0$ to $t = 1$ anticipated in period $t$ in response to the change in the nominal interest rate $\hat{i}_0$ is not at all sluggish.

Our solution of the model in this benchmark case with $N = 1$ is not yet complete as we have not solved for the equilibrium responses of money growth $\hat{\mu}_0$ and inflation $\hat{\pi}_0$ on impact, at date $t = 0$. It is well known that in this textbook cash-in-advance model ($N = 1$) this initial money growth rate and inflation rate are not determinate under an exogenous interest rate rule. We resolve the indeterminacy by choosing the particular path of money growth $\hat{\mu}_0$ so that, on impact, inflation from the last period to the current period does not respond to the change in the nominal interest rate in the current period. (That is, so that $\hat{\pi}_0 = 0$). In the model with $N = 1$, this is achieved by setting $\hat{\mu}_0 = 0$. This resolution of the indeterminacy is equivalent to assuming that the price level in period $t = 0$ does not respond to the change in the nominal interest rate and hence is consistent with the schemes used to identify shocks to monetary policy discussed in Christiano, Eichenbaum, and Evans (1999). Note that this resolution of the indeterminacy fixes the responses of money growth and inflation at date $t = 0$ by assumption. What is of interest are the equilibrium values of money growth and inflation at date $t = 1$, $\hat{\mu}_1$ and $\hat{\pi}_1$.

We now turn to the case of a general $N > 1$. At the end of this section, we show that this indeterminacy of the initial money growth rate $\hat{\mu}_0$ given the exogenous path of the nominal interest rate extends to our setting with $N > 1$. In particular, we show that, as in...
the case with \(N = 1\), there is a continuum of paths of money growth consistent with a given path of nominal interest rates. As in the case with \(N = 1\), with \(N > 1\), this continuum has only one dimension, that is, these paths can be indexed by their initial money growth rates \(\mu_0\) despite the fact that this model has a non-degenerate distribution of money holdings across households as a state variable that is absent from the model with \(N = 1\). Here, we again resolve this indeterminacy by examining the path of money growth consistent with \(\pi_0 = 0\). Given our assumptions about households’ preferences and \(\gamma = 0\) ensuring that individual velocities are constant over time, this path of money growth has initial money growth at its steady-state level \(\mu_0 = 0\).

Given this result that \(\mu_0 = 0\) under our resolution of the indeterminacy under an interest rate rule, we solve for the equilibrium responses of money growth \(\mu_1\), velocity \(v_1\), and inflation \(\pi_1\) to the change in the nominal interest rate \(\dot{i}_0\) in period \(t = 0\) by finding the value of money growth \(\mu_1\) such that the equilibrium responses of the real interest rate \(\dot{r}_0\) and inflation \(\pi_1\) are consistent with the assumed movement in the nominal interest rate. We solve for each of these responses in turn.

Consider first the response of the real interest rate \(\dot{r}_0\) to a change in money growth \(\mu_1\). This real interest rate is determined by the growth of the consumption of active households according to \(\dot{r}_0 = \dot{c}_1(0) - \dot{c}_0(0)\). Given that the individual velocity for active households \(v(0)\) is constant over time, the consumption of active households is given by \(c_t(0) = v(0)m_t(0)M_t/P_t\) where \(m_t(0) = M_t(0)/M_t\) is the share of the money supply held by active households. The real interest rate can therefore be written:

\[
\dot{r}_0 = \dot{m}_1(0) - \dot{m}_0(0) + \dot{\mu}_1 - \dot{\pi}_1. \tag{25}
\]

Given that initial inflation and money growth are at their steady-state values, and given our assumed initial conditions, the distribution of money holdings across households at date \(t = 0\) is equal to its steady-state value, and hence share of the money supply held by active households \(m_t(0)\) and velocity \(v_t\) are also equal to their steady-state values. Thus, we have:

\[
\dot{r}_0 = \left[ \frac{\partial \log(m(0))}{\partial \log(\mu)} + 1 - \frac{\partial \log(\pi)}{\partial \log(\mu)} \right] \dot{\mu}_1, \tag{26}
\]

where \(\partial \log(m(0))/\partial \log(\mu)\) and \(\partial \log(\pi)/\partial \log(\mu)\) are the elasticities of the share of money...
held by active households and of inflation with respect to money growth, both evaluated at the steady-state. These results then imply that the money growth required in period 1 to implement the nominal interest rate $\hat{i}_0$ in period 0 is given by:

$$\hat{\mu}_1 = \left[ \frac{1}{1 + \frac{\partial \log(m(0))}{\partial \log(\mu)}} \right] \hat{i}_0. \quad (27)$$

Thus, the real interest rate and inflation rate are given by:

$$\hat{r}_0 = \left[ 1 - \frac{\partial \log(\pi)}{\partial \log(\mu)} \right] \hat{i}_0, \quad \text{and} \quad \hat{\pi}_1 = \left[ \frac{\partial \log(\pi)}{\partial \log(\mu)} \right] \hat{i}_0. \quad (28)$$

As we can see from these formulas, the difference between our model with $N > 1$ and the standard model with $N = 1$ comes through the terms $\partial \log(m(0))/\partial \log(\mu)$ and $\partial \log(\pi)/\partial \log(\mu)$ reflecting the elasticities of the share of money held by active households and of inflation with respect to a money injection. In the standard model with $N = 1$, a money injection has no effect in terms of redistributing money holdings across households so that this elasticity is zero and the elasticity of inflation with respect to money growth is one. Thus, as we have seen, in this case, money growth and inflation respond one-for-one with the nominal interest rate and the real interest rate remains constant. In contrast, with $N > 1$, the elasticity of the share of money holdings of active households with respect to money growth is positive and grows large as $N$ gets large. Specifically, we show in the Appendix that, taking the limit as $\beta/\bar{\mu} \rightarrow 1$, the elasticity of the money share of active agents is approximately:

$$\frac{\partial \log(m(0))}{\partial \log(\mu)} = \frac{N - 1}{2}. \quad (29)$$

And, as we showed above, the elasticity of inflation is $\partial \log(\pi)/\partial \log(\mu) = (N + 1)/2N$ which is less than one for $N > 1$ and falls towards 1/2 as $N$ gets large. Plugging in these expressions for the elasticities gives:

$$\hat{\mu}_1 = \frac{2}{N + 1} \hat{i}_0, \quad \text{and} \quad \hat{\pi}_1 = \frac{1}{N} \hat{i}_0. \quad (29)$$
and that the real interest rate is:

\[ \hat{r}_0 = \frac{N - 1}{N} \hat{i}_0. \]  

(30)

The size of the response of real interest rates to a change in the nominal interest rate on impact is measured by \((N - 1)/N\) which is increasing in \(N\). For large \(N\), a given increase in the nominal interest rate gives rise to a nearly one-for-one increase in the real rate and almost no increase in expected inflation. The small response of inflation to a change in interest rates comes from segmented asset markets, i.e., only the fraction \(1/N\) of households that are active receive the entire increase in the money supply and so a given money injection has a disproportionally large impact on the marginal utility of a dollar for these households. Therefore, for large \(N\), a given change in nominal interest rates is obtained with a small change in money growth because that small change in the money supply has a large impact on real interest rates. Inflation is sluggish when \(N\) is large because this small change in money growth leads only to a small change in inflation.

In our model, increasing \(N\) has two effects that together contribute to the sluggish response of inflation — raising \(N\) increases the elasticity of the share of money held by active households and lowers the elasticity of inflation with respect to a change in money growth. The more important of these two effects is the first one. To see this, consider a constant velocity model in which agents are permanently divided into a fraction \(\lambda\) who are always active and a remaining fraction \(1 - \lambda\) who are never active (as in Alvarez, Lucas and Weber (2001)). Then, using the same resolution of the indeterminate price level, the relationship between real and nominal rates on impact is still given by (28) above. Since aggregate velocity is constant in this alternative model, \(\partial \log(\pi)/\partial \log(\mu) = 1\). It can also be shown that in this case the elasticity of the share of money held by the permanently active agents to money growth is \(\partial \log(m(0))/\partial \log(\mu) = (1 - \lambda)/\lambda\). Therefore the response of the real rate is:

\[ \hat{r}_0 = (1 - \lambda)\hat{i}_0. \]

So if the fraction of agents who are always active in this alternative model is \(\lambda = 1/N\), with \(N\) defined as in our model, then the alternative model with constant velocity gives the same response of inflation on impact to a change in the nominal interest rate as our model with
variable velocity. In this sense, our result that the response of inflation to a change in interest rates is sluggish is driven by mainly by asset market segmentation and not variable velocity.

We now present the indeterminacy result that holds in our model.

**Proposition.** Let \( \{i_t^*\}_{t=0}^{\infty} \) be a given sequence of interest rates and \( M_{t-1}^* (s) \) be the initial distribution of money holdings across households. Let \( \{M_t^*, M_t^* (s), c_t^* (s), P_t^*\}_{t=0}^{\infty} \) be an equilibrium corresponding to this sequence of interest rates and these initial conditions. Then, for each \( M_0 \) in an open neighborhood of \( M_0^* \), there exists a unique equilibrium \( \{M_t, M_t(s), c_t(s), P_t\}_{t=0}^{\infty} \) consistent with the same path of interest rates \( \{i_t^*\}_{t=0}^{\infty} \) and initial distribution of money holdings \( M_{-1}^* (s) \). In this alternative equilibrium, for \( t \geq N \), the distribution of consumption, money growth, and inflation are unchanged in that

\[
c_t(s) = c_t^*(s), \quad \frac{M_{t+1}}{M_t} = \frac{M_{t+1}^*}{M_t^*} \quad \text{and} \quad \frac{P_{t+1}}{P_t} = \frac{P_{t+1}^*}{P_t^*}.
\]

For periods \( t = 0, \ldots, N - 1 \), however, the distribution of consumption, money growth, and inflation depend on the value of \( M_0 \).

We prove this result in the Appendix.

This indeterminacy result reduces to the standard indeterminacy result when \( N = 1 \). (See, for example, Woodford 2003 chapter 2 for an extended discussion). And since for each \( M_0 \) there is a unique alternative equilibrium, even for \( N > 1 \) the indeterminacy is one-dimensional, as in the standard model. However, for \( N > 1 \), this indeterminacy result differs from the standard result in that the distribution of consumption across agents and the path of money growth and inflation differ across these equilibria for the first \( N \) periods.

The set-up used in this section, with \( u(c) = \log(c) \) and \( \gamma = 0 \), simplifies calculations since individual velocities \( v(s) \) are time invariant. In the case where \( \gamma > 0 \) or for general \( u(c) \) the dynamics are more complex, since households’ expenditure decisions will be forward-looking and consequently individual velocities will be time-varying. Below, we examine the quantitative implications of our model for alternative parameterizations numerically and find that our main result — that the response of inflation to a change in interest rates is small for large \( N \) — still holds. Moreover, in these quantitative examples the small impact effect on inflation persists for many periods.
4. Quantitative exercises

We have examined the predictions of our model in a simple version with log utility and paycheck parameter \( \gamma = 0 \). Under these assumptions, households act as if they are not forward-looking in choosing their nominal expenditure in the sense that the individual velocities of money \( v(s) \) are constant over time. In contrast, when \( \gamma > 0 \), the individual velocities of money \( v_t(s) \) chosen by households vary over time with changes in expectations of future inflation. In this case, we must solve the model numerically to characterize the responses of velocity to money growth and of inflation to interest rates. In this section, we characterize these responses numerically with values of the parameters \( N \) and \( \gamma \) chosen so that our model reproduces the average level of velocity for a broad monetary aggregate held by U.S. households. We then conduct two exercises with the model to illustrate its quantitative implications.

In the first exercise, we examine our model’s quantitative implications for the response of velocity to changes in money growth. In this experiment, we feed into the model the sequences of money growth and aggregate consumption shocks observed in U.S. data and compare the model’s implications for the short-run fluctuations in velocity with those observed in the data. We find that velocity in the model is highly correlated with velocity in the data. The magnitude of the fluctuations in the model, however, are significantly smaller than the magnitude of those observed in the data.

In the second exercise, we examine the responses of money, prices, and velocity in the model to a monetary policy shock represented here as a persistent movement in the nominal interest rate similar to those estimated as the response of the Federal Funds rate to a monetary policy shock in the VAR literature. Here we find that the corresponding impulse responses of money and prices implied by our model are similar to those estimated in the VAR literature. In particular, inflation in the model responds quite sluggishly to the change in interest rates.

A. Choosing \( N \) and \( \gamma \)

We interpret the separation between the household’s two financial accounts, its brokerage and bank accounts, as corresponding to a situation where financial intermediaries such as brokerage houses, mutual fund companies, pension and insurance funds, hold and manage portfolios of high-yield securities on behalf of individual households. Households make
infrequent transfers to and from this managed portfolios to a low-yield bank account. This differs from the traditional interpretation of Baumol-Tobin models, where withdrawals are made from a safe interest-bearing asset into cash. Instead, we interpret the bank accounts as a broader monetary aggregate, and the account from which these transfers are made as one with high-yield managed portfolios of risky and riskless assets. Our interpretation is the same as used in the models of Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007).

Based on this interpretation, for our quantitative exercises we choose the paycheck parameter to reflect regular wage and salary income automatically deposited in its bank account. We choose $\gamma = 0.6$ to match the fraction of personal income that is received as wage and salary disbursements observed in the data.$^6$

The steady-state velocity implied by our model is a simple function of the parameters $N$ and $\gamma$. In particular, holding $N$ fixed, the model's implications for steady-state velocity are an increasing function of the paycheck parameter $\gamma$ since the automatic deposit of paychecks into households' bank accounts allows for faster circulation of money. In the example with $u(c) = \log(c)$ and $\gamma = 0$ that we used for intuition in the previous sections, with $\beta/\bar{\mu}$ close to one, aggregate velocity is given by $\bar{v} = 2/(N+1)$. With $\gamma > 0$, for $\beta/\bar{\mu}$ close to one, aggregate velocity is well approximated by $\bar{v} = 2/(N+1)(1-\gamma)$, which increases as $\gamma$ increases.

We choose the parameters $N$ and $\gamma$ to match the average velocity of a broad money aggregate — the sum of U.S. households' holdings of currency plus demand, savings, and time deposits. This aggregate is essentially U.S. households’ holdings of M2 less retail money market funds. In choosing this money aggregate, we consider currency and bank accounts in the data as corresponding to funds held in households’ bank accounts in the model, while stocks, bonds, money market and other mutual funds in the data as corresponding to assets held in households’ brokerage accounts.

We choose a relatively broad concept of money because, as documented in the Appendix, U.S. households pay a substantial cost in terms of foregone interest to hold these assets relative to the interest available on retail money market mutual funds or short-term Treasury securities. The standard Baumol-Tobin framework has typically been applied to a narrow definition of money such as currency or M1, with households facing a substantial opportunity cost in terms of forgone interest to hold this narrow monetary aggregate. The

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$^6$From Table 2.1 of the National Income and Product Accounts, we observe that this fraction has been equal to 60% on average over the period from 1959-2001.
evidence that we present on the opportunity cost to households of various types of accounts indicates that there is not a substantial difference in the opportunity cost of M1 and the components of M2 that we consider.

From the Flow of Funds Accounts, we observe that U.S. households hold a large quantity of M2 less retail money market funds — on the order of 1.5 to 2 times annual personal consumption expenditure. We use this average level of velocity of 1.5 to guide our choice of \( N \) and \( \gamma \) for the quantitative results that follow. We match those observations in our model by assuming that households transfer money between their brokerage accounts and bank accounts very infrequently — on the order of only once every year to once every three years. While such an assumption may seem implausible, the microeconomic evidence summarized in Vissing-Jorgensen (2002) on the frequency with which households trade financial assets held outside of their bank accounts so defined is consistent with these assumptions.

To parameterize our model to reproduce an average annual velocity of money of 1.5, we choose the length of a period to be one month and use \( N = 38 \) so that with \( \gamma = 0.6 \), the model produces an average velocity of 1.5. These are the values that are required to account for the average level of low-yielding assets held by U.S. households given our assumption that the paycheck parameter \( \gamma = 0.6 \). These parameters imply, within the model, that households transfer funds between their brokerage accounts and bank accounts very infrequently. Now we argue that this assumption is not inconsistent with the available microeconomic evidence on the frequency with which agents trade financial assets held outside of their bank accounts.

The first set of such microeconomic data concerns the frequency with which households trade equity. Such data are relevant since a household would have to trade equity to rebalance its portfolio between funds held in its bank account and equity held in its brokerage account. The Investment Company Institute (1999) conducted an extensive survey of households’ holdings and trading of equity in 1998. They report on the frequency with which households traded stocks and stock mutual funds in 1998. They report that 48% of the households that held individual stocks outside of their retirement accounts neither bought nor sold any stock in 1998 and 63% of the households that held stock mutual funds outside of their retirement accounts neither bought nor sold mutual funds in 1998. Since a household would have to buy or sell some of these assets to transfer funds between these higher yielding assets held in a brokerage account and a lower yielding bank account, these data, interpreted in light of our model, would indicate choices of \( N \) ranging from roughly 24 (for roughly 1/2 of households...
trading these risky assets at least once within the year) to roughly 36 (for roughly 1/3 of households trading within the year).\(^7\)

The second set of microeconomic data is that presented by Vissing-Jorgensen (2002). She studies micro data on the frequency of household trading of stocks, bonds, mutual funds and other risky assets obtained from the Consumer Expenditure Survey. In figure 6 in her paper, she shows the fraction of households who bought or sold one of these assets over the course of one year as a function of their financial wealth at the beginning of the year. She finds that the fraction of agents who traded one of these assets ranges from roughly 1/3 to 1/2 of the households owning these assets at the beginning of the year. Again, given our interpretation that households hold stocks, bonds, mutual funds and other risky assets in their brokerage accounts, these data would lead us to choose \(N\) between 24 and 36.

**B. The response of velocity to U.S. money and consumption shocks**

We now study the implications of our model for velocity in the short run when we feed in the money growth and aggregate consumption shocks observed in the U.S. data. We use monthly data on M2 as our measure of the monetary aggregate \(M_t\), and we use monthly data on the deviation of the log of real personal consumption expenditure from a linear trend as our measure of the shocks to aggregate endowment \(y_t\). To solve for households’ decision rules in the model, we estimate a VAR relating the current money growth rate and aggregate consumption to 12 lags of these variables and use this VAR as the stochastic process governing the exogenous shocks. We then generate the model’s implications for velocity by feeding in the actual series for these shocks. To compare the implications of our model for the dynamics of money and velocity in the short-run to the data, we detrend the series implied by the model using the HP-filter.

Consider the implications of our model with \(N = 38\) months and \(\gamma = 0.6\). In Figure 8, we show the HP-filtered series for velocity implied by our model together with the corresponding HP-filtered series for velocity from the data. The correlation between velocity in

\(^7\)These data may also overstate the frequency with which households transfer funds between their equity accounts and their transactions accounts since some of the instances of equity trading are simply a reallocation of the equity portfolio. The Investment Company Institute reports that more than 2/3 of those households that sold individual shares of stock in 1998 reinvested all of the proceeds, while 57% of those households that sold stock mutual funds reinvested all of the proceeds. In the context of our model, reallocation of the household portfolio in the asset market is costless and does not generate cash that can be used to purchase goods.
the model and the data is 0.6. In the figure, we have used different scales in plotting the series from the model and the data. These different scales reflect the fact that the standard deviation of velocity in the data is 2.6 times larger than the standard deviation of velocity in the model.

Given that we have used nothing but steady-state information to choose the parameters of this model, we regard the high correlation between velocity from the model and the data as a remarkable success. Observe that if we had chosen \( N = 1 \), as in a standard cash-in-advance model, velocity as implied by the model would be constant at one regardless of the shock process and, hence, the correlation between velocity in the model and velocity in the data would be zero. We interpret this finding as offering support for the hypothesis that a substantial portion of the negative correlation between the short run movements of velocity and the ratio of money to consumption is due to the endogenous response of velocity to changes in the ratio of money to consumption.

C. The response to a shock to the interest rate

We now consider the response of inflation to a shock to the nominal interest rate. A large literature estimates the response of the macroeconomy to a monetary policy shock modeled as a shock to the Federal Funds rate. The consensus in this literature is that a monetary policy shock is associated with a persistent increase in the short-term nominal interest rate, a persistent decrease in the money supply and, at least initially, little or no response in the price level (Christiano, Eichenbaum, and Evans 1999).\(^8\)

To simulate the effects of a monetary policy shock, we solve for a money growth path consistent with an exogenous, persistent movement in the short-term nominal interest rate. This raises two technical issues. First, recall from Proposition 1 that there is an indeterminacy in this model if the nominal interest rate is exogenous. In equilibrium, there are many paths for money growth all consistent with the same exogenously specified path for nominal interest rates.\(^9\) In the quantitative experiment below, we resolve this indeterminacy in the same way that we did in Section 3. We choose the unique path for money growth that, on impact, leaves the price level unchanged. A second technical issue is that in this model

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\(^9\)The indeterminacy result of Section 3 is for \( u(c) = \log(c) \) and \( \gamma = 0 \) but extends to the case of general isoelastic preferences and \( \gamma > 0 \).
the endogenous dynamics last exactly $N$ periods. The matrix describing the equilibrium dynamics of endogenous variables has its $N$ eigenvalues all exactly equal to zero. This implies that, if the interest rate is set at its steady-state value but the initial distribution of money holdings is not, then steady-state will be reached in exactly $N$ periods. The repetition of the eigenvalues also implies that the matrix that described equilibrium dynamics is not diagonalizable, and hence, this model cannot be solved using standard methods such as those outlined by Blanchard and Kahn (1980) or Uhlig (1999). In an online technical appendix to this paper we develop a specific solution method for this model based on the use of the generalized Schur form that makes use of the information that the eigenvalues of the matrix describing equilibrium dynamics are all equal to zero.$^{10}$

We now study the quantitative implications of our model with $N = 38$ and $\gamma = 0.6$ having solved for money growth consistent with the log of the short-term gross interest rate following an AR(1) process with persistence $\rho = 0.87$. This persistence produces a response of the nominal interest rate to a monetary policy shock similar to that estimated by Christiano, Eichenbaum, and Evans (1999).

Figure 10 shows the impulse responses of inflation, money growth and velocity growth following a persistent increase in the nominal interest rate. The model produces a persistent liquidity effect both in the sense that an increase in the nominal interest rate is associated with a fall in money growth and in the sense that an increase in the nominal interest rate is associated, at least initially, with an increase in the real interest rate of roughly the same size. Inflation is sluggish, responding only slowly to the increase in the nominal interest rate.

Figure 11 shows the same impulses responses but for the levels of the variables rather than their growth rates. As a result of the negative co-movement of money and velocity, the aggregate price level appears “sticky,” showing little or no response to the shock to interest rates for at least the first 12 months. It is only after 12 months have passed that the money stock and the price level begin to rise together in the manner that would be expected in a flexible price model following a persistent increase in the nominal interest rate.

In this exercise, $N$ is large so that markets are highly segmented and on impact the increase in the nominal interest rate has a nearly one-for-one increase in the real interest rate

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$^{10}$This online technical appendix is available at www.atkeson.net\andy. We also found that direct methods based on use of the generalized Schur form, as suggested by Klein (2000) and others, did not correctly identify that the matrix describing equilibrium dynamics had eigenvalues all equal to zero. This appears to be a numerical issue since this methodology should work in cases with repeated eigenvalues.
but little effect on expected inflation. On impact, an active household holds a disproportion-
ately low share of the money supply and low share of real consumption but their expected
consumption growth is positive so that the real interest rate rises above steady-state. This
initial liquidity effect persists both because the persistent nature of the shock tilts the distri-
bution of money holdings and real consumption away from currently active households and
because large $N$ implies that the consequences of low money holdings stay with a household
for many periods after it has been active.

This quantitative exercise indicates that our model can account for a substantial delay
in the response of inflation to an exogenous shock to the nominal interest rate and it does so
simply because of the endogenous off-setting response of velocity to that interest rate shock.

5. Conclusion

In this paper, we have put forward a simple inventory-theoretic model of the demand
for money and shown, in that model, that the price level does not respond immediately to
an exogenous increase in the money supply and that expected inflation does not respond
immediately to an exogenous increase in the nominal interest rate. Instead, there is an
extended period of price sluggishness that occurs because the exogenous increase in the money
supply leads, at least initially, to an endogenous decrease in the velocity of money and an
extended period of inflation sluggishness that occurs because of asset market segmentation.
We have argued that if this simple model is used to analyze the dynamics of money and
velocity using a relatively broad measure of money, then it produces sluggish responses of
the price level and inflation similar to that estimated in the VAR literature on the response
of the economy to monetary policy shocks.

There is a huge literature that looks to model the sluggish responses of prices and
inflation in an alternative framework in which prices are sticky because firms adjust prices
infrequently. This literature includes models in which firms set prices according to time-
rules (Caplin and Leahy 1991, Dotsey, King and Wolman 1999, Golosov and Lucas 2007,
Midrigan 2006), or, more recently, on the basis of slowly updated information (Mankiw and
Reis 2002, Woodford 2003a). We see our results on the sluggish response of prices and
inflation as complementary to this literature.

Consider first the link between our results on the sluggishness of prices and the sticky-
price literature. The dynamics of money, velocity, and nominal expenditure are linked by the identity \( Mv = Pc \). If we consider as a benchmark a model that embeds a theory of money demand in which velocity is constant, then, in that model, the response of nominal expenditure \( Pc \) to an exogenous shock to money \( M \) is given directly by the dynamics of money following the shock. This is true whether prices are sticky or flexible. In the data shown in figure 1, however, we see instead a strong negative correlation between money and velocity suggesting sluggish dynamics of nominal expenditure. Neither a sticky-price nor a flexible price model can reproduce these dynamics of money, velocity, and hence nominal expenditure unless it embeds a more sophisticated theory of money demand than a constant-velocity benchmark. We see our inventory-theoretic model of money demand as a natural candidate for that theory.

Now consider the link between our results on the sluggishness of expected inflation in response to a change in the short-term nominal interest rate and the sticky-price literature. The short-term nominal interest rate, the real interest rate, and expected inflation are linked through the Fisher equation. Sticky-price models built on a representative household framework link the real interest rate to the growth of marginal utility for the representative household, and hence aggregate consumption, through an intertemporal Euler equation. Thus, in these models, if expected inflation responds sluggishly to a change in the nominal interest rate, then the growth rate of marginal utility for the representative household must respond strongly to a change in the nominal interest rate. Hence, capturing simultaneously a sluggish response of expected inflation and aggregate consumption to a change in the short-term nominal interest rate has been a challenge for these models. Our model complements this work by abandoning the assumption of a representative household for pricing assets. In our model, the real interest rate is linked to the growth of marginal utility for active households, not for a representative household consuming aggregate consumption. Hence, as we have seen in our model, we can produce a sluggish response of expected inflation to a change in the nominal interest rate even if aggregate consumption is constant and hence has no response at all to a change in the nominal interest rate.

In keeping this model simple, we have abstracted from a number of issues that might play an important role in the development of a more complete model. First, we have simply assumed that households have the opportunity to transfer funds between their brokerage and bank accounts only every \( N \) periods and have not allowed households to alter the timing of
these transactions after paying some fixed cost. For work along these lines see Khan and Thomas (2007). We have also assumed that output is exogenous. Could a version of our model with endogenous output generate the hump-shaped impulse response of output to a monetary policy shock as estimated in the VAR literature (e.g., by Christiano, Eichenbaum and Evans 2005)? Answering that question is a promising direction for future research.

Having abstracted from heterogeneity across households, differences among monetary aggregates, as well as other potentially important issues, we cannot draw many specific quantitative conclusions from this analysis. We do, however, conclude with the broader point that the dynamics of money demand may play an important role in accounting for the sluggish response of prices to changes in monetary policy.
References


Appendix

A1. Data

All data is monthly 1959:1-2006:12 and seasonally adjusted. We measure the price level $P$ as the personal consumption expenditures chain-type price index with a base year of 2000 from the Bureau of Economic Analysis (BEA). We measure real consumption $c$ as personal consumption expenditure on nondurables and services from the BEA deflated by $P$. We measure the money supply $M$ as the M2 stock from the Board of Governors of the Federal Reserve System. We define velocity as $v \equiv Pc/M$.

Alternative measures of the short-run correlation of money and velocity

Here we document the robustness of the negative correlation between $\log(M/c)$ and $\log(v)$ using alternative detrending methods to characterize the short-run fluctuations in money and velocity. We report statistics for HP-filtered data based on the usual smoothing parameter for monthly data $\lambda = 1600 \times 3^2$. These are the statistics reported in the main text. In addition, we compute statistics for HP-filtered data with the higher smoothing parameter $\lambda = 1600 \times 3^4$ recommended by Ravn and Uhlig (2002) and we report statistics for monthly differences and for annual differences. No matter how the short-run fluctuations are measured, we find that there is a pronounced negative correlation between $\log(M/c)$ and $\log(v)$ and that the standard deviation of $\log(v)$ is almost as high or higher than the standard deviation of $\log(M/c)$.

<table>
<thead>
<tr>
<th></th>
<th>HP-filtered 1600 × 3²</th>
<th>HP-filtered 1600 × 3⁴</th>
<th>differenced monthly</th>
<th>differenced annual</th>
</tr>
</thead>
<tbody>
<tr>
<td>correlation</td>
<td>–0.91</td>
<td>–0.86</td>
<td>–0.88</td>
<td>–0.63</td>
</tr>
<tr>
<td>standard deviation</td>
<td>1.25</td>
<td>1.33</td>
<td>1.01</td>
<td>0.98</td>
</tr>
</tbody>
</table>

Table 1: Correlation and relative standard deviation of $\log(v)$ to $\log(M/c)$ based on alternative measures of short-run fluctuations.

Evidence on opportunity cost of holding broad money

We measure the opportunity costs of monetary assets using data collected by the Monetary Services Index project of the Federal Reserve Bank of St Louis. We measure the opportunity cost of an asset as the short term treasury rate less the own rate of return on the asset in question. We take the short term treasury rate and own rates of return on currency and demand deposits from the spreadsheet ADJSAM.WKS available from the website of the Federal Reserve Bank of St Louis. We take the own rate of return on M2 from the Board of Governors of the Federal Reserve System. All opportunity cost data is monthly 1959:1-2006:2 and seasonally adjusted.

As is clear from Table 2, the average opportunity cost of holding demand deposits and M2 is roughly similar, on the order of 200 basis points. Both opportunity costs have fallen somewhat in recent years.
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
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<tr>
<td>currency</td>
<td>4.91</td>
<td>5.61</td>
<td>3.45</td>
</tr>
<tr>
<td>demand deposits</td>
<td>1.80</td>
<td>2.25</td>
<td>0.85</td>
</tr>
<tr>
<td>M2</td>
<td>2.08</td>
<td>2.30</td>
<td>1.64</td>
</tr>
</tbody>
</table>

Table 2: Opportunity costs of various monetary assets. All opportunity costs measured as the average short-term Treasury rate less own rate reported in percentage points.

**A2. Algebra of steady-state money distribution and elasticities**

Let period utility be \( u(c) = \log(c) \) and set the paycheck parameter to \( \gamma = 0 \). Individual velocity \( v(s) \) is time invariant and given by \( v(s) = (1 - \beta)/(1 - \beta^{N-s}) \) for \( s = 0, 1, \ldots, N-1 \). As \( \beta \to 1 \) these are just \( v(s) = 1/(N-s) \). For households \( s = 1, \ldots, N-1 \) the distribution of money holdings satisfies:

\[
\frac{M_t(s)}{M_t} = [1 - v(s - 1)] \frac{M_{t-1}(s - 1)}{M_t} \frac{1}{\mu_t}, \tag{A1}
\]

with money market clearing implying:

\[
\frac{1}{N} \frac{M_t(0)}{M_t} = 1 - \frac{1}{N} \sum_{s=1}^{N-1} [1 - v(s - 1)] \frac{M_{t-1}(s - 1)}{M_t} \frac{1}{\mu_t}. \tag{A2}
\]

**Steady-state money distribution**

Now consider a steady-state with \( \mu = 1 \) in the limit as \( \beta \to 1 \). Iterating on the steady-state version of (A1) and using the formula for individual velocity shows that the steady-state money holdings of household \( s \) are related to the holdings of an active household by:

\[
\frac{M(s)}{M} = \frac{N-s}{N} \frac{M(0)}{M}. \tag{A3}
\]

We solve for the money holdings of an active household using money market clearing:

\[
1 = \frac{1}{N} \sum_{s=0}^{N-1} \frac{M(s)}{M} = \frac{1}{N} \frac{M(0)}{M} \sum_{s=0}^{N-1} \frac{N-s}{N} = \frac{1}{N} \frac{M(0)}{M} \left[ N - \frac{1}{2} (N-1) \right],
\]

so that \( M(0)/M = 2N/(N + 1) \) and we have the complete solution for the distribution:

\[
\frac{M(s)}{M} = 2 \frac{N-s}{N+1}, \tag{A3}
\]
for \( s = 0, 1, \ldots, N - 1 \). We can now solve for steady-state aggregate velocity:

\[
\bar{v} = \frac{1}{N} \sum_{s=0}^{N-1} v(s) \frac{M(s)}{M} = \frac{1}{N} \sum_{s=0}^{N-1} \frac{2}{N-s} \frac{N-s}{N+1} = \frac{2}{N+1},
\]

as used in the text.

**Elasticities with respect to money growth**

In the text we derived the elasticity of aggregate velocity with respect to money growth. Evaluated at steady-state:

\[
\frac{\partial \log(v)}{\partial \log(\mu)} = \frac{v(0)}{\bar{v}} - 1 = -\frac{1}{2} N - 1.
\]

And since the aggregate endowment \( y \) is constant, the elasticity of inflation with respect to money growth evaluated at steady-state is:

\[
\frac{\partial \log(\pi)}{\partial \log(\mu)} = \frac{\partial \log(v)}{\partial \log(\mu)} + 1 = \frac{v(0)}{\bar{v}} + 1 = \frac{1}{2} N + 1.
\]

To derive the elasticity of the share of money held by active households with respect to money growth, multiply (A2) by \( M_t \) and differentiate both sides with respect to \( M_t \) to get:

\[
\frac{\partial M_t(0)}{\partial M_t} = N.
\]

So that evaluated at steady-state:

\[
\frac{\partial \log(M(0))}{\partial \log(\mu)} = N \frac{M}{M(0)} = N \frac{N+1}{2N} = \frac{N+1}{2}.
\]

Now let \( m(0) \equiv M(0)/M \) denote the steady-state money share. Then we have:

\[
\frac{\partial \log(m(0))}{\partial \log(\mu)} = \frac{\partial \log(M(0))}{\partial \log(\mu)} - 1 = \frac{N+1}{2} - 1 = \frac{N-1}{2},
\]

as used in the text.

**A3. Dynamic response of velocity to a money growth shock**

Here we analytically characterize the impulse response of velocity to a money growth shock. The dynamics of velocity following a money growth shock are determined by the subsequent evolution of the distribution of money over time. It is easiest to analyze the
dynamics of velocity following a shock in a log-linearized version of the model. We proceed
in two steps. First, we provide an ARMA representation of the dynamics of the money
distribution. Second, we map the ARMA representation into a formula for the impulse
response of velocity that is exact (up to the log-linearization) for the first \( N-1 \) periods after
a shock.

**ARMA representation**

Two sets of equations govern the dynamics of the distribution of money. First, there
is an equation requiring that the sum of the log deviations of the fractions of money held by
agents of type \( s \) is zero:

\[
0 = m(0)\hat{m}_t(0) + \sum_{s=1}^{N-1} m(s)\hat{m}_t(s)
\]

where steady-state money shares are \( m(s) \equiv M(s)/M \) and \( \hat{m}_t(s) \equiv \log[m_t(s)/m(s)] \). Second
there is a set of equations for \( s = 1, ..., N-1 \) governing the evolution of the money shares:

\[
\hat{m}_t(s) = \hat{m}_{t-1}(s-1) - \hat{\mu}_t,
\]

where these equations follow from the fact that individual velocities \( v(s) \) are time-invariant. Rearranging the first equation and using \( m(s) = 2(N-s)/(N+1) \) we have for active households:

\[
\hat{m}_t(0) = -\sum_{s=1}^{N-1} \frac{m(s)}{m(0)}\hat{m}_t(s) = -\sum_{s=1}^{N-1} \frac{N-s}{N}\hat{m}_t(s),
\]

and after iterating on the transitions for inactive households:

\[
\hat{m}_t(s) = \hat{m}_{t-s}(0) - \sum_{k=1}^{s} \hat{\mu}_{t-k+1},
\]

for \( s = 1, ..., N-1 \). Combining these gives an ARMA representation of the dynamics of the
money distribution:

\[
\hat{m}_t(0) = -\sum_{s=1}^{N-1} \frac{N-s}{N}\hat{m}_{t-s}(0) + \sum_{s=1}^{N-1} \frac{N-s}{N} \sum_{k=1}^{s} \hat{\mu}_{t-k+1}.
\]
Impulse response for velocity

The log deviation of velocity can be written:

\[ \hat{v}_t = \frac{1}{N} \sum_{s=0}^{N-1} \hat{m}_t(s). \]

using \( v(s) m(s) = 2/(N + 1) = \bar{v} \) for all \( s \). Differencing this once and simplifying gives

\[ \Delta \hat{v}_t = \frac{1}{N} \sum_{s=0}^{N-1} \Delta \hat{m}_t(s) = \frac{1}{N} \left[ \hat{m}_t(0) - \hat{m}_{t-N}(0) - (N - 1) \hat{\mu}_t + \sum_{s=1}^{N-1} \hat{\mu}_{t-s} \right], \]

which repeatedly uses \( \hat{m}_{t-1}(s - 1) = \hat{m}_t(s) + \hat{\mu}_t \) to cancel terms in the sum. Let the economy start in steady-state for \( t < 0 \) and consider a given shock \( \hat{\mu}_t \) at date \( t \) with \( \hat{\mu}_{t+k} = 0 \) for all \( k > 0 \). For the first \( N - 1 \) periods after a shock, the terms \( \hat{m}_{t-N}(0) \) and the sum \( \sum_{s=1}^{N-1} \hat{\mu}_{t-s} \) are zero so that \( \Delta \hat{v}_t = [\hat{m}_t(0) - (N - 1) \hat{\mu}_t]/N \). We can solve this for \( \hat{m}_t(0) = N \Delta \hat{v}_t + (N - 1) \hat{\mu}_t \) and use the ARMA representation for the money share of active households to get an ARMA representation of velocity growth that is exact for the first \( N - 1 \) periods:

\[ \Delta \hat{v}_t = -\sum_{s=1}^{N-1} \frac{N - s}{N} \Delta \hat{v}_{t-s} - \frac{1}{2} \frac{N - 1}{N} \hat{\mu}_t, \]

(using \( \hat{\mu}_{t-s} = 0 \) for the first \( N - 1 \) periods). Rearranging terms to write this in levels we get:

\[ \hat{v}_t = \frac{1}{N} \sum_{s=1}^{N-1} \hat{v}_{t-s} - \frac{1}{2} \frac{N - 1}{N} \hat{\mu}_t, \]

(this time using \( \hat{v}_{t-N} = 0 \) for the first \( N - 1 \) periods). When \( N \) is large so that \( (N - 1)/N \approx 1 \) this implies the impulse response of the log of velocity over the first \( N - 1 \) periods is given by

\[ \hat{v}_{t+k} = \frac{1}{2} \left( 1 + \frac{1}{N} \right)^{k+1} - 1. \]  

(A8)

This starts with \( \hat{v}_t = -1/2 \), for large \( N \) it crosses zero at roughly \( k = N \log(2) \) and then rises above zero until \( k = N \).
A4. Proof of indeterminacy proposition

Using that \( u(c) = \log(c) \) and \( \gamma = 0 \) so that \( P_t c_t(0) = v(0) M_t(0) \), and that:

\[
u'(c_t(0))/P_t = \frac{1}{c_t(0) P_t} = \frac{1}{v(0) M_t(0)},
\]

the sequence of \( M_t(0) \) that supports the interest rate \( \{i_t^*\}_{t=0}^{\infty} \) must satisfy:

\[
\frac{M_{t+1}(0)}{M_t(0)} = (1 + i_t^*) \beta, \quad t = 0, 1, ...
\]

or:

\[
M_{t+1}(0) = M_0(0) \beta^t \prod_{j=0}^{t} (1 + i_j^*).
\] (A9)

For future reference, we can write equation (A9) as:

\[
M_{t-1-s}(0) = M_0(0) \beta^{t-1-s-1} \prod_{j=0}^{t-1-s-1} (1 + i_j^*),
\] (A10)

which applies if \( t - 1 - s \geq 0 \) or \( s \leq t - 1 \).

Now again using that \( u(c) = \log(c) \) and \( \gamma = 0 \) we have:

\[
M_t(s) = (1 - v(s - 1)) M_{t-1}(s - 1), \quad s = 1, ..., N,
\]

which we can substitute into:

\[
M_t(0) = N M_t - \sum_{s=1}^{N-1} (1 - v(s - 1)) M_t(s - 1),
\]

to obtain:

\[
M_t(0) = N (M_t - M_{t-1}) + \sum_{s=0}^{N-1} \theta_s M_{t-1-s}(0),
\] (A11)

where the coefficients \( \theta_s \) are given by \( \theta_s \equiv v(s) \left[ \prod_{j=0}^{s-1} (1 - v(j)) \right] > 0 \).

It is easy to verify that any sequence of \( \{M_t - M_{t-1}\} \) for \( t \geq 0 \) and \( \{M_t(0)\} \) for \( t \geq -N + 1 \) that solves equation (A11) completely characterizes an equilibrium.
Now we specialize equation (A11) for three different types of time periods. For \( t = 0 \) we have:

\[
M_0(0) = N (M_0 - M_{s_1}) + \sum_{s=0}^{N-1} \theta_s M_{s-1-s}(0). 
\]  
(A12)

For \( t = 1, 2, ..., N - 1 \) we can break the sum in two parts and use the expression for \( M_{t-1-s}(0) \) in terms of interest rates, equation (A10), so we have:

\[
M_t(0) \quad \text{(A13)}
\]

\[
= N (M_t - M_{t-1}) + \sum_{s=0}^{t-1} \theta_s M_{t-1-s}(0) + \sum_{s=t}^{N-1} \theta_s M_{t-1-s}(0)
\]

\[
= N (M_t - M_{t-1}) + \sum_{s=0}^{t-1} \theta_s M_0(0) \beta^{t-1-s-1} \prod_{j=0}^{t-1-s-1} (1 + i_j^*) + \sum_{s=t}^{N-1} \theta_s M_{t-1-s}(0),
\]

and using the expression for the interest rate equation (A9) again:

\[
M_0(0) \beta^{t-1} \prod_{j=0}^{t-1} (1 + i_j^*) \quad \text{(A14)}
\]

\[
= N (M_t - M_{t-1}) + \sum_{s=0}^{t-1} \theta_s M_0(0) \beta^{t-1-s-1} \prod_{j=0}^{t-1-s-1} (1 + i_j^*) + \sum_{s=t}^{N-1} \theta_s M_{t-1-s}(0). 
\]

Finally, for \( t = N, N + 1, ... \) we have:

\[
M_t(0) = N (M_t - M_{t-1}) + \sum_{s=0}^{N-1} \theta_s M_0(0) \beta^{t-1-s-1} \prod_{j=0}^{t-1-s-1} (1 + i_j),
\]

and inserting the expression for \( M_t(0) \) based on the interest rates:

\[
M_0(0) \beta^{t-1} \prod_{j=0}^{t-1} (1 + i_j^*) \quad \text{(A15)}
\]

\[
= N (M_t - M_{t-1}) + \sum_{s=0}^{N-1} \theta_s M_0(0) \beta^{t-1-s-1} \prod_{j=0}^{t-1-s-1} (1 + i_j). 
\]

Now we are ready to construct the path of the remaining variables for an equilibrium that support the interest rate path \( \{i^*_t\}_{t=0}^{\infty} \). We do this in three steps, one for each type of time period. We do this for an arbitrary value of \( M_0 \).

**Step a.** Solve for \( M_0(0) \). For \( t = 0 \), \( M_0(0) \) is a function of predetermined variables,
$M^*_j, M^*_j(0)$ for $j < 0$, and $M_0$. Thus for the given value of $M_0$ there is a unique value of $M_0(0)$.

Step b. Solve for $M_t(0)$ and $M_t$ for $t = 1, ..., N-1$. Equation (A14) gives one equation in one unknown, namely $M_t - M_{t-1}$, given $M_0(0)$. Using these equations recursively, using the initial conditions $M_0$ found in step a, we can solve for $M_1, ..., M_{N-1}$.

Step c. Solve for $M_t$ for $t \geq N$. Given the initial condition $M_{N-1}$ found in step b, equation (A15) can be used to solve for $M_t$ for $t \geq N$.

Steps a-b-c show that for any given $M_0$ there is a unique way to construct an equilibrium that supports the path of interest rates $\{i_t^*\}_{t=0}^\infty$.

We now show that for any equilibrium that supports the interest rate sequence $\{i_t^*\}_{t=0}^\infty$, the distribution of cash $M_t(s)/M_t$ for $s = 0, ..., N-1$ for all $t \geq N$ is the same. Using equation (A9) for $t \geq N$ in:

$$M_t(0) = N M_t - \sum_{s=1}^{N-1} (1 - v (s - 1)) M_t (s - 1),$$

we obtain:

$$M_t(0) = N M_t - \sum_{s=1}^{N-1} (1 - v (s - 1)) \prod_{k=1}^{s-1} v (k) M_{t-k} (0),$$

and using equation (A9) we get:

$$M_0 (0) \beta^{t-1} \prod_{j=0}^{t-1} (1 + i_j^*)$$

$$= N M_t - \sum_{s=1}^{N-1} (1 - v (s - 1)) \prod_{k=1}^{s-1} v (k) M_0 (0) \beta^{t-k-1} \prod_{j=0}^{t-k-1} (1 + i_j^*),$$

which shows that the path of $M_t$ is proportional to $M_0 (0)$ for $t \geq N$. Finally, equation (A9) implies the path of $M_t(s)$ is proportional to $M_0 (0)$, which establishes the desired result. This in turn immediately implies that $M_t(s)/M_t = M^*_t(s)/M^*_t$ and $M_{t+1}/M_t = M^*_t/M^*_t$, and thus that $c_t (s) = c_t^* (s)$ $P_{t+1}/P_t = P^*_t/P^*_t$ for $t \geq N$.

Finally, the qualification that $M_0$ has to be close to $M_0^*$ insures that in the values constructed for $M_t (0)$ during the periods $t = 0, ..., N-1$ are all strictly positive.

QED.
Figure 1: Short run negative correlation of $M/c$ and $v$ log\(v\)

$$\text{log(M/c)}$$

Correlation between log\(M/c\) and log\(v\) = \(-0.90\)
Figure 2: Money up, velocity down, prices sluggish
Figure 3: Model and data velocity (deviations from HP trend)

- Correlation between data and model = 0.60
- Standard deviation of data relative to standard deviation of model = 2.6

N = 38, γ = 0.60
Figure 4: Large liquidity effects

Response to unit shock

- Nominal interest rate
- Inflation
- Money growth
- Velocity growth

Months after shock
Figure 5: Sluggish price response to persistent interest rate shock

- log(v)
- Nominal interest rate
- log(P)
- log(M/c)

Response to unit shock

Months after shock