19. SIMPLE LINEAR REGRESSION IV

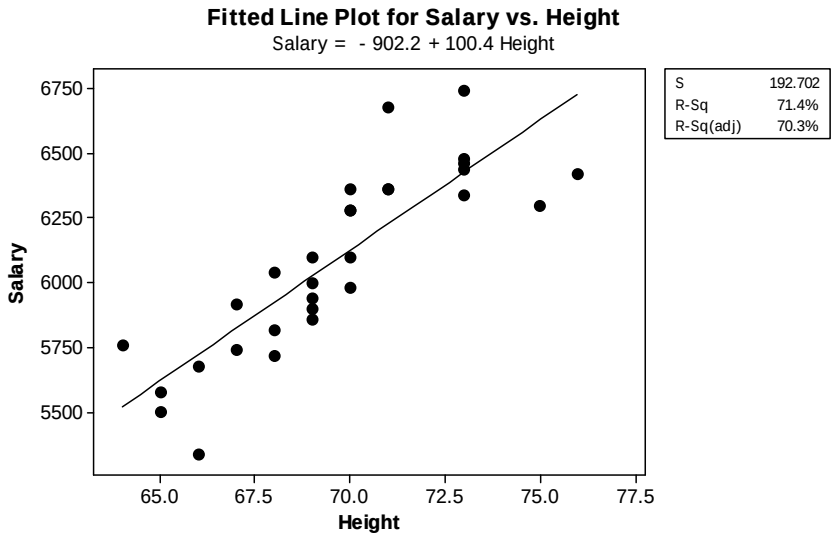
The Coefficient of Determination, R^2

Once we have decided that β is not zero, so that a linear relationship seems to exist between *x* and *y*, it is useful to measure the *strength* of this linear relationship. Such a measure is provided by the **coefficient of determination**, R^2.

To understand R^2, note that one of the aims of regression analysis is to study the relationship between *x* and *y*, i.e., to try to use the value of *x* to "explain" *y*.

• Keep in mind, though, that this "explanation" may not be one of cause and effect.

Recall the Salary vs. Height data.



Regression Analysis: Salary versus Height

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Analysis of Variance
Source         DF       SS       MS  F-Value  P-Value
Regression      1  2590433  2590433    69.76    0.000
Error          28  1039754    37134
Total          29  3630187

Model Summary
      S    R-sq  R-sq(adj)  R-sq(pred)
192.702  71.36%     70.34%      65.65%

Coefficients
Term       Coef  SE Coef  T-Value  P-Value
Constant   -902      837    -1.08    0.290
Height    100.4     12.0     8.35    0.000

Regression Equation
Salary = -902 + 100.4 Height
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If we look at the *y*'s (the salaries) as a data set, we note that they are not all the same; the *y*'s exhibit variability. A rough measure of this variability is the total sum of squares,

$$SST = \sum_{i=1}^{n} (y_i - \bar{y})^2 \ .$$

Note that SST is (*n*–1) times the sample variance of the *y*'s.

If there is a linear relationship between *x* and *y*, then the variability of the *y*'s is not due entirely to chance fluctuations.

Instead, the fact that the salaries are different can be partially "explained" by the fact that the heights (*x*) are different. Of course, salary is not completely explained by height, so part of the variability in the salaries remains unexplained.

• Interestingly, the variability in salaries can be broken into two parts, the first attributed to differences in height, and the second attributed to other factors not yet accounted for.

• We have the following important formula:

$$\sum_{i=1}^{n}(y_i-\bar{y})^2=\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2+\sum_{i=1}^{n}(y_i-\hat{y}_i)^2\ ,$$

or $SST = SSR + SSE$.

Interpretation:
The variability of the *y*'s (*SST*) can be broken into two parts, *SSR* + *SSE*.

• The first part is the regression sum of squares, $SSR=\sum_{i=1}^{n}(\hat{y}_i-\bar{y})^2$.

This is simply (n–1) times the sample variance of the fitted values.

Since $\hat{y}_i=\hat{\alpha}+\hat{\beta}\,x_i$ the fluctuation in the $\hat{y}_i$'s is completely "explained" by the fluctuation in the x_i's.

Thus, *SSR* is the part of the variability of *y* which can be "explained" by *x* (or, more precisely, by the *regression* of *y* on *x*).

• The second part is the residual sum of squares, $SSE=\sum_{i=1}^{n}(y_i-\hat{y}_i)^2$.

The residuals represent the part of *y* which remains after we try to "explain" *y* based on *x*.

(This is clear, since $y_i=\hat{y}_i+e_i$.)

Thus, SSE represents the part of the variability of *y* which is "unexplained" by *x*.

This unexplained variability is attributed to chance, or to other factors not yet considered.

Now, we define the **coefficient of determination** by $R^2=\frac{SSR}{SST}$.

• We see that R^2 measures the proportion of the variability of *y* that is "explained" by *x*.

An equivalent definition is $R^2=1-\frac{SSE}{SST}$.

• It can be shown that $0 \le R^2 \le 1$.

We will get $R^2 = 1$ if, and only if, all points lie exactly on a straight (non-horizontal) line.

The closer R^2 is to 1, the stronger the linear relationship between *x* and *y*.

If R^2 is near zero, then almost none of the variability of *y* is explained by *x*, so the linear relationship is weak.

We will get $R^2 = 0$ if, and only if, $\hat{\beta}=0$.

This can happen in a variety of ways, including:
(1) All *y*'s lie on a horizontal line;

(2) The data points lie on a parabola $y = a + b\,x^2$, which peaks in the middle of the range of the equally-spaced *x*'s.

• Note that in (2), there is a clear nonlinear relationship but no linear relationship whatsoever! So keep in mind that R^2 only measures the strength of the *linear* relationship.

If R^2 is large, we say that *x* and *y* are "highly correlated". In this case, there is a strong linear relationship between *x* and *y*.

If R^2 is near zero, we say that *x* and *y* are nearly "uncorrelated". In this case, the linear relationship is weak.

Note that R^2 is the square of the **correlation coefficient** r.
But the use of r is potentially misleading.

For example, if $r = 0.8$, then only 64% of the variability in y is "explained" by x.

The coefficient of determination R^2 contains the same information as r (except for the sign of the slope), and has the interpretation as the proportion of explained variability.

Note that a high correlation should not be taken as evidence of a causal relationship. Consider the Rotten Tomatoes example. The ultimate explanation of a high Audience Score is presumably that the movie was popular. This is presumably what causes the Critics Score to also be high. So it's not that high Critics Scores are the *cause* (or the explanation, in any meaningful sense) of high Audience Scores.

Eg: Since 71.36% of the variability in salary is "explained" by height, the linear relationship is strong. Height is a good predictor of salary. The other 28.64% of the variability in salary is unexplained, but we could try to include more variables in our regression. This would definitely improve the R^2.
(We will return to this point later.)

Eg: For the Stock Market example, the Minitab output shows that $R^2 = 0.0056$. Only 0.56% of the variability in Today's returns is "explained" by Yesterday's returns.

Although the linear relationship is statistically significant (low p-value), it is still quite weak (low R^2).

The forecast of today's return based on yesterday's return will not be very accurate.

Regression Analysis: Today versus Yesterday

Analysis of Variance

Source	DF	SS	MS	F-Value	P-Value
Regression	1	61.4	61.3510	65.00	0.000
Error	11534	10887.0	0.9439		
Total	11535	10948.3			

Model Summary

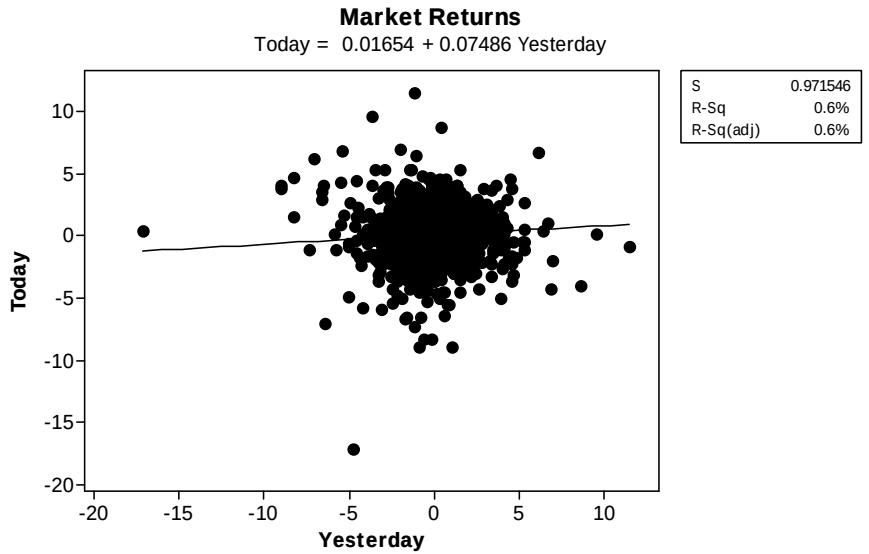
S	R-sq	R-sq(adj)	R-sq(pred)
0.971546	0.56%	0.55%	0.46%

Coefficients

Term	Coef	SE Coef	T-Value	P-Value
Constant	0.01654	0.00905	1.83	0.067
Yesterday	0.07486	0.00928	8.06	0.000

Regression Equation

Today = 0.01654 + 0.07486 Yesterday



The New York Times

Mutual Funds Report

SUNDAY, OCTOBER 10, 1999

Investors Pay Extra, But Some Managers Just Mimic the S.& P.

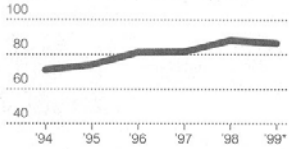
By Richard A. Oppel, Jr.

To get your arms around the idea, it helps to understand a measure called "R-squared," which gauges the correlation between a fund and whatever index — usually the Standard & Poor's 500-stock index — it is measured against. If a fund has an R-squared of, say, 90, that means that 90 percent of the fund's movement can be explained by movements in the compo-

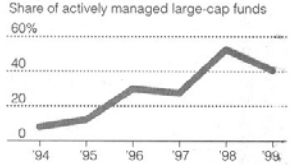
Closer Resemblance

More mutual funds are looking like index funds. To measure how closely a fund tracks an index, experts rely on a statistic called R-squared. The closer it is to 100, the closer a fund's correlation to the index. Lately, the average R-squared of actively managed large-capitalization equity funds, in relation to the Standard & Poor's 500 index, has risen well above 80, and the percentage of those over 90 has quintupled since 1994.

AVERAGE R-SQUARED
Actively managed large-cap equity funds



R-SQUARED OVER 90
Share of actively managed large-cap funds



Source: Morningstar Inc.

*As of August 31.

The New York Times