

10. EXPECTED VALUE AND VARIANCE FOR DISCRETE RANDOM VARIABLES

Eg 1: Consider all families in the world having three children. What should be the average number of girls in these families?

- Expected Value: $E(X) = \sum_{\text{all } x} xp(x)$ if X is a discrete RV.
- $E(X)$ is a weighted average of the possible values of X . The weights are the probabilities of occurrence of those values.
- $E(X)$ is the long run average value of X if the experiment is repeated many times. Therefore, $E(X)$ may be thought of as the theoretical mean of the random variable X .
- $E(X)$ is the mean of the population of values of X obtained over many repetitions of the experiment. So we can write

$$E(X) = \mu.$$

Eg 2: In the game of craps, for a pass-line bet of \$1 without odds, the player wins \$1 with probability 0.493. What is the average casino take, per dollar wagered?

Eg: Using the probability distribution for the duration of the World Series for two equally matched teams, the expected length of the series is

$$4(0.125)+5(0.25)+6(0.3125)+7(0.3125) \\ = 5.8125.$$

Eg: For the New York Lotto, we would need to play an average of 18,009,460 times to hit a First Prize jackpot. Thus, at 2 tickets a week, it would take an average of 173,167 years to win a jackpot.

(We will verify these numbers later).

Eg 3: Compute expected value for a Let's Make a Deal game where you win \$1 if you end up with the red card, and you lose \$1 if you end up with a black card. Try two strategies: Never Switch, Always Switch.

Eg 4: Consider a roulette doubling (“Martingale”) system.

Start by betting \$1 on Black.

Keep doubling your bet until Black is finally rolled.

Once this happens, you will have a net gain of \$1!

Because of table limits (Maximum Bet = \$1000), you can’t double more than 9 times. Is this a good system?

(Compute expected value).

Mini-Lab: Suppose a couple has one boy. How many more children will they need to have to get a girl?

Class: What do you think the answer is?

How do you allow for randomness?

Can you give a mathematical expression for the answer?

Try flipping a coin. What did you get, Heads or Tails?

Keep flipping. How many more flips did you need to get a different outcome from the first?

Can we compute an (approximate) expected value from this data? How?

- **Variance:** $\sigma^2 = V(X) = E[(X - \mu)^2] = \sum_{\text{all } x} (x - \mu)^2 p(x)$

if X is a discrete RV.

- The variance of X is the expected value of the RV $(X - \mu)^2$.

Thus, $V(X) = \text{Mean Squared Deviation of } X \text{ from its own mean, } \mu$.

- **Standard Deviation:** The standard deviation of X is $\sigma = \sqrt{V(X)}$.
- σ measures the amount of fluctuation in X over a large number of repetitions of the experiment.

Note: σ^2 and σ defined above are *theoretical* variance and standard deviation of X.

You don't need any *data* to compute them. You just need to know the distribution of X.

The mean of a random variable is NOT the same thing as a sample mean. The variance of a random variable is NOT the same thing as a sample variance.

Eg 5: What is the standard deviation of the player's profit for a \$1 craps bet?

Eg 6: What is the standard deviation of the number of girls in families with three children?

- **Covariance:** $Cov(X, Y) = E[(X - \mu_X)(Y - \mu_Y)]$

Measures how X and Y move together.

Here, $\mu_X = E[X]$ and $\mu_Y = E[Y]$.

- **Correlation:** $Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$

Standardized version of Covariance, always between -1 and 1 .
The closer $|Corr(X, Y)|$ is to 1 , the stronger is the linear relationship between X and Y.

- **Variance of a Sum of Two Random Variables:**

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y).$$

We can't just add the variances, in general.

Let's verify the formula: Since $E[X + Y] = \mu_X + \mu_Y$

we get $Var(X + Y) = E[\{(X - \mu_X) + (Y - \mu_Y)\}^2]$

$$= E[(X - \mu_X)^2] + E[(Y - \mu_Y)^2] + 2E[(X - \mu_X)(Y - \mu_Y)]$$

$$= Var(X) + Var(Y) + 2Cov(X, Y).$$

• **Variance of a Portfolio:** If a and b are constants (not random) and $Z=aX+bY$ then after a bit more work, we get

$$\text{Var}(Z) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y).$$

Example: X =Next Month's Return on Intel
 Y =Next Month's Return on Hershey.

If $a = b = .5$ then

Z =Next Month's Return on a Portfolio
(Equal Dollar Amount of Intel, Hershey).

How risky is the portfolio?

Suppose $\text{Var}(X)=(.13)^2$, $\text{Var}(Y)=(.07)^2$ and $\text{Cov}(X, Y)=.0019$.

Then there is a weak, positive linear relationship between X and Y ,

Since $\text{Corr}(X, Y)=.0019/[(.13)(.07)]=.21$.

The variance of the portfolio return is

$$\text{Var}(Z)=(.25)(.13)^2 + (.25)(.07)^2 + 2(.25)(.0019) = .0064.$$

The risk of the portfolio is measured by the standard deviation,

$$\sqrt{\text{Var}(Z)} = .08.$$