

11. COUNTING RULES

In order to calculate probabilities, we often need to count how many different ways there are to do some activity.

For example, how many different outcomes are there from tossing a coin three times? (Many people think that the answer is 6. Is this right?)

To help us to count accurately, we need to learn some counting rules.

Multiplication Rule: The number of ways of doing a compound activity is the product $n_1 n_2 \cdots n_K$, if the activity can be broken into K stages where the number of choices at the k^{th} stage is n_k , for each of the possible choices at the previous stages.

Eg: If we toss a coin 3 times, the total number of outcomes is $(2)(2)(2) = 8$:

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT.

Eg: If Race 1 has 10 horses and Race 2 has 9 horses, how many daily doubles are possible? (In a daily double, the bettor must correctly pick the winner of both races).

Eg: In a race with 9 horses, how many exactas are possible? Trifectas? (In an exacta, you must correctly pick the first two finishing horses, in order. In a trifecta, you must correctly pick the first three horses, in order).

Eg: How many 7-digit phone numbers are possible? (First digit cannot be 0 or 1). Do you see why we need area codes?

- **Factorial Notation:** If n is a positive integer, define $n!$ (“ n factorial”) by

$$n! = n (n-1) (n-2) \cdots (2) (1) .$$

Define $0! = 1$.

Eg: $3! = 3 \cdot 2 \cdot 1 = 6$.

- **Choice Rule:** The number of different ways of choosing x objects from a total of n objects is

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} = \frac{n(n-1)\cdots(n-x+1)}{x!}$$

if the order does not matter.

- $\binom{n}{x}$ (“ n choose x ”) is the number of subsets of size x which can be formed from a set of n objects. The order in which we list the elements is not important to us. The quantity $\binom{n}{x}$ is sometimes called a **binomial coefficient**. Since we can select x objects by *not* selecting $n-x$ objects, we see that $\binom{n}{x} = \binom{n}{n-x}$.

Eg: Consider the set of $n = 3$ stocks, {IBM, Digital, Apple}. There are three possible portfolios of two stocks from this set, namely {IBM, Digital}, {IBM, Apple}, {Digital, Apple}. This result is consistent with the Choice Rule, since $\binom{3}{2} = 3$.

- For a set of 1000 stocks on the NYSE there are more than 8.25 trillion mutual funds consisting of an equal number of shares of 5 stocks. We start to see why there are so many mutual funds. In fact, the existing funds don't even scratch the surface of what's possible.

Eg: The number of flights connecting n cities

(disregarding direction) is $\binom{n}{2} = \frac{n(n-1)}{2}$.

This may require a huge number of planes.

If all of these flights must originate at a hub city, the required number of flights becomes just $n - 1$. This is one big reason why hubs are used.

Arrangement Rule: The number of arrangements of n things taken x at a time is $n(n-1) \cdots (n-x+1)$. In this case, we care about the order in which the elements are listed.

(Both the Choice Rule and the Arrangement Rule can be proved from the Counting Rule).

Eg: In a team with 9 batters, there are a total of $9! = 362,880$ batting orders. For the “top of the batting order” (first three batters) there are $(9)(8)(7) = 504$ arrangements.

Eg: What is the probability of obtaining a flush (5 cards of the same suit) in a 5 card poker hand dealt from a fresh deck?

Eg: What's the probability of a full house in poker?
(A full house consists of 3 cards of one kind and two of another kind, for example, three kings and two aces).

Eg: [My poker trick. Do trick, calculate probability].

Eg: In the game of craps, we are interested in the total from tossing two dice. This total is a discrete random variable. Here is the probability distribution:

<u>x</u>	<u>p(x)</u>
2	$1/36 = 0.0278$
3	$2/36 = 0.0556$
4	$3/36 = 0.0833$
5	$4/36 = 0.1111$
6	$5/36 = 0.1389$
7	$6/36 = 0.1667$
8	$5/36 = 0.1389$
9	$4/36 = 0.1111$
10	$3/36 = 0.0833$
11	$2/36 = 0.0556$
12	$1/36 = 0.0278$

To see this, imagine that one die is red, the other is green. There are $36 = 6 \times 6$ equally likely simple events, consisting of the red and green outcomes (in order).

Note that 7 is the most likely outcome, while 2 and 12 are the least likely.

A “crap” is a 2, 3 or 12. The probability of throwing a “crap” on any given roll is $4/36 = 0.11$.

In a 5-dollar “Odds bet”, you can bet, for example, that a 6 will come up before a 7. Such a bet is paid at odds of 6 to 5, that is, a winning five dollar bet is paid six dollars, in addition to the original five dollars wagered.

These are actually the correct odds (the only completely fair bet in the casino!).

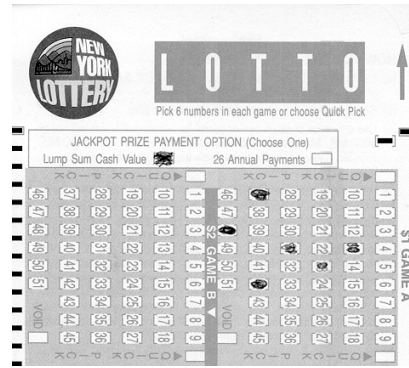
This follows since there are 5 ways to throw a six on any given roll, and 6 ways to throw a seven.

The expected loss on an odds bet is zero!

On the average, you will lose 6 times out of every 11 (total of 30 dollars, that is, six 5-dollar bets) and win 5 times out of every 11 (total of $5 \times 6 = 30$ dollars).

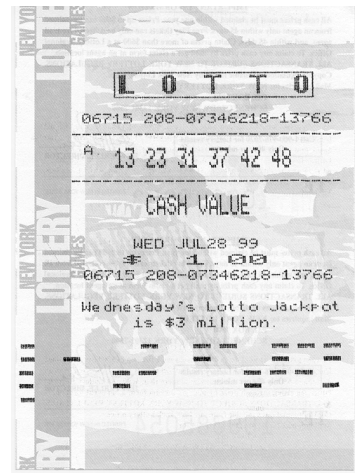
Lotteries

New York State runs a 6/51 Lotto. Twice each week, the Lottery Commission selects 6 winning numbers (and one bonus number) at random from 1,...,51. Prior to this drawing, players select 6 numbers from a ticket of 51.



CHANCES AND PRIZE LEVELS		
PRIZE LEVEL	NUMBER OF WINNING NUMBERS MATCHED ON ONE GAME PANEL	CHANCES OF WINNING ON ONE GAME PANEL
First	6	1: 18,009,460
Second	5 Plus Bonus	1: 3,001,577
Third	5	1: 68,218
Fourth	4 Plus Bonus	1: 27,287
Fifth	4	1: 1,269
Sixth	3 Plus Bonus	1: 952
Free Play	3	1: 68

Overall chances of winning on a \$1 play --- 1:60.25
NOTE: Prizes are paid on a pari-mutuel basis. The Free Play must be claimed at a retailer location within 45 days of the draw date.



To win the First Prize (main jackpot), a ticket must match all 6 winning numbers. Since there are $\binom{51}{6}$ possible selections, the probability of winning First Prize is

$$\frac{1}{\binom{51}{6}} = \frac{1}{18,009,460}$$

To win Second Prize, the player must match 5 of the 6 winning numbers as well as the bonus number. There are $\binom{6}{5}(1) = 6$ ways to make such a match, so the probability of winning Second Prize is

$$\frac{6}{18,009,460} = \frac{1}{3,001,577}$$

To win Third Prize, the player must match 5 of the 6 winning numbers without matching the bonus number. Since there are 44 non-winning non-bonus numbers, the number of ways to win Third Prize is

$$\binom{6}{5} \binom{44}{1} = (6)(44) = 264.$$

Thus the probability of winning Third Prize is

$$\frac{264}{18,009,460} = \frac{1}{68,218}$$

Since the winning numbers are selected at random, there is no strategy for number selection that will improve your odds of winning.

But you can still maximize the *amount* you win (if you ever do hit a jackpot) by avoiding popular choices such as the “lucky numbers” 7, 11, 21 or the “dates” 1,...,31, or any obvious patterns on the ticket.

The idea is to share the jackpot with as few others as possible. Other things to avoid are arithmetic progressions, previous winning combinations, and “hot” or “cold” numbers.

It's a lottery odd-ity

Wacky numbers pay off bigger, say pros

By **JON R. SORENSEN**

Daily News Albany Bureau Chief

Avoid traditional lucky numbers like 7, 11 and 21, said New York University business Prof. Clifford Hurvich. Lots of players select them.

Hurvich also advised against picking numbers under 31 — the better to avoid favorite birth dates often used by Lotto players.

And stay away from selections based on straight-line patterns or the corner num-

bers on Lotto tickets. For example, that rules out the combination 1, 10, 19, 28, 37 and 46 — the numbers on the left side of the ticket.

Thousands of hopeful Lotto players will let the state Lottery Division's computer select a Quick Pick number for them. Don't do it, said Hurvich. The computer may be fast, but it won't necessarily give you the unique combination needed to score lone Lotto loot.

Hurvich has been teaching these lessons to his students for years. It paid off big four years ago for one student, who won \$10,000 playing numbers seldom picked by the Lotto-playing public.

"I asked him, 'So where's my cut?' and he told me I had to wait until next time," Hurvich

said.

Of course, if you're really serious about winning, you can always try to scrape together \$12.9 million for Lotto tickets. At two games to a \$1 ticket, that gives you enough to cover each of the 25,827,165 possible combinations with the 54 Lotto numbers.

A \$25 million pay-off on a \$12.9 million investment sounds good. But if someone else hits the winning combo too, well, you figure it out.

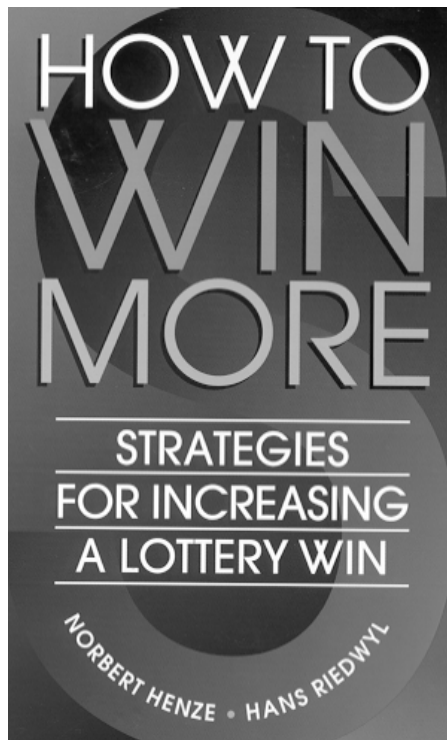
A final word of caution. Browne and Hurvich conceded they don't bother playing the lottery.

HOW TO WIN MORE

STRATEGIES
FOR INCREASING
A LOTTERY WIN

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- The most popular number combinations are also the most foolish. (Many are given in Chapter 5.)
- It is impossible to improve the *odds* of winning any legitimate lottery (for example, by charting numbers).
- The notion of picking so-called hot or cold numbers is nonsense. (You might as well throw your money away.)
- Although nice-looking tickets may be popular, they are also unprofitable.
- Never copy suggested numbers from "Nighthawks," lottery horoscopes ("Zodiac Best Bets"), or "Dr. X's Computer Generated Lotto Picks" (to name only a few supposed consulting experts).
- It is extremely important to avoid picking the winning combinations of past drawings (although each has the same odds of winning as any other combination for future drawings).



Combination						Frequency
7	14	21	28	35	42	16,771
1	2	3	4	5	6	15,415
1	4	15	22	26	46	11,821
5	10	15	20	25	30	10,789
3	8	13	18	23	28	9,034
8	16	24	32	40	48	6,823
3	13	23	33	43	48	6,629
5	10	13	22	27	33	6,340
1	6	11	16	21	26	6,093
18	20	29	32	33	35	5,456

Table 5.1 Frequencies of the 10 most popular number combinations for the October 29, 1988, draw of the California Lotto 6/49.

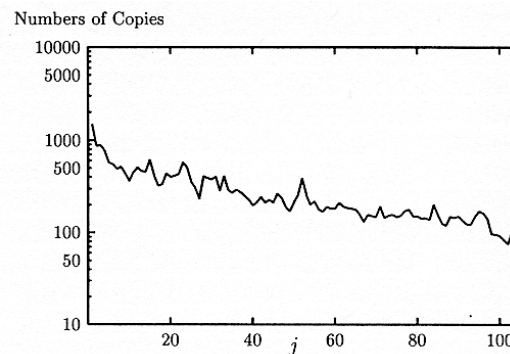


Figure 5.2 Number of combinations representing copies of the winning numbers of the j th preceding week for the August 27, 1994, draw of the Finnish Lotto 7/39.

A good (but not foolproof) way to avoid popular numbers is to use "Quick Pick": Instead of selecting your own numbers, let the computer select them at random.

Jackpot Winners			
	1993	1994	Total
Frequency			
Manual Play	20	16	36
Quick-Pick Play	29	33	62
Mean Payoff			
Manual Play	\$7,470,546	\$7,466,040	\$7,468,543
Quick-Pick Play	\$9,013,138	\$11,855,858	\$10,526,198

Table 6.1 Number of jackpot winners in the California Super Lotto 6/51, 1993 and 1994.

Quick Pick gave a higher average payoff than manual play, for the 98 winning tickets considered. About 63% (62/98) of the winning tickets were Quick Pick. This is very close to the percentage of Quick Pick sales for all tickets sold, which is about 62% (Data not shown).

So Quick Pick doesn't improve your chances of winning, but it does increase the average dollar amount of jackpot winnings.