Backus & Zin/April 14, 1999

Binomial Models 2

1. Interest rate caps

2. Black-Scholes and binomial valuation of caps

3. Swaps

4. Swaptions

5. Black-Scholes and binomial valuation of swaptions

6. Summary and final thoughts

1. Interest Rate Caps

- Terminology:
  - An *interest rate cap* pays the difference between a reference rate and the cap rate, if positive (a series of call options on an interest rate)
  - An *interest rate floor* pays the difference between the floor rate and a reference rate, if positive (a series of put options on an interest rate)
  - A *caplet* (*floorlet*) is a single payment in a cap (floor)
  - An *interest rate collar* is a long position in a cap plus a short position in a floor (it puts upper and lower bounds on floating interest payments)

- Example: a 5-year semiannual cap would typically pay the difference between 6-month LIBOR and the cap rate every six months starting in 12 months (the first payment is generally dropped) and ending in 60 months

- It's convenient to measure time in periods between payments: t is trade date ("now"), payments occur at $t+2$, $t+3$, etc

- Day counts (which we ignore) follow LIBOR conventions

- Timing: if there are m payments per year and the notional principal is 100, a caplet's cash flows of

$$m^{-1}(Y_{t+j} - K)^+$$

  are observed at $t+j$ but paid one period (six months?) later (think about how LIBOR is paid)

2. Black-Scholes Valuation of Caps

- The Black-Scholes formula for a caplet whose underlying rate is observed in j periods and paid in $j+1$ periods is

$$\text{Caplet Price} = m^{-1}\left[b^{j+1}F^j\Phi(d_j) - b^{j+1}K\Phi(d_j - (jh)^{1/2}v)\right]$$

$$\begin{aligned}
d_j &= \frac{\log(F^j/K) + (jh)v^2/2}{(jh)^{1/2}v} \\
F^j &= j\text{-period forward rate} \\
K &= \text{cap rate} \\
j &= \text{number of periods until rate is observed} \\
j+1 &= \text{number of periods until rate is paid} \\
m &= \text{number of payments per year} \\
h &= 1/m \ = \ \text{time between payments in years} \\
jh &= \text{number of years until rate is observed} \\
b^{j+1} &= (j+1)\text{-period discount factor} \\
\Phi &= \text{cumulative normal distribution function} \\
v &= \text{annualized volatility}
\end{aligned}$$

Comments:

- the only new issue here is the difference between when the rate is observed and when it's paid
- the forward rate F follows the same day count and compounding convention as Y
- the value of a cap is the sum of the values of its component caplets
- presumption: the underlying rate is lognormal

2. Black-Scholes Valuation of Caps (continued)

- Numerical example
  - Consider the following interest rate data:

Period (j)	Disc Factor	Spot Rate	Fwd Rate F^{j-1}
1	0.975365	4.989	5.052
2	0.949999	5.129	5.340
3	0.924837	5.209	5.442
4	0.899541	5.294	5.624
5	0.874550	5.362	5.715
6	0.849939	5.420	5.791

    (as usual, the data are based on the quote sheet)
    Keep these numbers in mind — we'll come back to them

  - Caplet prices for $K = 5.50\%$ are

Period ($j+1$)	Volatility	Caplet Price	Cap Price
1	none	none	none
2	12.50	0.0578	0.0578
3	15.00	0.1381	
4	16.50	0.2304	0.4264
5	17.00	0.2847	
6	17.50	0.3305	1.0414

  - Comments:
    * caps are sums of caplets
    * you might want to work through some of these calculations, but don't get bogged down

2. Black-Scholes Valuation of Caps (continued)

- Cap valuation:

$$\text{Cap Price} = m^{-1} \sum_j \left[b^{j+1} F^j \Phi(d_j) - b^{j+1} K \Phi(d_j - (jh)^{1/2} v) \right]$$

$$d_j = \frac{\log(F^j/K) + (jh)v^2/2}{(jh)^{1/2} v}$$

  – These things vary with j: forward rate F^j, discount factor b^{j+1}, "d_j"

  – These do not: volatility v, cap rate K

- Implied volatility: find the value of v that generates the observed price

  – Our example: if 2-year 5.5% cap price is 0.4264, implied volatility is 15.03%

  – Comment: this is a composite of the volatilities of the caplets, which are not generally the same for all maturities (in our example, they are 12.50, 15.00, and 16.5)

3. Black-Derman-Toy Valuation of Caps

- The short rate tree for BDT model

4.989	5.735	6.567	7.724	8.881	10.254
	4.806	5.312	6.117	6.983	8.006
		4.296	4.844	5.491	6.251
			3.836	4.318	4.880
				3.395	3.810
					2.975

- The state price tree (via Duffie's formula):

1.0000	.4877	.2369	.1146	.0552	.0264
	.4877	.4750	.3459	.2229	.1340
		.2381	.3478	.3375	.2718
			.1165	.2269	.2752
				.0571	.1391
					.0281

- The 6-month LIBOR tree:

5.051	5.818	6.676	7.875	9.081	10.521
	4.864	5.383	6.211	7.107	8.168
		4.343	4.903	5.567	6.349
			3.873	4.364	4.940
				3.424	3.847
					2.997

3. Black-Derman-Toy Valuation of Caps (continued)

- Where did this stuff come from?
  - Short rate tree:
    - volatilities are $(.125, .150, .165, .170, .175, .175)$
      (taken from caplet volatilities above)
    - drift parameters chosen to match current spot rates
      (same as those listed above)
  - State prices: computed using Duffie's formula
  - 6-month LIBOR:
    - one-period discount factor b related to short rate r by

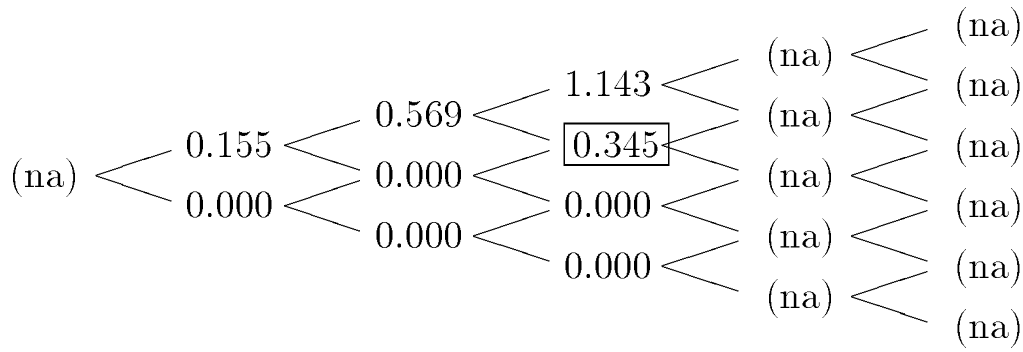
$$b \;=\; \exp(-rh/100) \quad\Longleftrightarrow\quad r \;=\; -(100/h)\log b$$

    - 6-month LIBOR Y related to b by

$$b \;=\; \frac{1}{1 + Yh/100} \quad\Longleftrightarrow\quad Y \;=\; (100/h)(1/b - 1)$$

3. Black-Derman-Toy Valuation of Caps (continued)

- 2-year semiannual interest rate cap at 5.5%
  - Cash flows are



Comments:

  * reminder: $(Y-K)^+$ paid one period later
  * if we push payments back a period, they don't fit into the tree (which node the following period?)
  * solution: discount the payments and shift them back a period
  * boxed node: payment is

$$0.5\frac{(Y-K)^+}{1+Y/200} \;=\; 0.5\frac{(6.211-5.500)^+}{1+6.211/200} \;=\; 0.345$$

    (think about this if it's not clear)
  * why only 3 periods? because we observe the final payment one period before it's scheduled to be paid

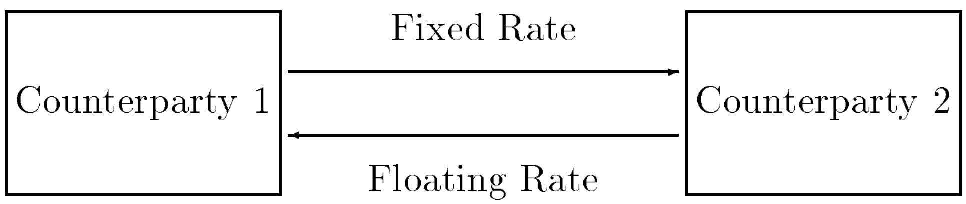
  - Value of option:
    * all-at-once method (multiply cash flows by state prices and add):

$$\text{Cap Price} \;=\; 0.461$$

    (similar to our earlier answer)

4. Swaps

- A (plain-vanilla) *interest rate swap* is an agreement between two parties to exchange fixed and floating interest payments:



  We say Counterparty 1 "pays fixed" and Counterparty 2 "receives fixed"

- Standard approach to valuation:
  - add principal to both sides
  - Counterparty 2 then has a long position in a bond and a short position in a floating rate note
  - bonds we value with discount factors and the FRN trades at par on reset dates

- Swap rates are par yields:
  - fixed payments of S/m are worth

$$\left(b_t^1 + \cdots b_t^\tau\right)(S/m) + b_t^\tau 100$$

    (τ is the tenor of the swap)
  - FRN worth 100 at start
  - if we choose the swap rate S to equate the initial values of the fixed and floating sides:

$$S \;=\; m \times 100 \times \frac{1 - b_t^\tau}{\sum_j b_t^j}$$

4. Swaps

- Numerical examples (same data as before):

Period (j)	Disc Factor	Swap Rate
1	0.975365	5.052
2	0.949999	5.194
3	0.924837	5.274
4	0.899541	5.358
5	0.874550	5.426
6	0.849939	5.483

Comments:

- swap rates are semi-annually compounded
- details for maturity = 4 periods (2 years):

$$\begin{aligned}
\sum_{j=1}^{4} b^j &= 0.975365 + 0.949999 + 0.924837 + 0.899541 \\
&= 3.7497 \\
1 - b^4 &= 1 - 0.899541 \;=\; 0.10045 \\
S &= 200 \times (0.10045/3.7497) \;=\; 5.358
\end{aligned}$$

5. Forward-Starting Swaps

- A *forward-starting swap* is an agreement to enter into a swap n periods (say) in the future

- Valuation follows a similar route:

  – fixed payments of F/m are worth

$$\left(b_t^{n+1} + \cdots b_t^{n+\tau}\right)(F/m) + b_t^{n+\tau}100$$

  – FRN worth 100 at start, b_t^n 100 now

  – the forward swap rate F equates the values of the fixed and floating sides:

$$F \;=\; m \times 100 \times \frac{b_t^n - b_t^{n+\tau}}{\sum_j b_t^{n+j}}$$

- Example: 2-year swap starting in 1 year

$$\begin{aligned}
\sum_{j=3}^{6} b^j &= 3.5489 \\
b^2 - b^6 &= 0.949999 - 0.849939 \;=\; 0.10006 \\
F &= 200 \times (0.10006/3.5489) \;=\; 5.639
\end{aligned}$$

  This is a little higher than either the 2- or 3-year swap rates, since it's based on the "3 to 6" part of the forward rate curve

- Forward-starting swaps are the underlying assets for common swaptions

6. Swaptions

- Terminology for common swaptions
  - A *payer swaption* is an option to enter into a pay fixed swap: a call option on a pay fixed swap
  - A *receiver swaption* is an option to enter into a receive fixed swap: a call option on a receive fixed swap or a put option on a pay fixed swap
  - Typically European
  - A "1 into 5" is a one-year option to enter into a 5-year swap
  - Strike generally quoted as a rate
  - General notation: n is the maturity of the option and τ is the tenor or term of the underlying swap
- Other structures
  - American or Bermudan call features
  - Extendible: the option to extend the maturity of an existing swap
  - Cancellable: the option to cancel an existing swap

6. Swaptions (continued)

- One view of swaptions (option on a bond):
  - a claim to a swap at rate K in n periods:

    $$V_{t+n}(K)^+$$

    where V_{t+n} is the value of the swap in n periods
  - requires only the ability to value fixed-rate bonds (the floating side trades at par)
  - we generally use this approach when we value swaptions in binomial models

6. Swaptions (continued)

- Another view of swaptions (option on the swap rate):
  - the owner of a payer swaption has a claim to the stream of equal payments
  $$m^{-1}(S_{t+n} - K)^+$$
  in periods $t+n+1$, $t+n+2$, ... , $t+n+\tau$
  - why?
    * an optional short position in a "rate-K" swap
    * ... is equivalent to a short position in a rate-K swap and a long position in a swap at the market rate S at time $t+n$ (since the latter is priced to trade at zero)
    * each swap position has a fixed rate bond on one side and a floating rate note on the other
    * the floating rate notes cancel (one is long, the other short, and they have the same value)
    * ... leaving us with a short position in a rate-K bond and a long position in a rate-S_{t+n} bond
    * ... which generates cash flows of $m^{-1}(S_{t+n} - K)^+$ at dates $t+n+1$, ... , $t+n+\tau$
  - we use this approach with Black-Scholes valuation

7. Black-Scholes Valuation of Swaptions

- The Black-Scholes formula for a payer swaption is

$$
\begin{aligned}
\text{Swaption Price} &= m^{-1}\left[BF\Phi(d) - BK\Phi(d-(nh)^{1/2}v)\right] \\
d &= \frac{\log(F/K) + (nh)v^2/2}{(nh)^{1/2}v} \\
F &= \text{forward-starting swap rate} \\
K &= \text{strike rate} \\
m &= \text{number of payments per year} \\
h &= 1/m \;=\; \text{time between payments in years} \\
n &= \text{maturity of swaption in length-}h\text{ periods} \\
nh &= \text{maturity of swaption in years} \\
B &= b_t^{n+1} + b_t^{n+2} + \cdots + b_t^{t+\tau} \\
\Phi &= \text{cumulative normal distribution function} \\
v &= \text{annualized volatility}
\end{aligned}
$$

Comments:

- Think of this as a call option on S with strike K
- B values the series of payments of $(S_{t+n} - K)^+$
- Common variants express B in different ways
- By expressing F as a percentage we get the price per 100 notional

7. Black-Scholes Valuation of Swaptions (continued)

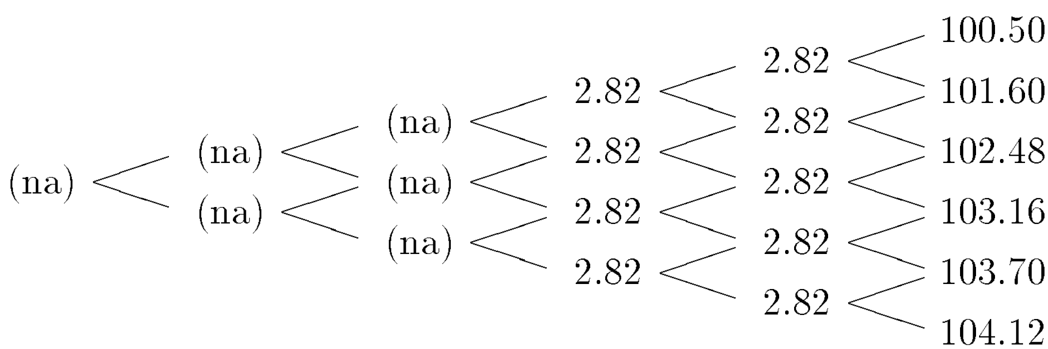
- Numerical example: 1-year option on 2-year swap
  - Recall from forward-starting swap: $F = 5.639$, $B = 3.5489$
  - Volatility (from quote sheet): $v = 15.55\%$
  - Swaption prices for various strikes

Strike	Swaption Price
5.639	0.6201
5.750	0.5326
6.000	0.3698
7.000	0.0648

(prices are in dollars per hundred notional)

8. Black-Derman-Toy Valuation of Swaptions

- Value a 1-year payer swaption on a 2-year swap
- Underlying: a forward-starting swap with maturity $n = 2$ and tenor $\tau = 4$ (both measured in half-years)
  - Regard f-s swap as short position in fixed rate bond and long position in floating rate note
  - Floating rate note trades for 100 in 2 periods, when the swap starts
  - Fixed rate bond has cash flows of $F/2 = 2.8195$ in all states in periods (3,4,5,6) plus principal of 100 in period 6:



Comments:

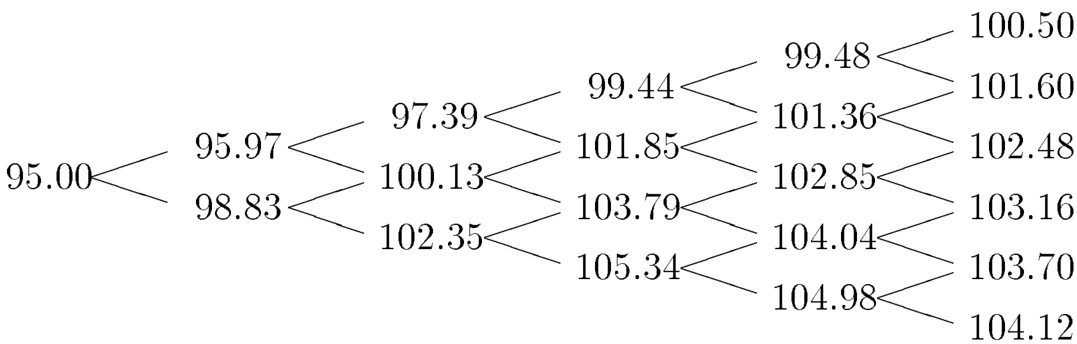
* last row: take cash flows from following period, and discount them back one period using the appropriate short rate; eg,

$$104.12 \ = \ 2.82 + \exp(-2.975/200)(102.82)$$

* earlier periods: fixed payments during the period of the swap

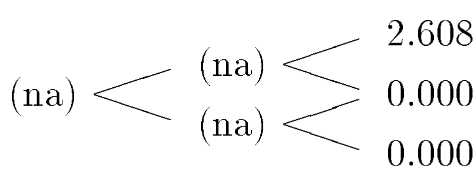
8. Black-Derman-Toy Valuation of Swaptions (continued)

- Valuation of underlying (continued)
  - Price path for fixed rate bond:



  - Swap includes short position in bond (above) plus long position in floating rate note (100 in period 2)

- Cash flows for swaption are therefore:



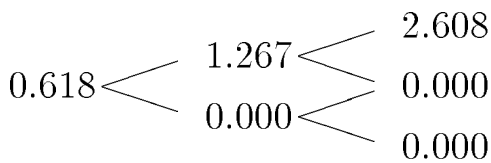
Comment:

$$2.608 \;=\; 100 - 97.392$$

(long position in note, short position in bond)

8. Black-Derman-Toy Valuation of Swaptions (continued)

- Recursive swaption valuation:



Comment: the usual approach

- All-at-once valuation:

$$0.618 \; = \; 0.2369 \times 2.608$$

(0.2369 is the state price for node (2,2))

- Value (0.618) similar to Black-Scholes calculation (0.620)

Summary

1. Common options on fixed income instruments include caps (and floors) and swaptions

2. Dealers often quote implied volatilities, which are based on Black-Scholes applied to the underlying *rates*

3. Valuation is often done with binomial models, which are valued the same way we value any derivative instrument

4. The Black-Derman-Toy model is based on lognormal rates, and in that sense is similar to applications of Black-Scholes