American Options

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1. **Introduction/Motivation**

   - Many popular options allow for "early" exercise, *i.e.*, can be exercised before the option expires.
   
   - Common for exchange-traded options and embedded options
   
   - Less common for OTC options

   - **Examples:**
     
     - Embedded options
       
       * callable (putable) bonds
       * sinking funds
       * mortgage-backed securities
     
     - Bond and bond futures options
     
     - American swaptions

   - Slightly different than American options on stocks → stocks don’t mature but bonds typically do
2. Problems with Black-Scholes

- Key assumption in deriving Black-Scholes is European option

- If you knew the exact date an American option will be exercised, then it is equivalent to a European options $\rightarrow$ B-S formula applies

- Value clearly changes as the exercise date changes

- Terminology: “exercise policy” is a rule that specifies the option holder’s actions at each time and at each state

- Cash flows of the option depend critically on the option holder’s exercise policy

- Is knowledge of the exercise policy sufficient for the use of Black-Scholes?
  
  - **No.** Since the exercise policy depends on the future state, it does not predict with certainty the date the option will be exercised.
  
  - $\rightarrow$ cash flows are “path dependant”
2. Problems... cont’d

- We will assume “optimal exercise,” i.e., the option holder chooses the action (exercise or not exercise) that maximizes the option’s value

  - Note: As usual, this abstracts from tricky issues like liquidity, incentive issues, etc.

- Bottom Line: Black-Scholes can at best be used to place a lower bound on an American option, but not to accurately value the option (even if all of the other assumptions are satisfied)

  - since the holder of an American option always has the option of waiting until expiration to exercise (effectively converting the American option to a European option), an American option can’t be worth less than an otherwise identical European option (which can be prices with the B-S formula)
3. Put-Call Non-Parity

- Put-call parity built on the idea of simultaneously buying an call and selling a put on the same underlying with the same strike, K, and maturity
  
  - this locks in the future price of the underlying at K

- With the possibility of early exercise, this logic breaks down

- Example:
  
  - 1 year to maturity
  - option to exercise puts and calls in 6 months or 1 year
  - price falls dramatically over the first 6 months
  - → induces the holder of the put to exercise and the writer of the put to finance the cash flow
  
  - no guarantee that the price will rise over the second 6 months to offset this loss with a profit from the call

- → put-call strategy with American options is risky
4. Payoff and Price of an American Option

- At each point in time during the life of an American option, the holder can exercise the option or leave it alone.

- Optimal exercise implies that they will take the strategy that is worth more.

- Cash flow at date $\tau$ of an American call with $n$-period left until expiration:
  \[
  \max\{S_\tau - K, C_{n-1}^\tau\}
  \]
  - $S$ is the price of the underlying.
  - $K$ is the strike price.
  - $C_{n-1}^\tau$ is the price of an American call at strike $K$ that expires in $n - 1$ periods.

- American call price:
  \[
  C_n^t = E_t [M_{t,n+1} \max\{S_{t+1} - K, C_{t+1}^{n-1}\}]
  \]

- American call is found “recursively”
  - $n = 1$: (one-period European)
    \[
    C_{1,n+1}^1 = E_{t+1} [M_{t+1,n+1} (S_{t+1} - K)^+]
    \]
  - $n = 2$:
    \[
    C_{2,n+2}^2 = E_{t+2} [M_{t+2,n+2} \max\{S_{t+2} - K, C_{t+2}^1\}]
    \]
  - and so on ...
Cash flow at date $\tau$ of an American put with $n$-period left until expiration:
$$\max\{K - S_\tau, P_{\tau}^{n-1}\}$$
- $S$ is the price of the underlying
- $K$ is the strike price
- $P^{n-1}$ is the price of an American put at strike $K$ that expires in $n - 1$ periods

American put price:
$$P_t^n = E_t \left[M_{t,t+1} \max\{S_{t+1} - K, P_{t+1}^{n-1}\}\right]$$

American put is also found “recursively”
- $n = 1$: (one-period European)
  $$P_{t+n-1}^1 = E_{t+n-1} \left[M_{t+n-1,t+n} (K - S_{t+n})^+\right]$$
- $n = 2$:
  $$P_{t+n-2}^2 = E_{t+n-2} \left[M_{t+n-2,t+n-1} \max\{K - S_{t+n-1}, P_{t+n-1}^1\}\right]$$
- and so on ...
4. Payoff and Price... cont’d

- Note: It is easy to see why Black-Scholes breaks down

- B-S works for $n = 1$:

$$C^1_{t+n-1} = b^1_{t+n-1} F^1_{t+n-1} \Phi(d) - b^1_{t+n-1} K \Phi(d - \omega)$$

- recall that $d = \log(F^1_{t+n-1}/K)/\omega + \omega/2$

- this implies that when trying to calculate the expectation at date $t + n - 2$ to calculate $C^2_{t+n-2}$, we are trying to evaluate an expectation of some highly nonlinear functions of random variables, e.g., $\Phi(d)$

- the “normality” assumptions that help for deriving B-S are of little use here
5. Valuation with Trees

- Backward recursions are very easy to calculate on trees

- Accounts for the popularity of discrete methods in general and binomial models in particular

- Example: 2-year, 5.5% coupon bond
  - Interest rate tree:

    $\begin{align*}
    5.5400 & \leftarrow 6.0040 & \leftarrow 6.9150 & \leftarrow 7.8640 \\
    4.7210 & \leftarrow 5.4370 & \leftarrow 4.8620 & \leftarrow 3.8230 \\
    \end{align*}$

  - 6-month zeros (discount factors):

    $\begin{align*}
    0.9730 & \leftarrow 0.9709 & \leftarrow 0.9666 & \leftarrow 0.9622 \\
    0.9769 & \leftarrow 0.9735 & \leftarrow 0.9763 & \leftarrow 0.9812 \\
    \end{align*}$

  - Coupon-bond prices (ex-coupon):

    $\begin{align*}
    100.0019 & \leftarrow 99.089 & \leftarrow 98.606 & \leftarrow 98.863 \\
    100.955 & \leftarrow 100.021 & \leftarrow 100.311 & \leftarrow 100.823 \\
    \end{align*}$
5. Valuation... cont’d

- Value European call, $n = 3$, $K = 100$

- Value American call, $n = 3$, $K = 100$

- Call Premium?

- **Note:** This methodology is the same for all trees!
6. Example: Callable bond

- The buyer of a callable bond may be viewed as being:
  - long a noncallable bond with the same maturity as the callable one
  - short an option on this bond

- Price of a callable bond

\[ P^{(callable)} = P^{(non-call)} - C \]

- In the example, at time-0, the callable bond is worth

\[ 100.0019 - 0.5003 = 99.5016 \]

- Interest rate delta of a callable bond is equal to the delta of the noncallable minus the delta on the option → callable bond has less interest rate sensitivity than the noncallable