Fixed Income Models: Assessment and New Directions

1. Uses of models

2. Assessment criteria

3. Assessment

4. Open questions and new directions
   - stochastic volatility (Cox-Ingersoll-Ross)
   - volatility inputs (Heath-Jarrow-Morton)
   - multiple factors
   - pricing kernels and risk-neutral probabilities
1. Use of Models

- Valuation of new structures
  - approach 1: look at comparable assets in the market
  - approach 2: models guarantee consistency with other assets, but not necessarily good answers
    (“even a bad model can be tuned to get some prices right”)

- Hedging and sensitivity
  - what’s the DV01 of a swaption?
  - what’s the sensitivity to a 20 bp rise in 5-7 year spot rates?
  - models can be used to do “partial derivative” calculations, but the answer depends on the model

- Why different models for different purposes?
  - modeling = finding useful short cuts
  - you take different short cuts depending on the purpose
  - the key is to understand where short cuts hurt you
2. Assessment

- Questions for models:
  - does it reproduce current spot rates?
  - does it reproduce current term structure of volatility?
  - are sensitivities and hedge ratios reasonable?
  - does it reproduce swaption volatility matrix?
  - can volatility change randomly?
  - can it reproduce volatility smiles and skews?
  - does it allow “twists” in spot rate curve?

- Summary assessment:

<table>
<thead>
<tr>
<th></th>
<th>Ho-Lee</th>
<th>Black-Derman-Toy</th>
<th>Hull-White</th>
</tr>
</thead>
<tbody>
<tr>
<td>spot rates?</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>vol term str?</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>sensitivities?</td>
<td>no</td>
<td>no?</td>
<td>maybe</td>
</tr>
<tr>
<td>vol matrix?</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>random vol?</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>smiles?</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>twists?</td>
<td>no</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

- All models have strengths and weaknesses
2. Assessment (continued)

- Ho and Lee
  - the innovation was to match current spot rates
  - didn’t do the next step: volatilities

- Black, Derman, and Toy
  - matches volatility term structure
  - short rate or log of short rate? (not clear)

- Hull and White
  - premise: lack of mean reversion above is a mistake
  - but where does it hurt us?
    (sensitivities? volatilities on long bonds/swaps?)
  - technical trick: build mean reversion into trinomial tree

- For all the above:
  - future volatility known now
  - volatility matrix implied by models, not a flexible input
  - no volatility smiles
  - one-factor structures
3. Vasicek Revisited

- The model
  \[-\log m_{t+1} = \frac{\lambda^2}{2} + z_t + \lambda \varepsilon_{t+1}\]
  \[z_{t+1} = (1 - \varphi)\theta + \varphi z_t + \sigma \varepsilon_{t+1}\]

- Pricing relation:
  \[b_{t+1}^{n+1} = E_t (m_{t+1} b_t^n)\]

- Solution is log-linear:
  \[-\log b_t^n = A_n + B_n z_t\]

  with
  \[A_{n+1} = A_n + \frac{\lambda^2}{2} + B_n (1 - \varphi)\theta - (\lambda + B_n \sigma)^2 / 2\]
  \[B_{n+1} = 1 + B_n \varphi\]

  (start with \(A_0 = B_0 = 0\))

- Forward rates
  - definition is
    \[f_t^n = \log(b_t^n / b_t^{n+1})\]
  - Vasicek solution is
    \[f_t^n = \varphi^n z_t + \text{constant}\]

  - note: sensitivity to \(z\) declines with \(n\) if \(0 < \varphi < 1\)
    (think: Ho-Lee and BDT set \(\varphi = 1\))
4. Stochastic Volatility

- Cox-Ingersoll-Ross model

\[- \log m_{t+1} = (1 + \lambda^2/2)z_t + \lambda z_t^{1/2} \varepsilon_{t+1} \]
\[ z_{t+1} = (1 - \varphi)\theta + \varphi z_t + \sigma z_t^{1/2} \varepsilon_{t+1} \]

Comments:
- conditional variance varies with \( z \): \( \text{Var}_t z_{t+1} = \sigma^2 z_t \)
- square root keeps \( z \) positive
- mean reversion

- Solution is log-linear (like Vasicek):

\[- \log b^n_t = A_n + B_n z_t \]

with
\[ A_{n+1} = A_n + B_n (1 - \varphi)\theta - (\lambda + B_n \sigma)^2 / 2 \]
\[ B_{n+1} = 1 + \lambda^2 / 2 + B_n \varphi - (\lambda + B_n \sigma)^2 / 2 \]

(start with \( A_0 = B_0 = 0 \))

- Illustrates a standard trick for allowing volatility to vary “stochastically”
5. Heath, Jarrow, and Morton

- Overview:
  - approach based on forward rates
  - allows input of volatility matrix
  - many variants — we focus on a linear one

- Pricing relation for forward rates:
  \[ 1 = E_t(m_{t+1}R_{t+1}) \]
  \[ \log R_{t+1} = r_t - \sum_{j=1}^{n} (f_{t+1}^{j-1} - f_t^j) \]

  (nothing here but basic algebra)

- Behavior of forward rates (we start to get specific here):
  \[ f_{t+1}^{n-1} = f_t^n + \alpha nt + \sigma nt \varepsilon_{t+1} \]

  Comments:
  - note the volatility input \( \sigma_{nt} \): varies with date \( t \) and maturity \( n \) (a matrix!)
  - linear structure common, not necessary
  - Vasicek is similar:
    \[ f_{t+1}^{n-1} = f_t^n + \text{constant} + \varphi^{n-1} \varepsilon_{t+1} \]
    (note the specific relation of volatility to maturity)
  - return is
    \[ \log R_{t+1} = r_t - \sum_{j=1}^{n} \alpha_{jt} - \sum_{j=1}^{n} \sigma_{jt} \varepsilon_{t+1} \]
    \[ = r_t - A_{nt} - S_{nt} \varepsilon_{t+1} \]
    (\( A \) and \( S \) are partial sums)
  - other versions have multiple \( \varepsilon \)'s
5. Heath, Jarrow, and Morton (continued)

- Arbitrage-free pricing
  - a pricing kernel:
    \[- \log m_{t+1} = \delta_t + \lambda_t \varepsilon_{t+1}\]
  - pricing relation imposes restrictions:
    \[A_{nt} - \lambda_t S_{nt} - \frac{S_{nt}^2}{2} = 0\]

- Calibration
  - inputs:
    * current forward rates: \(\{f^n_t\}_n\)
    * volatility matrix: \(\{\sigma_{nt}\}_{n,t}\)
  - drift parameters \(\{\alpha_{nt}\}\) chosen to satisfy arbitrage relation
  - open question: set \(\lambda_t = 0\)? (more shortly)

- Implementation on trees
  - a Wall Street standard
  - at each node we have complete forward rate curve
  - branches don’t typically “recombine”
6. Volatility Smiles and Skews

- Approaches:
  - fit smooth curve to observed implied volatilities
    (but: not all smooth curves are arbitrage-free)
  - “implied binomial trees” that allow volatility to vary across
    states as well as dates (see Chriss’s book)
  - continuous models that extend Black-Scholes logic to
    non-normal distributions

- Gram-Charlier smiles
  - in Vasicek, normal ε leads to Black-Scholes prices for zeros
  - Gram-Charlier expansion adds terms for skewness (γ₁) and
    kurtosis (γ₂) to the normal (where γ₁ = γ₂ = 0)
  - Wu’s approximation of a volatility smile:
    \[ v = \sigma \left[ 1 - \frac{\gamma_1}{3!} d - \frac{\gamma_2}{4!} (1 - d^2) \right] \]
    where
    \[ d = \frac{\log(F/K) + v^2/2}{v} \]
  - intuition:
    * positive skewness raises value of out-of-the-money calls
    * positive kurtosis raises probability of extreme events
      and value of out-of-the-money puts and calls
7. Multi-Factor Models

- Problems with one-factor models
  - changes in rates of all maturities tied to a single random variable ($\varepsilon$)
  - shifts in spot rate curve come in only one type (parallel?)
  - correlation of spot rates across maturities restricted
  - spreads typically not variable enough

- Example: Two-factor Vasicek
  - the model:
    \[
    \begin{align*}
    \log m_{t+1} &= \delta + \sum_i (\lambda_i^2/2 + z_{it} + \lambda_i \varepsilon_{i,t+1}) \\
    z_{i,t+1} &= \varphi z_{it} + \sigma_i \varepsilon_{i,t+1}
    \end{align*}
    \]
  - solution includes
    \[
    f^n_i = \delta + \frac{1}{2} \sum_i \left[ \lambda_i^2 - \left( \lambda_i + \frac{1 - \varphi^n_i}{1 - \varphi_i} \sigma_i \right)^2 \right] + \sum \varphi^n_i z_{it}.
    \]
  - standard solution: $\varphi_1$ close to one, $\varphi_2$ smaller $\Rightarrow$
    * $z_1$ generates almost parallel shifts, $z_2$ “twists”
    * long rates dominated by $z_1$, spreads by $z_2$
  - generates better behavior of spreads
8. Pricing Kernels and Risk-Neutral Probabilities

- We’ve taken two approaches to valuation
  - a pricing kernel (Vasicek, for example)
  - risk-neutral probabilities (binomial models)

  How are they related?

- Two-period state prices
  - $q_u$ is value now of 1 dollar in up state next period
  - $q_d$ is value now of 1 dollar in down state next period
  - valuation of cash flows $(c_u, c_d)$ follows
    \[ p = q_u c_u + q_d c_d \]
  - one-period discount factor is $b = q_u + q_d = \exp(-rh)$

- Risk-neutral probabilities
  - define $\pi_u^* = q_u/(q_u + q_d)$, $\pi_d^* = q_d/(q_u + q_d)$
  - note: $(\pi_u^*, \pi_d^*)$ are positive and sum to one (they’re probabilities!)
  - valuation follows
    \[ p = \exp(-rh)(\pi_u^* c_u + \pi_d^* c_d) \]

- Pricing kernel
  - define $q_u = \pi_u m_u$, $q_d = \pi_d m_d$ (“true probabilities”)
  - valuation follows
    \[ p = \pi_u m_u c_u + \pi_d m_d c_d \]
8. Pricing Kernels and Risk-Neutral Probabilities (cont’d)

- Where’s the pricing kernel in a binomial model?
  - figure it out:
    
    $$-\log m_{t+1} = \delta^* + r_t + \lambda \varepsilon_{t+1},$$
    
    with
    
    $$\delta^* = \pi \log(\pi_u/\pi_u^*) + \pi_d \log(\pi_d/\pi_d^*)$$
    
    $$2\lambda = \log(\pi_u/\pi_d^*) - \log(\pi_d/\pi_d^*)$$
    
    - summary: $\lambda$ is built into the difference between true and risk-neutral probabilities (and is zero when the two are equal)

- Is $\lambda = 0$ a mistake?
  - in principle yes: if kernel is wrong, valuation is wrong
  - caveat: Black-Scholes pricing doesn’t depend directly on $\lambda$
    (so maybe it’s not a bad mistake in general)
  - bottom line: who knows?
Summary

1. Models are
   - simplifications of reality
   - ways to ensure consistency of pricing across assets
   - only as good as (and floors) and swaptions

2. Binomial models capture some of the elements of observed asset prices, but most versions leave some issues open:
   - stochastic volatility
   - volatility smiles (again, more work)
   - multiple factors (possible, just more work)
   - can we ignore $\lambda$?

3. Modeling remains as much art as science