Start out by talking about examples of situations in which contracts (either explicit or implicit) are in place and point to the particular form of asymmetric information.

1) PhD student and GSIA. The school devotes faculty time and money to the student. In exchange, the student produces papers and interacts with the faculty. There are two types of uncertainty here: the school does not know how smart and skilled is the student. Moreover, there are random shocks to the productivity of the student himself. In general, the school bases the allocation on the results (grades, papers, participation...)

2) Car Insurance (Collision insurance). The contract works in such a way that you have the accident and make the claim. Based upon the claim, the insurance pays. However, the insurance does not really know whether the guy had an accident. Example of Naples.

The objective of contract theory is to model formally this kind of relationships, by assuming:

1) objective functions for the agents involved (payoff function)
2) distribution of bargaining power
3) distribution of information
4) an equilibrium solution

and then figure out the arrangement that satisfy that definition of equilibrium. Everything will be clearer as we go along.

Positive approach: under which condition the equilibrium contract is what we observe?

Normative approach: what is the equilibrium contract under some conditions?

The models of the theory of contracts can be distinguished along several dimensions. A crucial distinction is between complete and incomplete contracts.

In the case of complete contracts, all variables that may have an impact on the conditions of the contractual relationship during its whole durantion (i.e. all variables on which the provisions of the contract may depend) are taken into account when the contract is negotiated and signed. Thus the contract itself may be contingent on a very large number of variables. This assumption implies that no unforeseen contingency may arise as the relationship evolves.

Any change in the economic environment just activates the ad hoc provisions of the contract.

When the above assumption does not hold, we say that the contract is incomplete. There are several reasons why, in the real world, a contract may be incomplete.

- Negotiating a contract is often a costly business, which mobilizes managers and lawyers. It must therefore be that at some point the cost of taking into account an improbable contingency outweigh the benefits of wrting a specific cluase in the contract. The contract should then be signed without this clause.

- The unability (or unwillingness) of courts or other third parties to verify ex post the values taken by certain variables observed by all contractants is another reason why contract will be incompleted. It is no use conditioning the contract on a variable if nobody can settle the disputes that may arise.

- Finally, bounded rationality may force the parties to neglect some variables whose effect on the relationship they find difficult to evaluate.

For all of these reasons, contracts typically only take into account a limited number of variables that may be the most relevant ones, or simply those that are most easily verifiable by a court. During the relationship, some unforeseen contingencies may arise, that have an impact on the conditions of the relationship and the contract gives no clue as to how the parties should react. As a result, the parties may want to renegotiate the contract.

In this course we will consider exclusively complete contracts. In particular, we will focus on situations in which there are two economic agents: an informed party, whose information is relevant for the common welfare, and the uninformed party. In general, this is a bilateral monopoly[1] situation. This implies that we cannot go very far unless we specify how the parties are going to bargain over the terms of exchange. In this respect , we will make quite a drastic assumption, by allocating all bargaining power to one of the parties. This party, known as *principal*, will make a *take it or leave it* offer

[1]As we will see a the course progresses, principal-agent models also apply to situations with a continuum of infinitesimal agents, each of whom interacts with the principal but not with the other agents.

to the other party, known as *agent*. The agent will either accept or reject. If the agent accepts, the parties execute the contract.

According to how the asymmetric information is modeled, we distinguish two kinds of principal-agent contracts. Adverse Selection and Moral Hazard. **Adverse Selection** is the case in which the uncertainty is with respect to some inherent characteristic of the agent. i.e. there is asymmetric information before the contract is signed. Example: health insurance.

In the case of Moral Hazard instead, there is no ex-ante asymmetry. The asymmetry arises after the contract is signed. We distinguish two cases.

Hidden Action. Here the agent has to take an action that as an effect on the Principal's payoff function, but the action itself is not observable to the principal. Example: Firm vs CEO.

***************** Time Line from the book by Macho-Stadler

Hidden information.

Here after the contract is signed, some event occurs, that is observable to the agent only. Example: a contract between the owner of a plot of land and the farmer that actually cultivates the land. The harvest is stochastic, and is not observable to the owner.

Since I have never taught this class before, I know only the lower bound on what we will be doing. Hopefully, there is going to be time to do more. At the beginning I will go slow. Then I will go faster and faster.

We will proceed as follows:
- Start with Hidden Information model.
- Then we will see and Hidden Action model.
- Finally (ad the end of the course) the adverse selection case.
References:
Salanie (1997), chapter 1.
Macho-Stadler and Perez-Castrillo (1997), chapter 1.

Reference: Townsend (1982)

Mechanism Design and Revelation principle: Fudenberg and Tirole (1991), section 7.3.

Bottom line.

With public information about individuals' endowments, the optimal insurance arrangement is to have full-insurance. Moreover, this outcome remains the same when individuals are allowed to repeat arrangements over time.

With private information, there is no insurance in the one-shot game: agents will always claim to have received a low endowment and thus receive a constant payment every period. However, as opposite to the public information case, the repetition of the arrangement can now yield some insurance by exploring long-term relationship.

The model.

We consider an economy with two agents. One, which we call principal, is risk-neutral. The other, which we call agent, is risk-averse. The principal receives a constant endowment of $\overline{y}$ every period. His preferences are represented by the linear utility in consumption: $v(c) = c$. The risk averse agent receives an endowment $y_i \in Y$ (Y finite set) in each period with probability π_i (where $\sum_i \pi_i = 1$) and has a utility function $u(c)$ that satisfies $u' > 0$ and $u'' < 0$, $u^{'}(0) = \infty$.

The one-period problem with symmetric information.

Consider first the one-period Pareto problem with public information about the endowments. The efficient allocation is given by solving the following problem:

$$\max_{\{T_i\}} \sum_i \pi_i(\overline{y} - T_i) \tag{1}$$

subject to:

$$\sum_i \pi_i u(y_i + T_i) = w \tag{2}$$

where T_i is the transfer that the risk neutral agent makes to the risk averse one when the state is i.

Assuming an interior solution (alternatively, considering $u(c) : \Re \to \Re$, of which an example is the negative exponential utility function, $u(c) =$

$-\exp(-Ac)$ with $A > 0$), the solution for the problem above can be characterized by:

$$
\begin{aligned}
-\pi_i + \lambda \pi_i u'(y_i + T_i) &= 0 && (3) \\
u'(y_i + T_i) &= \frac{1}{\lambda} && (4)
\end{aligned}
$$

and we are able to conclude that:

$$y_i + T_i = c \quad \text{for all } i \tag{5}$$

which is to say that agents get full-insurance from the efficient risk allocation.

In other words, here the contract is a schedule $T_i = u^{-1}(w) - y_i$. By moving w we can figure out the set of Pareto-Optimal Allocations. For every w in fact the payoffs are $\sum_i \pi_i(\overline{y} - u^{-1}(w) + y_i)$ for the principal and w for the agent.

The one-period problem with asymmetric information.

Contract theory formalizes this problem as three-step game of incomplete information, where the income of the agent is private information.

In step 1, the principal designs a 'mechanism' or contract, or incentive scheme. A mechanism is a game in which the agent send a costless message, and the principal provides an allocation that depends on the realized messages. The allocation is a decision about the level of some observable variable (in this case the transfer). In step 2, the agent either accepts or rejects the contract. An agent who rejects the contract gets some exogenously specified "reservation utility". In step 3, if the agent accepted the mechanism, plays the game specified by the mechanism itself.

A contract here defines a message space Ω and a game form to announce the messages. In this case (if we limit to pure strategies) the message is a mapping $Y \rightarrow \Omega$. The set Ω can be anything. For simplicity here we assume that is bounded. Because income is private information, the allocation T can depend on income only through the agent's message. Thus the allocation will be a function $T : \Omega \rightarrow \Re$. The Bayesian Equilibrium for this game is going to be functions $\{T^*(\varpi), \varpi^*(y_i)\}$ such that:

$$\{T^*(\varpi)\} = argmax \ \sum_i \pi_i(\overline{y} - T(\varpi^*(y_i)))$$

Here it seems important to me that the function $T^*(\varpi)$ is defined for every ϖ.

$$\forall i\colon\ u(y_i + T^*(\varpi^*(y_i))) \geq u(y_i + T^*(\varpi)) \quad \forall \varpi \neq \varpi^*(y_i)\,.$$

Revelation principle (Myerson (1979)). It states that the principal can content herself with 'direct' mechanisms, in which the message space is the income space and all the agents announce their type truthfully. ****** Change notation, in particular ϖ. Do not confuse ϖ and w.

Proposition. Suppose that a mechanism with message space Ω and allocation function $T^*(\varpi)$ has a Bayesian Equilibrium

$$\varpi^*(\cdot) = \{\varpi^*(y_i)\}_{y_i \in Y}$$

Then there exists a direct-revelation mechanism ($\overline{T}^* = T^* \circ \varpi^*$) such that the message space is the income space Y and such that there exists a Bayesian equilibrium in which all agents announce their types truthfully.

Consider the new message space Y, so that the agent gives a message $\widehat{y}_i$ which in principle is different from the true y_i. A Bayesian equilibrium for the new game is given by functions $\left\{\overline{T}^*(\widehat{y}_i), \widehat{y}^*(y_i)\right\}$ such that

$$\left\{\overline{T}^*(\widehat{y}_i)\right\} = argmax \ \sum_i \pi_i(\overline{y} - \overline{T}_i(\widehat{y}_i(y_i)))$$

$$\forall i\colon\ u(y_i + \overline{T}^*(\widehat{y}_i(y_i))) \geq u(y_i + \overline{T}^*(\widehat{y}_j)) \quad \forall \widehat{y}_j \neq \widehat{y}_i.$$

Basically the Revelation Principle says that $\widehat{y}^*(y_i) = y_i$ is a Bayesian equilibrium of the new game.

Proof. Consider the allocation rule $\overline{T}^*(\widehat{y}_i) \equiv T^*(\varpi^*(\widehat{y}_i))$.

Straight by definition:

$$\forall i\colon\ u\left[y_i + \overline{T}^*(y_i)\right] = u\left[y_i + T^*(\varpi^*(y_i))\right]$$

The condition for Bayesian equilibrium in the original message game:

$$\forall i\colon\ u(y_i + T^*(\varpi^*(y_i))) = \underset{\varpi\in\Omega}{sup}\ u\left[y_i + T^*(\varpi(y_i))\right].$$

Finally:

$$\forall i\colon\ \underset{\varpi\in\Omega}{sup}\ u(y_i + T^*(\varpi(y_i))) \geq \underset{\widehat{y}_i\in Y}{max}\ u(y_i + \overline{T}^*(\widehat{y}_i)).$$

, since $\{\varpi^*(\widehat{y}_i)\}_{\widehat{y}_i\in Y} \subset \Omega$.

This last weak inequality expresses the fact that in the direct-revelation mechanism everything is as if the agent picked an announcement in the subset of messages $\{\varpi^*(\widehat{y}_i)\}_{\widehat{y}_i\in Y}$, which is a subset of Ω. The agent thus has, at most, as many posssibilities for deviating as in the original game. In other words:

$$\underset{\widehat{y}_i\in Y}{max}\ u(y_i + \overline{T}^*(\widehat{y}_i)) = \underset{\varpi\in\{\varpi^*(\widehat{y}_i)\}_{\widehat{y}_i\in Y}}{max}\ u(y_i + T^*(\varpi))$$

End of First Class.

To analyze the one-period game with private information, we make use of the revelation principle and include the truth-telling constraints in the Pareto problem. The game is:

Message space: Y.

Allocation: $\{T(\widehat{y}_i)\}_{\widehat{y}_i \in Y} = \{T_i\}$. The allocation is what is commonly referred to as the contract.

$$\max_{\{T_i\}} \sum_i \pi_i(\overline{y} - T_i) \tag{6}$$

subject to:

$$\begin{aligned} \sum_i \pi_i u(y_i + T_i) &= w \qquad (7) \\ u(y_i + T_i) &\geq u(y_i + T_j) \quad \forall i, j. \qquad (8) \end{aligned}$$

No insurance result.

Note that, since $u(\cdot)$ is strictly monotone, we obtain from (8) that $T_i \geq T_j$, $\forall j$. By interchanging i and j in (8), we are also able to conclude that $T_j \geq T_i$, $\forall i$.

Hence, and without further considerations, we obtain that $T_i = T_j = T$ for all i and for all j, which is to say that individuals do not get any insurance in the one-period game with private information.

The two-period problem with symmetric information.

Consider now the two-period version of the above environments.

Here the contract is simply given by functions $T{:}Y \to \Re$ and $T'{:}Y \times Y \to \Re$ or schedules $\{T_i, T_{ij}\}$.

The Pareto problem under public information is the solution to the following problem:

$$\max_{\{T_i, T_{ij}\}} \sum_i \pi_i \left[(\overline{y} - T_i) + \beta \sum_j \pi_j(\overline{y} - T_{ij})\right] \tag{9}$$

subject to the consistency requirement:

$$\sum_i \pi_i \left[u(y_i + T_i) + \beta \sum_j \pi_j u(y_j + T_{ij})\right] = w \tag{10}$$

where T_{ij} is the transfer from the risk neutral agent to the risk averse agent when the first period state was i and the second period state is j.

We thus allow agents to engage in long-term arrangements. That is: we allow for transfer to depend on current and past outcomes. Notice that $T_{ij} = T_j$ for all i is a particular case of the transfer system that the economy is allowed to choose. (That is: it is the case of no long-term arrangements).

Also, this formulation assumes that both agents have the same discount factor $0 < \beta < 1$.

Ignoring nonnegativity constraints (or with a negative exponential utility for the risk averse agent), the solution to the problem can be characterized by:

$$-\beta\pi_i\pi_j + \lambda\beta\pi_i\pi_j u'(y_j + T_{ij}) = 0 \Leftrightarrow u'(y_j + T_{ij}) = \frac{1}{\lambda} \tag{11}$$

$$-\pi_i + \lambda\pi_i u'(y_i + T_i) = 0 \Leftrightarrow u'(y_i + T_i) = \frac{1}{\lambda} \tag{12}$$

and hence we can conclude that:

$$y_i + T_i = y_j + T_{ij} = c \quad \forall i, j \tag{13}$$

where the fact that we get exactly $c_i = c_j$ has to do with the assumption of equal discount factors for both agents.

We thus obtain the *full-insurance* result, despite the fact that the game is repeated one more time. Observe that (13) implies that allowing for second period transfers to depend on the first period state is irrelevant: the left-hand-side of this equation says that the same level of (constant) consumption can be obtained with repeated static arrangements.

Bottom line: repetition does not add anything. Exactly the same allocation can be achieved by repeating the static contract twice.

The two-period problem with asymmetric information.

Message space: Y.

Allocations: $T : Y \to \Re$ and $T' : Y \times Y \to \Re$. Realization are iid. We will use the following notation:

$\{T(\widehat{y}_i)\}_{\widehat{y}_i \in Y} = \{T_i\}$ and $\{T'(\widehat{y}_i, \widehat{y}_j)\}_{(\widehat{y}_i, \widehat{y}_j) \in Y \times Y} = \{T_{ij}\}$

You have to tell that a modified version of the Revelation Principle holds. You have to make sure it is optimal to reveal truthfully at any information node. The first IC below is standard. The second imposes that the agent won't lie in the first period, given that he won't lie in the last.

The allocation is what is commonly referred to as the contract.On the other hand, the Pareto problem under private information is the solution to:

$$\max_{\{T_i,T_{ij}\}} \sum_i \pi_i \left[(\overline{y} - T_i) + \beta \sum_j \pi_j (\overline{y} - T_{ij}) \right] \tag{14}$$

subject to:

$$\sum_i \pi_i \left[u(y_i + T_i) + \beta \sum_j \pi_j u(y_j + T_{ij}) \right] = w \tag{15}$$

$$u(y_j + T_{ij}) \geq u(y_j + T_{ik}) \quad \forall i, j, k \tag{16}$$

$$u(y_i + T_i) + \beta \sum_j \pi_j u(y_j + T_{ij}) \geq \tag{17}$$

$$\geq u(y_i + T_k) + \beta \sum_j \pi_j u(y_j + T_{kj}) \quad \forall i, k$$

where (17) is the first period truth-telling constraint and (16) the second period one. This last constraint says that it is always better to tell the truth in the second period, no matter the first period state (i), for any actual second period endowment (j) and for any *potential first period report* (k).

Following the same reasoning as before, we can interchange j and k in (16) and use the strict monotonicity of $u(\cdot)$ to conclude that $T_{ij} = T_{ik} = T_i$ for each i and for all j and k. That is, agents receive no insurance in the second period. This is really a result we should expect, since the second period is essentially like a static one-shot arrangement: there is no next period to make its outcome contingent on this period's reports.

Using this result, we can simplify the notation by defining $T_i' \equiv T_{ij}$, the second period transfer as a function of the first period state only. Additionally, since second period transfers are independent of the second period reports, the constraint (16) actually becomes irrelevant under this notation.

The above problem can thus be rewritten as:

$$\max_{\{T_i,T_i'\}} \sum_i \pi_i \left[(\overline{y} - T_i) + \beta \sum_j \pi_j (\overline{y} - T_i') \right] \tag{18}$$

subject to:

$$\begin{aligned} \sum_i \pi_i \left[u(y_i + T_i) + \beta \sum_j \pi_j u(y_j + T_i') \right] &= w & (19) \\ u(y_i + T_i) + \beta \sum_j \pi_j u(y_j + T_i') &\geq & (20) \\ &\geq u(y_i + T_k) + \beta \sum_j \pi_j u(y_j + T_k') \quad \forall i, k. \end{aligned}$$

If we prevent long-term arrangements in this problem, by constraining $T_i' = T_k' = T' \ \forall i, k$, then the summations in (20) cancel. We can again interchange i and k in this constraint and, by using $u' > 0$, we conclude that $T_i = T_k$ and so agents get no insurance.

Therefore, we have to allow for long-term arrangements for some insurance to be possible here.

Consider a particular case of this problem. We assume there are two states only, with $y_1 > y_2$.

The incentive constraint (20) can be now written as:

$$\begin{aligned} u(y_1 + T_1) + \beta \sum_j \pi_j u(y_j + T_1') &\geq u(y_1 + T_2) + \beta \sum_j \pi_j u(y_j + T_2') & (21) \\ u(y_2 + T_2) + \beta \sum_j \pi_j u(y_j + T_2') &\geq u(y_2 + T_1) + \beta \sum_j \pi_j u(y_j + T_1'). & (22) \end{aligned}$$

It is also useful to rewrite the above problem as one of choosing directly utility levels (current and future) instead of transfers. Hence, define the following variables (for $i = 1, 2$):

$$\begin{aligned} u_i &\equiv u(y_i + T_i) \\ c_i &\equiv y_i + T_i \\ w_i' &\equiv \sum_j \pi_j u(y_j + T_i') \end{aligned}$$

and define also $c \equiv u^{-1}(\cdot)$, the level of consumption required to attain a given level of utility. From the properties of u we can readily conclude that $c' > 0$ and $c'' > 0$.

The original choice variables in terms of the transformed problem can be recovered by (for $i = 1, 2$):

$$\begin{aligned} T_i &= c(u_i) - y_i \\ T_i' &\equiv V(w_i') \end{aligned}$$

where we can conclude from the properties of c that $V' > 0$ and $V'' > 0$.

The Pareto problem written in terms of choosing (u_1, u_2, w_1', w_2') instead of (T_1, T_2, T_1', T_2') is then:

$$\max_{\{u_1,u_2,w_1',w_2'\}} \sum_i \pi_i \left[\overline{y} - c(u_i) + y_i + \beta\left(\overline{y} - V(w_i')\right)\right] \tag{23}$$

subject to:

$$\begin{aligned} \sum_i \pi_i \left(u_i + \beta w_i'\right) &= w && (24) \\ u_1 + \beta w_1' &\geq u\left(y_1 + c(u_2) - y_2\right) + \beta w_2' && (25) \\ u_2 + \beta w_2' &\geq u\left(y_2 + c(u_1) - y_1\right) + \beta w_1' && (26) \end{aligned}$$

or still, by defining $\Delta \equiv y_1 - y_2 > 0$ ($\overline{y}$ and y_i are constants):

$$\min_{\{u_1,u_2,w_1',w_2'\}} \sum_i \pi_i \left[c(u_i) + \beta V(w_i')\right] \tag{27}$$

subject to:

$$\begin{aligned} \sum_i \pi_i \left(u_i + \beta w_i'\right) &= w && (28) \\ u_1 + \beta w_1' &\geq u\left(c(u_2) + \Delta\right) + \beta w_2' && (29) \\ u_2 + \beta w_2' &\geq u\left(c(u_1) - \Delta\right) + \beta w_1' && (30) \end{aligned}$$

where we will attach a multiplier λ to (28) and multipliers μ_1 and μ_2 to the IC constraints, respectively (29) and (30). It follows from the properties of c and V that the objective function in problem P is strictly convex.

The two benchmark solutions in terms of this transformed problem are:

- Full-insurance:

When the endowments are publicly observable, the two IC constraints become irrelevant and the (interior) solution is $u_1 = u_2$ (implying $c_1 = c_2$, since $u'' < 0$) and $w_1' = w_2'$.

- No insurance:

When we constrain transfers to be noncontingent, i.e. $T_1 = T_2 = T$, the solution is trivially given by $c_1 = y_1 + T$ and $c_2 = y_2 + T$, which means that $c_1 = c_2 + \Delta$.

Therefore, in the intermediate case where endowments are private information but we allow for long-term arrangements, we also expect an intermediate solution c_1 such that $c_1^{FI} = c_2 < c_1 < c_1^{NI} = c_2 + \Delta$.

This is exactly the content of the next proposition.

In the solution to problem P, $c_1 > c_2$, $w_1' > w_2'$, (29) binds and (30) is slack.

$w_1' > w_2'$ and (29) binding imply that $c_1 < c_2 + \Delta$, as we want to show.

The proof follows in 5 steps.

Step 1. Both (29) and (30) cannot be slack.

Suppose they are both slack. Then the solution to problem P can be obtained without these two constraints. However, as we saw above, this yields the full-insurance solution, $u_1 = u_2$ and $w_1' = w_2'$. By replacing this solution in (29), we find that $u(c_1) > u(c_1 + \Delta)$, a contradiction since $\Delta > 0$ and $u' > 0$.

Step 2. (29) must bind.

By contradiction, assume (29) is slack. Then (30) must bind by Step 1.

Hence, (29) and (30) together imply that $w_2' \geq w_1'$. Otherwise, if $w_1' > w_2'$, we could slightly increase w_2' and decrease w_1' such that $\sum_{i=1}^{2} \pi_i w_i'$ is constant, so (28) and both (29) and (30) still hold. Strict convexity of V would in turn imply that we were not at a minimizer in the first place, a contradiction.

Also, if (29) is slack we can conclude that $u_2 \geq u_1$. Otherwise, if $u_1 > u_2$, we could slightly increase u_2 and decrease u_1 such that $\sum_{i=1}^{2} \pi_i u_i$ is constant, so (28) and both (29) and (30) still hold. Strict convexity of c would in turn imply that we were not at a minimizer in the first place, a contradiction.

Therefore, $w'_2 \geq w'_1$ and $u_2 \geq u_1$ if (29) is slack.

However, if (29) is slack, then:

$$\begin{aligned} u_1 + \beta w'_1 &> u\left(c(u_2) + \Delta\right) + \beta w'_2 \\ &\geq u\left(c(u_1) + \Delta\right) + \beta w'_1 \\ &> u\left(c(u_1)\right) + \beta w'_1 \end{aligned}$$

where the first step follows from $c' > 0$ and since $w'_2 \geq w'_1$ and $u_2 \geq u_1$ and the second step follows from $\Delta > 0$ and $u' > 0$. This is obviously a contradiction, since $u\left(c(u_1)\right) = u_1$.

Hence, (29) must bind, whereas (30) may or may not bind.

Suppose for now that (30) actually binds.

Step 3. $u_1 > u_2$.

The Lagrangean:

$$\sum_i \pi_i \left[c(u_i) + \beta V(w'_i)\right] + \lambda \left[\sum_i \pi_i \left(u_i + \beta w'_i\right) - w\right] + \\ \mu_1 \left[u_1 + \beta w'_1 - u\left(c(u_2) + \Delta\right) - \beta w'_2\right] + \mu_2 \left[u_2 + \beta w'_2 - u\left(c(u_1) - \Delta\right) - \beta w'_1\right]$$

The first order conditions for problem P are:

$$(u_1) \quad : \quad -\pi_i c'(u_1) + \lambda \pi_1 + \mu_1 - \mu_2 u'(c_1 - \Delta) c'(u_1) = 0 \qquad (31)$$

$$(u_2) \quad : \quad -\pi_i c'(u_2) + \lambda \pi_2 + \mu_2 - \mu_1 u'(c_2 + \Delta) c'(u_2) = 0 \qquad (32)$$

$$(w'_1) \quad : \quad -\pi_1 V'(w'_1) + \lambda \pi_1 + \mu_1 - \mu_2 = 0 \qquad (33)$$

$$(w'_2) \quad : \quad -\pi_2 V'(w'_2) + \lambda \pi_2 + \mu_2 - \mu_1 = 0. \qquad (34)$$

Notice that $c'(u_1) = \frac{1}{u'(c_1)}$.
By adding (29) and (30):

$$u_1 + u_2 = u(c_2 + \Delta) + u(c_1 - \Delta). \tag{35}$$

Define $f(x) \equiv u(c_2 + x) + u(c_1 - x)$. Some useful properties of f are that it is strictly concave since $f'(x) = u'(c_2 + x) - u'(c_1 - x)$ and $f''(x) = u''(c_2 + x) + u''(c_1 - x) < 0$ and also that $f(0) = f(\Delta)$, since $f(0) = u_1 + u_2$. Hence, $f(x)$ must look like:

which implies that we must have $f'(0) > 0$ (otherwise we violate strict concavity).

But then $f'(0) = -u'(c_1) + u'(c_2) > 0$ implies that

$$u'(c_2) > u'(c_1).$$

Since $u'' < 0$, u' is decresasing and thus $c_1 > c_2$. In turn, by monotonicity of u, we have that $u_1 > u_2$, which is what we wanted to show.

Step 4. $w_1' > w_2'$.

From (**??**) and (**??**) we get:

$$c'(u_1) = \lambda + \frac{1}{\pi_1}\left[\mu_1 - \mu_2 \frac{u'(c_1 - \Delta)}{u'(c_1)}\right] \tag{36}$$

$$c'(u_2) = \lambda + \frac{1}{\pi_2}\left[\mu_2 - \mu_1 \frac{u'(c_2 + \Delta)}{u'(c_2)}\right]. \tag{37}$$

Since $u_1 > u_2$ from Step 3, $c'' > 0$ implies that $c'(u_1) > c'(u_2)$ and this is equivalent to:

$$\frac{1}{\pi_1}\left[\mu_1 - \mu_2 \frac{u'(c_1 - \Delta)}{u'(c_1)}\right] > \frac{1}{\pi_2}\left[\mu_2 - \mu_1 \frac{u'(c_2 + \Delta)}{u'(c_2)}\right] \tag{38}$$

where $\frac{u'(c_1-\Delta)}{u'(c_1)} > 1$ and $\frac{u'(c_2+\Delta)}{u'(c_2)} < 1$ by virtue of strict concavity of u.

Since $\mu_1 \leq \mu_2$ makes the LHS of (**??**) negative and the RHS positive, the only way this inequality may hold is when $\mu_2 < \mu_1$.

Using this fact on (**??**) and (**??**) we conclude that $V'(w_1') > \lambda > V'(w_2')$, and from strict convexity of V we finally get that $w_1' > w_2'$.

Now, recall we concluded that $u_1 > u_2$ and $w_1' > w_2'$ by assuming that (30) actually binds.

If it slacks instead we have $\mu_2 = 0$. However, we can check by replacing $\mu_2 = 0$ in the proofs of steps 3 and 4 that they will still follow, and hence we have $u_1 > u_2$ and $w_1' > w_2'$ irrespective of (30) being binding or slack.

Step 5. (30) is slack.

Suppose instead that (30) binds.

Then we saw that, by adding (29) and (30) we get:

$$u_1 + u_2 = u(c_2 + \Delta) + u(c_1 - \Delta) \tag{39}$$

which is equivalent to:

$$u(c_2) - u(c_2 - (c_2 + \Delta - c_1)) = u(c_2 + \Delta) - u(c_2 + \Delta - (c_2 + \Delta - c_1)) \tag{40}$$

Define $\Delta' \equiv c_2 + \Delta - c_1$. Then this equation can rewritten as:

$$u(c_2) - u(c_2 - \Delta') = u(c_2 + \Delta) - u(c_2 + \Delta - \Delta') \tag{41}$$

Recall from our previous remark that $w_1' > w_2'$ and (29) binding imply that $c_1 < c_2 + \Delta$, and so $\Delta' > 0$.

However, (**??**) contradicts strict concavity of u, as can be seen from the next figure:

and hence (30) must be slack.

This completes the proof of the Proposition.

Note that the above Proposition says that truthful revelation in the present entails giving a smaller transfer in the future if an agent reveals low endowment today (truthfully or not). That is, $w_1' > w_2'$ means $T_1' > T_2'$.

End of Second Class

Borrowing and Lending Scheme.

A relevant question is whether the apparent similarity between the efficient insurance contract and the pure borrowing and lending scheme actually means that the two are equivalent. It turns out that they are not, which means the pure borrowing and lending scheme is not optimal.

The pure borrowing and lending scheme would equalize marginal utility of consumption across the two periods, that is:

$$u'(c_i) = \beta(1+r)\sum_j \pi_j u'(c_{ij}) \tag{42}$$

where i and j are the states in the first and second period.

By evaluating (**??**) at first period states $i = 1, 2$

$$\frac{u'(c_1)}{u'(c_2)} = \frac{\sum_j \pi_j u'(c_{1j})}{\sum_j \pi_j u'(c_{2j})}. \tag{43}$$

Recall that $w_i' \equiv \sum_j \pi_j u(y_j + V(w_i'))$. By totally differentiating:

$$dw_i' = dV \sum_j \pi_j u^{'}(y_j + V(w_i'))$$

we get $V'(w_i') = \left[\sum_j \pi_j u'(c_{ij})\right]^{-1}$, and so (**??**) can be rewritten as:

$$\frac{u'(c_1)}{u'(c_2)} = \frac{V'(w_2')}{V'(w_1')} \Leftrightarrow \frac{V'(w_1')}{c'(u_1)} = \frac{V'(w_2')}{c'(u_2)}.$$

In the efficient contract, we have (by replacing $\mu_2 = 0$ in the first order conditions for problem P, (**??**)-(**??**)):

$$\begin{aligned} c'(u_1) &= \lambda + \frac{\mu_1}{\pi_1} \\ c'(u_2) &= \lambda - \frac{\mu_1}{\pi_2}\frac{u'(c_2+\Delta)}{u'(c_2)} \\ V'(w_1') &= \lambda + \frac{\mu_1}{\pi_1} \\ V'(w_2') &= \lambda - \frac{\mu_1}{\pi_2}. \end{aligned}$$

Hence, $c'(u_1) = V'(w_1')$ and $c'(u_2) > V'(w_2')$, implying that the efficient contract is characterized by:

$$\frac{V'(w_1')}{c'(u_1)} > \frac{V'(w_2')}{c'(u_2)}$$

where we had an equality in the pure borrowing and lending scheme.

The departure from efficiency in the pure borrowing and lending scheme occurs because agents would like to transfer income from the low endowment state to the high endowment one. Instead, the efficient contract makes consumption in the bad state relatively less desirable so that agents have the correct incentives not to lie when y_2 occurs in order to obtain the high transfer.

An illustration can be given for $\beta(1+r) = 1$.

In this case, the borrowing and lending scheme is characterized by

$$\begin{aligned} u'(c_1) &= (V'(w_1'))^{-1} \\ u'(c_2) &= (V'(w_2'))^{-1} \end{aligned}$$

and the efficient contract by

$$\begin{aligned} u'(c_1) &= (V'(w_1'))^{-1} \\ u'(c_2) &< (V'(w_2'))^{-1}. \end{aligned}$$

If agents were allowed to borrow or lend on top of the efficient contract, they would rather lend part of the transfer they get in the bad state (while the borrowing and lending possibility would not be used in the good state). The borrowing and lending scheme does not provide enough insurance since individuals get an higher c_2 in the efficient contract.

An alternative way to show the nonequivalence between the optimal and the borrowing and lending contracts is to note that the solution to the latter is obtained by replacing the incentive compatibility constraint in the Pareto problem by a "zero expected transfers" constraint: $T_1 + \beta T_1' = T_2 + \beta T_2' = 0$. It can be shown that the incentive constraints for both states are satisfied on inequality when agents can freely borrow or lend at rate r, which implies that we can improve on this allocation by, namely, increasing c_2 until the incentive constraint for the good state is satisfied with equality.

1 The infinite horizon case - The study of wealth distribution.

A final note concerns the study of wealth distribution. Since it is appropriate to think of expected utility as being approximately equal to wealth, this setup can be used to study the properties of the distribution of wealth across agents. Notice, for instance, that in the efficient contract incentives provide an intertemporal dependence of expected utilities: if an agent reports a low endowment today, expected utility must be lower, and conversely if an agent reports an high endowment.

A question of interest is to consider the infinite horizon version of the previous setup and to investigate whether there exists a limiting distribution of expected utilities. We know that this distribution, if it exists, will not be degenerate since different agents have different histories of shocks.

The **full insurance** contract when the time horizon is infinite yields $w = u(c) + \beta w$ or $w = \frac{u(c)}{1-\beta}$.

A parametric case that is widely used since it provides a closed-form solution is the negative exponential one. That is, $u(c) = -e^{-Ac}$ with $A > 0$ and $c \in \Re$, implying that consumption has a logarithmic form: $c(u) = -\frac{1}{A}\ln(-u)$.

Sequence Problem.

The risk-averse agent is offered some utility ω.

We use $\widehat{y}_t(y^t)$ to denote the report an agent plans (at date 0) to give about his date t endowment if he has actually experienced the history $y^t = \{y_0, y_1, ..., y_t\}$.

A **reporting strategy** is thus $\widehat{y} = \{\widehat{y}_t(y^t)\}_{t=0}^{\infty}$, where for all t, $\widehat{y}_t : Y^{t+1} \to Y$.

The truthful reporting strategy is is denoted by $\widehat{y}^* = \{\widehat{y}_t^*(y^t)\}_{t=0}^{\infty}$ where $\widehat{y}_t^*(y^t) = y_t$ for all t and $y^t \in Y^{t+1}$.

Let $T(\widehat{y}^t)$ denote the transfer that the individual w receives at date t on the basis of the reporting history $\widehat{y}^t$.

Let $-e^{-A\left(y_t+T\left(\widehat{y}^t\right)\right)} = -e^{-A[y_t-\widehat{y}_t]}e^{-A\left(\widehat{y}_t+T\left(\widehat{y}^t\right)\right)} \equiv e^{-A[y_t-\widehat{y}_t]}u\left[\widehat{y}^t\right]$ be the utility he receives from his consumption.

Thus the principal will assign a sequence $u = \{u(\widehat{y}^t)\}_{t=0}^{\infty}$ to the risk-averse agent.

If the principal chooses the sequence u and the agent chooses the reporting strategy $\widehat{y}$, then the agent receives a total discounted expected utility given by

$$U(u, \widehat{y}) = \sum_{t=0}^{\infty} \int_{Y^{t+1}} \beta^t u_t \left[\widehat{y}^t \left(y^t\right)\right] e^{-A[y_t-\widehat{y}_t]} d\mu^{t+1}.$$

Define an **allocation** as a sequence u such that

$$w = U(u, \widehat{y}^*)$$

(i.e. the allocation awards utility w if truthful reporting is chosen)
and

$$U(u, \widehat{y}^*) \geq U(u, \widehat{y}) \ \ \forall \widehat{y}.$$

(truthful reporting is optimal for the agent).

The principal will choose the allocation u that minimizes

$$\sum_{t=0}^{\infty} \beta^t \int_{Y^{t+1}} c\left[u\left(w, y^t\right)\right] d\mu^{t+1}.$$

Recursive Formulation.

Think now of the principal as choosing a sequence of functions (f_t, g_t) , functions of $(w, \widehat{y})$.

$f_t(w, \widehat{y})$ is the current utility an agent receives if his expected utility entitlement **from** t **on** is w and he announces the income $\widehat{y}$.

$w_t(w, \widehat{y})$ is the expected utlity entitlement an agent receives if his expected utility entitlement from t on is w and he announces the income $\widehat{y}$.

That is, we can think of the planner as summarizing a consumer's entire report history and his initial entitlement in a single

number w that represents his expected utility entitlement from the current period on.

Let

$$\sigma = \{f_t, g_t\}_{t=0}^{\infty}$$

denote a sequence of such functions.

A sequence σ defines a sequence u as follows. Let the sequence $\{w_t\}_{t=0}^{\infty}$ solve the difference equation $w_{t+1} = g_t(w_t, \widehat{y}_t)$, with initial value w_0. Then

$$u_t\left(w_0, \widehat{y}^t\right) = f_t\left[w_t\left(w_0, \widehat{y}^{t-1}\right), \widehat{y}_t\right]$$

In other words, a sequence σ generates a plan u. **Do the sequence thing.**

We call the sequence σ an allocation rule if it satisfies the two following conditions:$\forall t$ and $\forall w$

$$w = \sum_{i=1}^{2} \pi_i \left[f_t(w, y_i) + \beta g_t(w, y_i)\right] \ ,$$

$$f_t(w, y_1) + \beta g_t(w, y_1) \geq f_t(w, y_2) e^{-A[y_1 - y_2]} + \beta g_t(w, y_2)$$

$$f_t(w, y_2) + \beta g_t(w, y_2) \geq f_t(w, y_1) e^{-A[y_2 - y_1]} + \beta g_t(w, y_1)$$

Obviously an allocation rule will also generate a cost for the principal, given by

$$\begin{aligned} V(w_0) &= \sum_{i=1}^{2} \pi_i c\left[f_0(w_0, y_i)\right] + \sum_{t=1}^{\infty} \beta^t \sum_{i=1}^{2} \pi_i c\left[f_t\left(g_{t-1}(w_{t-1}, y_{t-1}), y_i\right)\right] \\ V(w_0) &= \sum_{i=1}^{2} \pi_i \left\{c\left[f_0(w_0, y_i)\right] + \beta V\left(g_0(w_0, y_i)\right)\right\} \end{aligned}$$

What we can prove in this environment:

a) If the allocation u attains w with total cost C, then there is an allocation rule σ that attains w with total cost C.

b) Suppose the allocation rule σ that attains w with total cost C and u is the utility plan generated by σ. Then u is an allocation, and u attains w with total cost C.

c) The function $V(w)$, that gives the minimum cost of achieving a given value w, solves the following Bellman equation.

d) The functions f_t, g_t are stationary. That is $f_t = f$ and $g_t = g$ for every t.

We can characterize the efficient contract, heuristically from the finite horizon setup, as the solution to the following Bellman equation:

$$V(w) = \min_{\{u_1, u_2, w_1', w_2'\}} \sum_i \pi_i \left[c(u_i) + \beta V(w_i') \right] \tag{44}$$

subject to:

$$\sum_i \pi_i \left(u_i + \beta w_i' \right) = w \tag{45}$$

$$u_1 + \beta w_1' \geq u\left(c(u_2) + \Delta \right) + \beta w_2' \tag{46}$$

$$u_2 + \beta w_2' \geq u\left(c(u_1) - \Delta \right) + \beta w_1'. \tag{47}$$

Here you have to say that: $f(w, \widehat{y}_1) = u_1$, $f(w, \widehat{y}_2) = u_2$, $g(w, \widehat{y}_1) = w_2'$, $g(w, \widehat{y}_2) = w_2'$.

In the finite horizon problem, the form of the value function was known to be given by $w_i' \equiv \sum_j \pi_j u(y_j + V(w_i'))$, whereas we do not know the general form of $V(w)$ in the above problem.

Some care is needed to formally justify the legitimacy of using the recursive formulation above.

Two interpretation of the above problem were given in the literature, both leading to this same formulation. The setup used in each of them differs in some respects:

- [Thomas and Worral (JET, 1990)] Two agents, one risk neutral and the other risk averse, leading to a principal-agent formulation. The risk-averse future utility becomes arbitrarily negative with probability one. The borrower gets deeper and deeper into debt (exponential utility allows for negative consumption), and consumption moves down as debt increases. The contract is therefore not very good at stabilizing consumption over time; nevertheless what apperas to be happening is that making future utilities low reduces the cost of inducing incentive compatibility, which is obtained by variations in future utility. So stability in consumption in the initial periods is obtained at a cost of variation in consumption over time. Thomas and Worral show that as $\beta \to 1$, the Second-Best Pareto frontier or (Costrained-Pareto optimal utilities) converge pointwise to the first-best Pareto Frontier (first-best utilitites), but not uniformly. **The principal becomes richer and richer. The agent becomes poorer and poorer. Crucial role for negative consumption. Ayiagari & Alvarez impose non-negative consumption and obtain a non-degenerate limiting distribution.**

- [Green (1987)] Large number of risk averse agents plus an intermediary which can borrow/lend from some outside source at the rate $\beta^{-1} - 1$. The risk averse agents do not have access to neither a storing technology nor to (say, foreign) borrowing/lending possibilities. The community of individuals is rich at the beginning, because $w > -1$, but it becomes poor over time. That implies that the intermediary must be borrowing from the outside source at the beginning of time and then paying back.

Differently from Atkeson and Lucas, a period-by-period feasibility constraint is not imposed.

An educated guess for V is of the same form as c, that is,

$$V(w) = \kappa_1 - \frac{1}{\kappa_0} \ln(-w)$$

.

Since V is strictly convex and strictly increasing, we can use the previous results to conclude that (**??**) binds and (**??**) is slack, and can be ignored. After replacing the guess for V in the Bellman equation, the first order conditions are: (maximize the negative of the objective function)

$$\begin{aligned}
\frac{\pi_1}{Au_1} + \lambda\pi_1 + \mu &= 0 \\
\frac{\pi_2}{Au_2} + \lambda\pi_2 - \mu e^{-A\Delta} &= 0 \\
\frac{\pi_1}{\kappa_0 w_1'} + \lambda\pi_1 + \mu &= 0 \\
\frac{\pi_2}{\kappa_0 w_2'} + \lambda\pi_2 - \mu &= 0.
\end{aligned}$$

****

Adding the first two equations (multiply and divide the two conditions by u_1 and u_2, respectively):

$$\frac{\pi_1}{A} + \lambda\pi_1 u_1 + \mu u_1 + \frac{\pi_2}{A} + \lambda\pi_2 u_2 - u_2\mu e^{-A\Delta} = 0.$$

and thus:

$$\frac{1}{A} + \lambda\left(\pi_1 u_1 + \pi_2 u_2\right) + \mu\left(u_1 - u_2 e^{-A\Delta}\right) = 0$$

Adding the last two equations (multiply the two conditions by w_1' and $w_2^{'}$, respectively):

$$\frac{\beta\pi_1}{k_0} + \lambda\beta\pi_1 w_1' + \beta\mu w_1' + \frac{\beta\pi_2}{k_0} + \lambda\beta\pi_2 w_2^{'} - \mu\beta w_2^{'} = 0$$

and thus:

$$\frac{\beta}{\kappa_0} + \lambda\beta\left(\pi_1 w_1' + \pi_2 w_2'\right) + \beta\mu\left(w_1' - w_2'\right) = 0.$$

Adding the two resulting equations:

$$\frac{1}{A} + \frac{\beta}{k_0} + \lambda\left[\sum_i \pi_i\left(u_i + \beta w_i'\right)\right] + \mu\left[u_1 + \beta w_1' - u\left(c(u_2) + \Delta\right) - \beta w_2'\right] = 0.$$

Using the incentive compatible and promise keeping constraints:

$$\frac{1}{A} + \frac{\beta}{\kappa_0} + \lambda w = 0$$

and we can solve for

$$\lambda = -\frac{1}{w}\left(\frac{1}{A} + \frac{\beta}{\kappa_0}\right)$$

****

By the Envelope theorem, $V'(w) = \lambda$.

Thus

$$-\frac{1}{w\kappa_0} = -\frac{1}{w}\left(\frac{1}{A} + \frac{\beta}{\kappa_0}\right)$$

and thus

$$\kappa_0 = A(1-\beta).$$

We have obtained the first parameter of the value function.

And also

$$\lambda = -\frac{1}{w}\frac{1}{A(1-\beta)}$$

****

Now we want to figure out the parameter μ.

Replacing κ_0 in the first order conditions:

$$u_1 = -\frac{\pi_1}{A(\lambda\pi_1 + \mu)} \quad (48)$$

$$u_2 = -\frac{\pi_2}{A(\lambda\pi_2 - \mu e^{-A\Delta})} \quad (49)$$

$$w_1' = -\frac{\pi_1}{\kappa_0(\lambda\pi_1 + \mu)} \quad (50)$$

$$w_2' = -\frac{\pi_2}{\kappa_0(\lambda\pi_2 - \mu)} \quad (51)$$

and replacing these in the promise keeping constraint:

$$-\frac{\pi_1^2}{A(\lambda\pi_1 + \mu)} - \frac{\pi_2^2}{A(\lambda\pi_2 - \mu e^{-A\Delta})} - \beta\left[\frac{\pi_1^2}{\kappa_0(\lambda\pi_1 + \mu)} + \frac{\pi_2^2}{\kappa_0(\lambda\pi_2 - \mu)}\right] = w.$$

Divide everywhere by w:

$$-\frac{\pi_1^2}{A(\lambda\pi_1 w + \mu w)} - \frac{\pi_2^2}{A(\lambda\pi_2 w - w\mu e^{-A\Delta})} - \beta\left[\frac{\pi_1^2}{\kappa_0(\lambda\pi_1 w + \mu w)} + \frac{\pi_2^2}{\kappa_0(\lambda\pi_2 w - \mu w)}\right] = 1.$$

To solve for μ, we guess it to be of the same form as λ: $\mu = -\frac{1}{w}\mu^*$ with $\mu^* > 0$ being a constant. Now use $\lambda = -\frac{1}{w}\frac{1}{A(1-\beta)}$ and $\mu = -\frac{1}{w}\mu^*$. Replacing this guess above,

$$-\frac{\pi_1^2}{A\left(\frac{-\pi_1}{A(1-\beta)} - \mu^*\right)} - \frac{\pi_2^2}{A\left(\frac{-\pi_2}{A(1-\beta)} + \mu^* e^{-A\Delta}\right)} - \beta\left[\frac{\pi_1^2}{\kappa_0\left(\frac{-\pi_1}{A(1-\beta)} - \mu^*\right)} + \frac{\pi_2^2}{\kappa_0\left(\frac{-\pi_2}{A(1-\beta)} + \mu^*\right)}\right] = 1$$

and (since $k_0 = A(1-B)$)

$$-\frac{\pi_1^2}{A\left(\frac{-\pi_1 - \mu^* k_0}{A(1-\beta)}\right)} - \frac{\pi_2^2}{A\left(\frac{-\pi_2 + k_0\mu^* e^{-A\Delta}}{A(1-\beta)}\right)} - \beta\left[\frac{\pi_1^2}{\kappa_0\left(\frac{-\pi_1 - \mu^* k_0}{A(1-\beta)}\right)} + \frac{\pi_2^2}{\kappa_0\left(\frac{-\pi_2 + \mu^* k_0}{A(1-\beta)}\right)}\right] = 1$$

or, equivalently,

$$\frac{\pi_1^2(1-\beta)}{\pi_1 + \mu^*\kappa_0} + \frac{\pi_2^2(1-\beta)}{\pi_2 - k_0\mu^* e^{-A\Delta}} + \beta\left[\frac{\pi_1^2}{\pi_1 + \mu^* k_0} + \frac{\pi_2^2}{\pi_2 - \mu^*\kappa_0}\right]$$

or

$$\frac{\pi_1^2}{\pi_1 + \mu^*\kappa_0} + \frac{\pi_2^2(1-\beta)}{\pi_2 - \mu^*\kappa_0 e^{-A\Delta}} + \beta\frac{\pi_2^2}{\pi_2 - \mu^*\kappa_0} = 1. \tag{52}$$

****

Define

$$\phi(\mu^*) \equiv \frac{\pi_1^2}{\pi_1 + \mu^*\kappa_0} + \frac{\pi_2^2(1-\beta)}{\pi_2 - \mu^*\kappa_0 e^{-A\Delta}} + \beta\frac{\pi_2^2}{\pi_2 - \mu^*\kappa_0}$$

and note that $\phi(0) = 1$, $\phi'(0) < 0$ and that $\phi'' > 0$ since ϕ is the sum of three convex functions in μ^*. Also $\phi(\mu^*) \to +\infty$ as $\mu^* \to \frac{\pi_2}{\kappa_0} > 0$.

The graph of this function $\phi(\mu^*)$ thus looks like:

The function $\phi(\mu^*)$ is continuous. It follows that there exists $0 < \mu^* < \frac{\pi_2}{\kappa_0}$

that solves (??). Note that $\mu^* = 0$ (and hence $\mu = 0$ as well) is not a solution to (??) since it implies that the IC constraint is slack and thus the full-insurance obtains. We know that it cannot be.
****

Using $\mu = -\frac{1}{w}\mu^*$ to solve for current and expected future utilities in (??)-(??):

$$u_1 = \frac{\pi_1 w (1-\beta)}{\pi_1 + \mu^* \kappa_0} \quad (53)$$

$$u_2 = \frac{\pi_2 w (1-\beta)}{\pi_2 - \mu^* \kappa_0 e^{-A\Delta}} \quad (54)$$

$$w_1' = \frac{\pi_1 w}{\pi_1 + \mu^* \kappa_0} \quad (55)$$

$$w_2' = \frac{\pi_2 w}{\pi_2 - \mu^* \kappa_0} \quad (56)$$

and it follows that

$$u_1 > u_2$$

and that

$$w_2' < w < w_1'$$

.

We can replace (??)-(??) in the Bellman equation and solve for κ_1. We do not do it here.
****

To characterize the limiting distribution of expected utilities, observe that:

$$w_{t+1} = \begin{cases} \frac{\pi_1 w_t}{\pi_1 + \mu^* \kappa_0} & \text{if } y_t = y_1 \\ \frac{\pi_2 w_t}{\pi_2 - \mu^* \kappa_0} & \text{if } y_t = y_2 \end{cases}$$

and so,

$$\ln(-w_{t+1}) = \begin{cases} \ln(-w_t) + \ln \frac{\pi_1}{\pi_1 + \mu^* \kappa_0} & \text{if } y_t = y_1 \\ \ln(-w_t) + \ln \frac{\pi_2}{\pi_2 - \mu^* \kappa_0} & \text{if } y_t = y_2 \end{cases} .$$

We can write that

$$\ln(-w_{t+1}) = \ln(-w_t) + \varepsilon \ , \ E(\varepsilon) = \pi_1 \ln \frac{\pi_1}{\pi_1 + \mu^* \kappa_0} + \pi_2 \ln \frac{\pi_2}{\pi_2 - \mu^* \kappa_0}$$

Therefore,

$$\begin{aligned} E \ln(-w) &= \pi_1 \ln(-w_1') + \pi_2 \ln(-w_2') \\ &= \ln(-w) + \pi_1 \ln \frac{\pi_1}{\pi_1 + \mu^* \kappa_0} + \pi_2 \ln \frac{\pi_2}{\pi_2 - \mu^* \kappa_0} \\ &\equiv \ln(-w) + \psi(\mu^*) . \end{aligned} \tag{57}$$

Since endowment shocks are i.i.d., $\ln(-w_t)$ is i.i.d. as well and $\ln(-w_t)$ is a random walk with positive drift ($\psi(\mu^*) > 0$). The drift is positive since :

$$\begin{aligned} \psi'(\mu^*) &= -\frac{\pi_1 \kappa_0}{\pi_1 + \mu^* \kappa_0} + \frac{\pi_2 \kappa_0}{\pi_2 - \mu^* \kappa_0} \Rightarrow \psi'(0) = 0 \\ \psi''(\mu^*) &= \frac{\pi_1 \kappa_0^2}{\pi_1 + \mu^* \kappa_0} + \frac{\pi_2 \kappa_0^2}{\pi_2 - \mu^* \kappa_0} > 0. \end{aligned}$$

Since $\ln(-w_t)$ is a random walk with positive drift (sub-martingale), it follows that $\ln(-w_t) \to +\infty$ as $t \to +\infty$ and hence that $w_t \to -\infty$ and $u_t \to -\infty$.

We thus conclude that there does not exist a stationary distribution of expected utilities, as risk-averse agents get increasingly poorer over time, while the risk neutral agent gets increasingly richer (so that the aggregate resource constraint keeps balanced).

The results will change dramatically if we impose $c \geq 0$ (or some other lower bound for consumption). In particular, a stationary distribution for utilities obtains in this case (see Aiyagari and Alvarez (1995)).

A final point is due about the relationship between the finite horizon and the infinite horizon setups.

In both cases, $\beta = 0$ leads to no insurance. This follows directly from the incentive compatible constraints.

However, as $\beta \to 1$ we converge to the full-insurance result in the infinite horizon setup, since we can check from (**??**)-(**??**) that $w_1' \to w$, $w_2' \to w$ and $\frac{u_1}{u_2} \to 1$.

Radner and Townsend provide an interpretation of this result. They set $\beta = 1$ and let $T \to \infty$. The economy is then able to sustain full-insurance allocations as the time horizon extends to infinity. An important point is that agents only care about in how many periods they will receive a compensation when signing the contract. If $\beta < 1$, the argument of Radner and Townsend does not work anymore.